THE RATIONALITY PROBLEM FOR FORMS OF $\overline{M}_{0,n}$

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Abstract. Let $X$ be a del Pezzo surface of degree 5 defined over a field $F$. A theorem of Yu. I. Manin and P. Swinnerton-Dyer asserts that every Del Pezzo surface of degree 5 is rational. In this paper we generalize this result as follows. Recall that del Pezzo surfaces of degree 5 over a field $F$ are precisely the $F$-forms of the moduli space $\overline{M}_{0,5}$ of stable curves of genus 0 with 5 marked points. Suppose $n \geq 5$ is an integer, and $F$ is an infinite field of characteristic $\neq 2$. It is easy to see that every twisted $F$-form of $\overline{M}_{0,n}$ is unirational over $F$. We show that

(a) If $n$ is odd, then every twisted $F$-form of $\overline{M}_{0,n}$ is rational over $F$.
(b) If $n$ is even, there exists a field extension $F/k$ and a twisted $F$-form $X$ of $\overline{M}_{0,n}$ such that $X$ is not retract rational over $F$.

1. Introduction

Let $X$ be a del Pezzo surface of degree 5 defined over a field $F$. Yu. I. Manin [Man63, Theorem 3.15] showed that if $X$ has an $F$-point, then $X$ is rational over $F$. P. Swinnerton-Dyer [SD72] then proved that $X$ always has an $F$-point; for alternative proofs of this assertion, see [SB92] and [Sko93]. In summary, one obtains the following result, published earlier by F. Enriques [E1897] (with an incomplete proof).

Theorem 1.1. (Enriques, Manin, Swinnerton-Dyer) Every del Pezzo surface of degree 5 defined over a field $F$ is $F$-rational. Equivalently, every $F$-form of $\overline{M}_{0,5}$ is $F$-rational.

The purpose of this paper is to generalize this celebrated theorem as follows. As usual, we will denote the moduli space of smooth (respectively, stable) curves of genus $g$ with $n$ marked points by $M_{g,n}$ (respectively, $\overline{M}_{g,n}$). Recall that these moduli spaces are defined over the prime field. A form of a scheme $X$ defined over a field $F$ is an $F$-scheme $Y$, such that $X$ and $Y$ become isomorphic over the separable closure $F^{sep}$. We will use the terms “form”, “$F$-form” and “twisted form” interchangeably throughout this paper. For a discussion of this notion and further references, see Section 2.

We now recall that $\overline{M}_{0,5}$ is a split del Pezzo surface of degree 5, and $F$-forms of $\overline{M}_{0,5}$ are precisely the del Pezzo surfaces of degree 5 defined over $F$. The main result of this paper is Theorem 1.2 below.

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Theorem 1.2. Let \( n \geq 5 \) be an integer and \( F \) be an infinite field of characteristic \( \neq 2 \).

(a) Assume \( n \) is odd. Then every \( F \)-form of \( \overline{M}_{0,n} \) is rational over \( F \).

(b) Assume \( n \) is even. If \( \text{Br}_2(F) \neq 0 \), then there exists an \( F \)-form \( X \) of \( \overline{M}_{0,n} \) such that \( X \) is not retract rational over \( F \).

Several remarks are in order.

(1) If \( F \) is assumed to be an infinite field of characteristic different from 2, Theorem 1.1 is recovered from Theorem 1.2(a) by setting \( n = 5 \).

(2) In part (b), \( \text{Br}_2 \) denotes the 2-torsion of the Brauer group. The condition that \( \text{Br}_2(F) \neq 0 \) is fairly mild; it is equivalent to the existence of a non-split quaternion algebra over \( F \). In particular, \( \text{Br}_2(F) \neq 0 \) if \( F = k(t_1,t_2) \) where \( k \) is an arbitrary field of characteristic \( \neq 2 \) and \( t_1,t_2 \) are independent variables.

(3) The assumption that \( F \) is infinite is only used in part (a); see Section 6. In part (b) it is automatic. Indeed, by a theorem of J. Wedderburn, \( \text{Br}_2(F) = (0) \) for any finite field \( F \); see e.g., [GS06, Remark 6.2.7].

(4) For \( n \geq 5 \), all \( F \)-forms of \( \overline{M}_{0,n} \) are unirational over \( F \); see [DR15, Theorem 6.1] or Proposition 4.4(a) below.

(5) It is natural to ask if similar rationality results hold for forms of \( \overline{M}_{g,n} \) for \( g \geq 1 \). Theorems of J. Harris, D. Mumford, D. Eisenbud and G. Farkas [HM82, EH87, Fa00, Fa11], assert that \( M_{g,0} \) is not unirational for any \( g \geq 23 \), and hence, neither is \( M_{g,n} \) for any \( n > 0 \). Moreover, A. Logan [Lo03] exhibited an explicit integer \( f(g) \), for each \( 1 < g \leq 22 \) such that \( M_{g,n} \) is not unirational as long as \( n \geq f(g) \). Deciding for which of the finitely many remaining pairs \((g,n)\) the moduli space \( M_{g,n} \) is rational, stably rational or unirational, over \( \mathbb{C} \) is a problem of ongoing interest; see, e.g., [CF07]. We have shown that in some cases (for small \( n, g \geq 1 \)), every form of \( \overline{M}_{g,n} \) is stably rational; see [FloR17].

The remainder of this paper will be devoted to proving Theorem 1.2.

2. Preliminaries on moduli spaces of curves and their twisted forms

The \( F \)-forms of a quasi-projective variety \( X \) are in a natural bijective correspondence with \( H^1(F, \text{Aut}(X)) \); see [Se97, II.1.3]. Here \( \text{Aut}(X) \) is a functor which associates to the scheme \( S/F \) the abstract group \( \text{Aut}(X_S) \). This functor is not representable by an algebraic group defined over \( F \) in general. If it is, one usually says that \( \text{Aut}(X) \) is an algebraic group. In the case, where \( \text{Aut}(X) \) is an algebraic group, the bijective correspondence between \( H^1(F, \text{Aut}(X)) \) and the set of \( F \)-forms of \( X \) (up to \( F \)-isomorphism) can be described explicitly by using the twisting operation. That is, to an \( \text{Aut}(X) \)-torsor \( \tau : Y \to \text{Spec}(F) \), we associate the \( F \)-variety \( \tau X := (X \times Y)/\text{Aut}(X) \), which is a twisted form of \( X \). Up to \( F \)-isomorphism, \( \tau X \) depends only on the class \( \alpha \) of \( \tau \) in \( H^1(F, \text{Aut}(X)) \); see [Se97, Section II.1.3]. By abuse of notation, we will sometimes write \( \alpha X \) in place of \( \tau X \). For the definition and basic properties of the twisting operation we refer the reader to [Flo08, Section 2] or [DR15, Section 3]. Conversely, to a twisted form \( X' \) of \( X \) defined over \( F \), we associate the \( \text{Aut}(X) \)-torsor \( \text{Isom}_F(X,X') \to \text{Spec}(F) \).

The following recent result is the starting point for our investigation.

Theorem 2.1. Let \( F \) be a field of characteristic \( \neq 2 \). If \( 2g + n \geq 5 \), then the natural embedding \( S_n \to \text{Aut}_F(\overline{M}_{g,n}) \) is an isomorphism.
In the case \( g = 0 \) and \( F = \mathbb{C} \), Theorem 2.1 was proved by A. Bruno and M. Mella [BM13]. In the more general situation, where \( F = \mathbb{C} \) but \( g \geq 0 \) is arbitrary, it is due to A. Massarenti [Mas14], and in full generality to B. Fantechi and A. Massarenti [FM17, Theorem A.2 and Remark A.4]. As an immediate consequence, we obtain the following.

**Corollary 2.2.** Let \( F \) be a field of characteristic \( \neq 2 \), and \( g, n \) be non-negative integers such that \( 2g + n \geq 5 \). Then every \( F \)-form of \( \overline{M}_{g,n} \) is isomorphic to \( \alpha \overline{M}_{g,n} \) for some \( \alpha \in H^1(F, S_n) \). □

**Remark 2.3.** If \( S_n \) is the full automorphism group of the moduli space \( M_{0,n} \) of smooth marked curves, then \( M_{0,n} \) can be replaced by \( M_{0,n} \) in the statement of Theorems 1.2. The proof remains unchanged. In particular, by [Lin11, Section 4.10, Corollary 7], this is the case if \( F \) is a subfield of \( \mathbb{C} \).

**Remark 2.4.** Recall that \( \overline{M}_{g,n} \) is, by definition, the coarse moduli space of the functor which assigns to a scheme \( X \), defined over \( F \), the set of isomorphism classes of pairs \( (C, s) \), where \( C \to X \) is a stable curve of genus \( g \) over \( X \) and \( s = (s_1, \ldots, s_n) \) is an \( n \)-tuple of disjoint sections \( s_i : X \to C \). Equivalently, we may view \( s \) as a single closed embedding \( s : X^{(n)} \hookrightarrow C \) (over \( X \)), where \( X^{(n)} = X \times_{\text{Spec}(F)} \text{Spec}(F^n) \) is the disjoint union of \( n \) copies of \( X \). To place our results into the context of moduli theory, we remark that if \( 2g + n \geq 5 \), then every form of \( \overline{M}_{g,n} \) admits a similar functorial interpretation. Suppose \( \alpha : Y \to \text{Spec}(F) \) is an \( S_n \)-torsor represented by an \( n \)-dimensional étale algebra \( E/F \). Then \( \alpha \overline{M}_{g,n} \) is the coarse moduli space for the functor

\[
X \mapsto \{ \text{isomorphism classes of pairs } (C, s) \},
\]

where \( C \to X \) is a stable curve of genus \( g \), and \( s \) is an embedding \( X \times_{\text{Spec}(F)} \text{Spec}(E) \to C \) (over \( X \)). We will not use this functorial description of \( \alpha \overline{M}_{g,n} \) in the sequel.

### 3. Preliminaries on the Noether problem

Let \( G \) be a linear algebraic group, and \( G \to \text{GL}(V) \) be a finite-dimensional representation of \( G \), both defined over a field \( F \). We will assume that this representation is generically free, i.e., there is a dense open subset \( U \subset V \) such that the scheme-theoretic stabilizer of every point of \( U \) is trivial.

The following questions originated in the work of E. Noether. Here (R) stands for rationality, (SR) for stable rationality and (RR) for retract rationality.

Noether’s problem (R): Is \( F(V)^G \) rational over \( F \)?

Noether’s problem (SR): Is \( F(V)^G \) stably rational over \( F \)? That is, is there a field \( E/F(V)^G \) such that \( E \) is rational over both \( F(V)^G \) and \( F \)?

Noether’s problem (RR): Is \( F(V)^G \) retract rational over \( F \)?

Recall that an irreducible variety \( Y \) defined over \( F \) is called *retract rational* if the identity map \( Y \to Y \) factors through the affine space \( A^n_F \) for some \( n \geq 1 \):

\[
\begin{array}{ccc}
Y & \xrightarrow{id} & Y \\
\downarrow i & & \downarrow j \\
A^n_F
\end{array}
\]
Here $i$ and $j$ are composable rational maps, i.e., the image of $i$ and the domain of $j$ intersect non-trivially. A finitely generated field extension $L/F$ is called retract rational if some (and thus any) model $Y$ of $L/F$ is retract rational. Here by a model of $L/F$ we mean an irreducible variety $Y$ defined over $F$ such that $F(Y) = L$.

Noether’s original paper [Noe13] only considered problem (R) (and only in the case, were $G$ is a finite group and $V$ is the regular representation of $G$). Subsequent attempts to solve problem (R) naturally led to problems (SR) and (RR). Note, in particular, that the answers to problems (SR) and (RR) depend only on the group $G$ and not on the choice of generically free representation $V$. For this reason we will refer to these problems as Noether’s problems (SR) and (RR) for $G$ in the sequel. The answer to problem (R) may a priori depend on the choice of $V$.

**Remark 3.2.** (see [CTS08, Section 4.2]) Suppose $G$ is a special group defined over $F$, i.e., $H^1(K,G) = \{1\}$ for every field extension $K/F$. Recall that a special group is always linear and connected; see [Se58, Theorem 4.4.1.1].

Let $\pi: V \to V/G$ be the rational quotient map. That is, $V/G$ is any variety defined over $F$ whose function field in $F(V)^G$, and $\pi$ is induced by the inclusion of fields $F(V)^G \to F(V)$. If $G$ is special, $\pi$ has a rational section and thus $V$ is birationally isomorphic to $V/G \times G$ over $F$. Consequently, Noether’s problem (SR) has a positive solution for $G$ if $G$ is itself stably rational over $F$, and similarly for Noether’s problem (RR).

**Definition 3.3.** We will say that a $G$-torsor $\alpha$ over a field $K$ is $r$-trivial if it can be connected to the trivial torsor by a rational curve. In other words, $\alpha$ is $r$-trivial if there exists an open subset $C \subset \mathbb{A}^1$ defined over $K$, a $G$-torsor $Y \to C$, and $K$-points $p_1, p_2: \text{Spec}(K) \to C$ such that $p_1^*(Y) \simeq \alpha$ and $p_2^*(Y)$ is split.

Note that our notion of $r$-triviality is a minor variant of the more commonly used notion of $R$-triviality, introduced by Manin [Man72]. A $G$-torsor $\alpha$ over $K$ is called $R$-trivial if it can be connected to the trivial torsor by a chain of rational curves defined over $K$.

**Lemma 3.4.** Suppose Noether’s problem (RR) has a positive solution for an affine algebraic group $G/K$. Then every $G$-torsor $\alpha: X \to \text{Spec}(K)$ is $r$-trivial, for every infinite field $K$ containing $F$.

**Proof.** There is a dense $G$-invariant open subset $V_0 \subset V$ which is the total space of a $G$-torsor $\pi: V_0 \to Y$; see [Se03, Section 5]. Here $\pi^*F(Y) = F(V)^G$. Recall that we are assuming $Y$ is retract rational. After replacing $Y$ by a dense open subset, we may further assume that $Y$ is a locally closed subvariety of $\mathbb{A}^n$, $i: Y \hookrightarrow \mathbb{A}^n$ in (3.1) is the inclusion map, and $j: \mathbb{A}^n \dashrightarrow Y$ is regular on some dense open subset $U$ of $\mathbb{A}^n$ containing $Y$.

It is well known that $\pi$ is a versal torsor; once again, see [Se03, Section 5] or [DR15]. In particular, there is a $K$-point $p_1: \text{Spec}(K) \to Y$ such that $\pi$ restricts to $\alpha$ over $p_1$, i.e., $p_1^*(\pi) = \alpha$. Similarly, there is a point $p_2: \text{Spec}(K) \to Y$ such that $\pi$ splits over $p_2$. It now suffices to connect $p_1$ and $p_2$ by an affine rational curve $C \subset Y$, defined over $K$, which is smooth at $p_1$ and $p_2$. After removing a closed subset from $C$ away from $p_1$ and $p_2$, we may assume that $C$ is isomorphic to an open subset of $\mathbb{A}^1_k$. Then we obtain a torsor $T \to C$ with the desired properties by pulling back $\pi$ to $C$. 
To construct $C$, we first connect $p_1$ and $p_2$ by a rational curve $C_0$ in $\mathbb{A}^n$, then set $C := j(C_0)$. Note that since $j: U \to Y$ is the identity map on $Y$, the differential $d_j\phi$ is surjective for every $p \in Y$. Hence, we can choose $C_0$ so that $C$ is smooth at $p_1$ and $p_2$. $\square$

4. THE NOETHER PROBLEM FOR A CLASS OF TWISTED GROUPS

Let $G_0 := G(F^n/F) = (GL_2 \times \mathbb{G}_m^n)/\mathbb{G}_m$, where $\mathbb{G}_m$ is centrally embedded into $GL_2 \times \mathbb{G}_m^n$ by $t \mapsto (t^{-1} \text{Id}, t, \ldots, t)$. The group $G_0$ and its twisted forms,

\[ G(E/F) := (GL_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m, \]

where $E/F$ is an étale algebra of degree $n$, will play a prominent role in the sequel.

Recall that $\overline{M}_{0,n}$ is $S_n$-equivariantly birationally isomorphic to $(\mathbb{P}^1)^n/PGL_2$. In turn, $(\mathbb{P}^1)^n/PGL_2$ is $S_n$-equivariantly birationally isomorphic to

\[ (\mathbb{A}^2)^n/(GL_2 \times (\mathbb{G}_m)^n). \]

Here we identify $\mathbb{G}_m^n$ with the diagonal maximal torus in $GL_n$, and $(\mathbb{A}^2)^n$ with the affine space $Mat_{2,n}$ of $2 \times n$ matrices. The group $GL_2$ acts on $Mat_{2,n}$ via multiplication on the left, and the torus $\mathbb{G}_m^n$ acts via multiplication on the right. These two commuting linear actions give rise to a linear representation

\[ GL_2 \times \mathbb{G}_m^n \to GL(Mat_{2,n}). \]

One readily checks that the kernel of this representation is

\[ H = \{(t^{-1} \text{Id}, t, \ldots, t) \in GL_2 \times \mathbb{G}_m^n \mid t \in \mathbb{G}_m\} \simeq \mathbb{G}_m \]

and that the induced representation

\[ \phi: G_0 = (GL_2 \times \mathbb{G}_m^n)/\mathbb{G}_m \to GL(Mat_{2,n}) \]

is generically free (recall that we are assuming that $n \geq 5$ throughout). Now identify $S_n$ with the subgroup of permutation matrices in $GL_n$, and let this group act on $Mat_{2,n}$ linearly, via multiplication on the right. In summary,

\[ \overline{M}_{0,n} \simeq (\mathbb{P}^1)^n/PGL_2 \simeq Mat_{2,n}/G_0, \]

where $\simeq$ denotes an $S_n$-equivariant birational isomorphism.

Let $\tau$ be an $S_n$-torsor over $\text{Spec}(F)$. Since $S_n$ normalizes $\mathbb{G}_m^n$ in $GL_n$, we can twist the group $G_0$ and the representation $\phi$ by $\tau$ and obtain a new group

\[ \tau G_0 := \tau(GL_2 \times R_{E/F}(\mathbb{G}_m))/\tau H := G(E/F) \]

and a new representation $\tau \phi: \tau G_0 \to GL(\alpha \text{Mat}_{2,n})$ defined over $F$. Note that $S_n$ acts trivially on $H$, and thus $\tau H \simeq H \simeq \mathbb{G}_m$ over $F$. Moreover, by Hilbert’s Theorem 90, $\tau \text{Mat}_{2,n}$ is isomorphic to $\text{Mat}_{2,n}$ as an $F$-vector space. Explicitly,

\[ \tau G_0 \simeq G(E/F) := (GL_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m, \]

$\tau \text{Mat}_{2,n}$ is the affine space $\mathbb{A}(F^2 \otimes_F E)$, where $GL_2$ acts $F$-linearly on $F^2 \otimes_F E$ via multiplication on $F^2$ and $R_{E/F}(\mathbb{G}_m)$ acts via multiplication on $E$. We have thus proved the following:
Proposition 4.4. Let $F$ be a field, $\tau$ be an $S_n$-torsor over $\text{Spec}(F)$, and $E/F$ be the étale algebra associated to $\tau$.

(a) (cf. [DR15, Theorem 6.1]) $\overline{M}_{0,n}$ is unirational.

(b) $\overline{M}_{0,n}$ is rational over $F$ if and only if Noether’s problem $(R)$ for the representation $\tau \phi$ of the group $G(E/F)$ has a positive solution.

(c) $\overline{M}_{0,n}$ is stably rational over $F$ if and only if Noether’s problem $(SR)$ for the group $G(E/F)$ has a positive solution.

(d) $\overline{M}_{0,n}$ is retract rational over $F$ if and only if Noether’s problem $(RR)$ for the group $G(E/F)$ has a positive solution. □

5. The Galois cohomology of $G(E/F)$

Let $E/F$ be a finite-dimensional étale algebra and $G := G(E/F) := (\text{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$ be the algebraic group we considered in the previous section; see (4.1).

Lemma 5.1. Let $\mathcal{F} : \text{Fields}_F \to \text{Sets}$ be the functor from the category of field extensions of $F$ to the category of sets, defined as follows:

$$\mathcal{F}(K) := \{\text{isomorphism classes of quaternion } K\text{-algebras } A \text{ such that } A \text{ is split by } E \otimes_F K\}.$$  

Then the functors $\mathcal{F}$ and $H^1(\ast, G)$ are isomorphic.

Proof. Consider the the short exact sequence

$$1 \to R_{E/F}(\mathbb{G}_m) \to G \to \text{PGL}_2 \to 1$$

of algebraic groups and the associated long exact sequence

$$H^1(K, R_{E/F}(\mathbb{G}_m)) \to H^1(K, G) \overset{\alpha}{\to} H^1(K, \text{PGL}_2) \overset{\delta}{\to} H^2(K, R_{E/F}(\mathbb{G}_m))$$

of Galois cohomology sets. By Shapiro’s Lemma,

$$H^1(K, R_{E/F}(\mathbb{G}_m)) \simeq H^1(K \otimes_F E, \mathbb{G}_m) = \{1\},$$

and $H^2(K, R_{E/F}(\mathbb{G}_m)) \simeq H^2(K \otimes_F E, \mathbb{G}_m)$ is in a natural bijective correspondence with the Brauer group $\text{Br}(K \otimes_F E)$. Thus the long exact sequence (5.3) simplifies to

$$\{1\} \to H^1(K, G) \overset{\alpha}{\to} H^1(K, \text{PGL}_2) \overset{\delta}{\to} \text{Br}(K \otimes_F E).$$

Here $H^1(K, \text{PGL}_2)$ is the set of isomorphism classes of quaternion algebras $A/K$. The connecting map $\delta$ takes an algebra $A/K$ to $A \otimes_K (K \otimes_F E)$. By [Se97, Proposition 42], $\alpha$ is injective.\footnote{Note that a priori the exact sequence (5.4) only tells us that $\alpha$ has trivial kernel. Injectivity is not automatic, since $H^1(K, R_{E/F}(\mathbb{G}_m))$ and $H^1(K, G)$ are pointed sets with no group structure.} Hence, we can identify $H^1(K, G)$ with the kernel of $\delta$, and the lemma follows. □
Remark 5.5. When $n$ is odd, Lemma 5.1 tells us that $H^1(K, G) = \{1\}$ for every field $K/F$. In other words, $G(E/F)$ is a special group. Using the short exact sequence (5.2) one readily checks that $G(E/F)$ is rational over $F$. By Remark 3.2, we conclude that the Noether problem (SR) for this group has a positive solution. In other words, every infinite field $F$ and every $\tau \in H^1(F, S_n)$ has a positive solution. In other words, every infinite field $F$ and every $\tau \in H^1(F, S_n)$ has a positive solution. In other words, every infinite field $F$ and every $\tau \in H^1(F, S_n)$ has a positive solution.

6. Proof of Theorem 1.2(a)

Suppose $n = 2s + 1 \geq 5$ is odd. Our goal is to show that $\overline{\mathcal{M}}_{0,n}$ is rational over $F$ for every infinite field $F$ and every $\tau \in H^1(F, S_n)$. Let $E/F$ be the étale algebra representing $\tau$. In view of Proposition 4.4(b), it suffices to show that Noether’s problem (R) for the representation $\tau \phi$ of the group $G(E/F)$ has a positive solution.

Recall that $\tau \phi$ is the natural representation of $G(E/F)$ on $F^2 \otimes_F E$ of $G(E/F)$. The quotient $A(F^2 \otimes_F E)/GL_2$ is the Grassmannian $Gr(2, E)$ (up to birational equivalence). Thus the quotient $A(F^2 \otimes_F E)/G(E/F)$ is birational to the quotient $Gr(2, E)/R_{E/F}(G_m)$. Our proof of the rationality of the quotient variety $Gr(2, E)/R_{E/F}(G_m)$ below is inspired by the arguments in [Flo13].

Fix an $F$-vector subspace $W$ of $E$ of dimension $s$, and define the rational map

$$f_W: Gr(2, E) \dashrightarrow \mathbb{P}(E)$$

$$V \mapsto V \cdot W,$$

where $V \cdot W$ is the $F$-linear span of elements of the form $v \cdot w$ in $E$, as $v$ ranges over $V$ and $w$ ranges over $W$. Here $v \cdot w$ stands for the product of $v$ and $w$ in $E$, and $\mathbb{P}(E)$ denotes the dual projective space to $\mathbb{P}(E)$. In other words, points of $\mathbb{P}(E)$ are $2s$-dimensional $F$-linear subspaces of $E$.

Lemma 6.1. (a) The dual projective space $\overline{\mathbb{P}}(E)_0$ has a point $H$ whose orbit with respect to the natural action of $R^{0}_{E/F}(G_m)$ is dense and whose stabilizer is trivial.

(b) Suppose $W \in Gr(s, E)$ is such that $f_W$ is well defined (i.e., $\dim(V \cdot W) = 2s$ for general $V \in Gr(2, E)$). Then $f_W$ is equivariant with respect to the natural action of $R^{0}_{E/F}(G_m)$ on $Gr(2, E)$ and $\overline{\mathbb{P}}(E)$.

(c) There exists $W \in Gr(s, E)$ defined over $F$ such that $f_W$ is well defined and dominant.

Proof. The assertions of parts (a) and (b) can be checked after passing to the separable closure of $F^{sep}$ of $F$. In other words, we may assume that $F = F^{sep}$. In this case $E$ is the split algebra $E = R_{E/F}(G_m) = G_m^{n}/G_m$, and $\overline{\mathbb{P}}(E) = \mathbb{P}^{n-1}$.

(a) $(t_1, \ldots, t_n) \in G_m^{n}/G_m$ takes the hyperplane $H \in \mathbb{P}(E)$ given by $c_1 x_1 + \cdots + c_n x_n = 0$ to the hyperplane given by $(t_1^{c_1} c_1 x_1 + \cdots + (t_n^{c_n} c_n x_n = 0$. Thus any $H$ with $c_1, \ldots, c_n \neq 0$ has a dense orbit in $\mathbb{P}(E)$ with trivial stabilizer. In fact, all such $H$ lie in the same dense orbit; for future reference, we will denote this dense orbit by $\mathbb{P}(E)_0$. 
for any $v = (a_1, \ldots, a_n) \in V$ and $w = (b_1, \ldots, b_n) \in W$. Hence, $(tV) \cdot W = t(V \cdot W)$, as desired.

(c) Recall that the eigenvalues of $a \in E$ are the eigenvalues of the multiplication map $E \to E$ given by $x \mapsto ax$. They are elements of $F^{sep}$. Under an isomorphism between $E \otimes_F F^{sep}$ and $(F^{sep})^n$ (over $F^{sep}$), $a$ will be identified with an element of $(F^{sep})^n$ of the form $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $a$.

Choose $a \in E$ with distinct eigenvalues in $F^{sep}$. Elements of $E$ with distinct eigenvalues form a Zariski open subvariety $U$ of $\mathbb{A}(E)$ defined over $F$. Passing to $F^{sep}$, we see that $U \neq \emptyset$. Since $F$ is assumed to be infinite, $F$-points are dense in $U$. We choose $a$ to be one of these $F$-points, and set $W = \text{span}_F(1, a, \ldots, a^{s-1})$. We claim that for this choice of $W$, the rational map $f_W$ is well defined and dominant.

First let us show that $f_W$ is well defined. From the definition of $V \cdot W$ it is clear that $\dim(V \cdot W) \leq 2s$ for any $V \in \text{Gr}(2, E)$ and that equality holds for $V$ in a Zariski open subset of $\text{Gr}(2, E)$. Thus in order to show that $f_W$ is a well-defined rational map, it suffices to exhibit one element $V \in \text{Gr}(2, E)$ such that $\dim(V \cdot W) = 2s$. We claim that $V = \text{span}_F(1, a^s)$ has this property, i.e.,

$$V \cdot W = \text{span}_F(1, a, \ldots, a^{s-1}, a^s, \ldots, a^{2s-1})$$

is a $2s$-dimensional subspace of $E$. It suffices to show that $1, a, \ldots, a^{2s}$ are linearly independent over $F$. Passing to $F^{sep}$, we can write $a = (\lambda_1, \ldots, \lambda_{2s+1})$, where $\lambda_1, \ldots, \lambda_{2s+1}$ are distinct elements of $F^{sep}$. (Recall that $n = 2s + 1$ throughout.) Since the $(2s+1) \times (2s+1)$ Vandermonde matrix

$$
\begin{pmatrix}
1 & \cdots & 1 \\
\lambda_1 & \cdots & \lambda_{2s+1} \\
\vdots & \ddots & \vdots \\
\lambda_1^s & \cdots & \lambda_{2s+1}^s
\end{pmatrix}
$$

is non-singular, we conclude that $1, a, \ldots, a^{2s}$ are linearly independent over $F^{sep}$ and hence, over $F$, as desired. This shows that $f_W$ is well defined.

It remains to show that $f_W$ is dominant. By part (b), the image of $f_W$ is an $\mathbb{P}^0_{E/G}(\mathbb{G}_m)$-invariant subvariety of $\tilde{\mathbb{P}}(E)$. In view of part (a), it suffices to show that this subvariety intersects the dense open orbit $\tilde{\mathbb{P}}(E)_0$. In fact, it suffices to show that $V \cdot W \in \tilde{\mathbb{P}}(E)_0$ for $V = \text{span}_F(1, a^s)$, as above. To do this, we may pass to $F^{sep}$ and thus identify $E \otimes_F F^{sep}$ with $(F^{sep})^n$. Then $a = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ are distinct non-zero elements of $F^{sep}$. Recall from part (a) that the complement of $\tilde{\mathbb{P}}(E)$ consists of hyperplanes of the form $c_1x_1 + \cdots + c_{2s}x_{2s} = 0$, where $c_i = 0$ for some $i$ but $(c_1, \ldots, c_{2s}) \neq (0, \ldots, 0)$. It remains to show that the hyperplane $V \cdot W = \text{span}_F(1, a, \ldots, a^{s-1}, a^s, \ldots, a^{2s-1})$ is not of this form. Indeed, assume the contrary. By symmetry we may assume that the equation of the hyperplane $\text{span}(1, \ldots, a^{2s-1})$ in $E$ is $c_1x_1 + \cdots + c_{2s}x_{2s} = 0$, with $c_{2s+1} = 0$. Since $a^i \in V \cdot W$, this means that

$$c_1\lambda_1^i + \cdots + c_{2s}\lambda_{2s}^i = 0 \text{ for } i = 0, 1, \ldots, 2s - 1.$$
Since $\lambda_1, \ldots, \lambda_{2s}$ are distinct, the $2s \times 2s$ Vandermonde matrix
\[
\begin{pmatrix}
1 & \ldots & 1 \\
\lambda_1 & \ldots & \lambda_{2s} \\
\ldots & \ldots & \ldots \\
\lambda_1^{2s-1} & \ldots & \lambda_{2s}^{2s-1}
\end{pmatrix}
\]
is non-singular. This implies that $c_1 = \cdots = c_{2s} = 0$, a contradiction. We conclude that $V \cdot W \in \overline{\mathbb{P}(E)}_0$, as desired. \qed

We are now ready to finish the proof of Theorem 1.2(a).

Let $W \in \text{Gr}(s, E)$ be the $s$-dimensional $E$-vector subspace of $E$ given by Lemma 6.1. Choose a dense open $R^0_{E/F}(\mathbb{G}_m)$-invariant subvariety $U \subset \text{Gr}(2, E)$ defined over $F$ such that $f_W^*: \text{Gr}(2, E) \to \overline{\mathbb{P}(E)}_0$ restricts to a regular map on $U$, and the rational quotient map $\text{Gr}(2, E) \to \text{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$ restricts to a $R^0_{E/F}(\mathbb{G}_m)$-torsor $\pi: U \to \text{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$ (over a suitably chosen birational model of $\text{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$). In summary, we obtain the following diagram of $R^0_{E/F}(\mathbb{G}_m)$-equivariant dominant rational maps:

\[
\begin{array}{ccc}
\text{Gr}(2, E) & \xrightarrow{f_W} & \overline{\mathbb{P}(E)}_0 \\
\downarrow \quad \text{open} & & \downarrow \quad \text{open} \\
U & \xrightarrow{f_W} & \overline{\mathbb{P}(E)}_0 \\
\downarrow \pi & & \downarrow \sim \\
\text{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m) & & R^0_{E/F}(\mathbb{G}_m)
\end{array}
\]

Now choose an $F$-point $H \in \overline{\mathbb{P}(E)}_0$; this can be done because we are assuming that $F$ is an infinite field. From the diagram, we see that $f_W^{-1}(H) \subset U$ is a section of $\pi$. In particular, $\text{Gr}(2, E)/R^0_{E/F}(\mathbb{G}_m)$ is birationally isomorphic to $f_W^{-1}(H)$ over $F$. It thus remains to show that $f_W^{-1}(H)$ is rational over $F$.

Let $Z$ be the $E$-vector subspace of $E$ given by $Z = \{a \in E \mid a \cdot W \subset H\}$. Clearly $V \in U$ belongs to $\phi^{-1}(H)$ if and only if $V \subset Z$ or equivalently, $V \in \text{Gr}(2, Z)$. Thus $f_W^{-1}(H) = \text{Gr}(2, Z) \cap U$ is a dense open subset of $\text{Gr}(2, Z)$. Clearly $f_W^{-1}(H)$ is non-empty. Since $\text{Gr}(2, Z)$ is rational over $F$, we conclude that $f_W^{-1}(H)$ is also rational over $F$, as desired. \qed

7. PROOF OF THEOREM 1.2(b)

We will deduce Theorem 1.2(b) from the following proposition.

**Proposition 7.1.** Suppose $F$ is a field of characteristic $\neq 2$ and $A = (a_1, a_2)$ is a quaternion division algebra over $F$, for some $a_1, a_2 \in F^*$. Set $a_3 = a_1a_2$ and $E_i = F(\sqrt[n_i]{a_i})$, for $i = 1, 2, 3$. Consider the étale $F$-algebra
\[
E = E_1^{n_1} \times E_2^{n_2} \times E_3^{n_3},
\]
for some $n_1, n_2, n_3 \geq 1$. Then Noether’s problem (RR) has a negative solution for the group $G(E/F) = (\text{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$. 

Assuming that Proposition 7.1 is established, we can complete the proof of Theorem 1.2 as follows. By a theorem of A. S. Merkurjev [Mer81], $\text{Br}_2(F)$ is generated, as an abelian group by classes of quaternion algebras. Since we are assuming that $\text{Br}_2(F) \neq 0$, one of these classes, say, $(a_1, a_2)$ is non-split. That is, $(a_1, a_2)$ is a division algebra. Since we are assuming that $n \geq 6$ is even, we can choose $n_1, n_2, n_3 \geq 1$ so that $n_1 + n_2 + n_3 = n$. For example, we can take $n_1 = \frac{n}{2} - 2$, $n_2 = 1$ and $n_3 = 1$. By Proposition 7.1, Noether’s problem (RR) has a negative solution for the group $G(E/F) = (\text{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$. By Proposition 4.4(d), the $F$-form $\mathbb{M}_{0,n}$ of $\mathbb{M}_{0,n}$ is not retract rational over $F$, where $\tau \in H^1(K,S_n)$ is the class of the étale algebra $E/F$. This completes the proof of Theorem 1.2(b).

Proof of Proposition 7.1. Since $E_1$, $E_2$, $E_3$ are maximal subfields of $A$,

$$A \otimes_F E_i \cong \text{Mat}_2(E_i)$$

for $i = 1, 2, 3$. In other words, $A$ is split by $E/F$. Thus by Lemma 5.1, $A$ corresponds to a class in $H^1(F,G)$, where $G := G(E/F)$. Denote this class by $\alpha$.

Our assumption that there exists a non-split quaternion algebra over $F$, implies that $F$ is an infinite field; see Remark (3) in the Introduction. Thus Lemma 3.4 applies: it suffices to show that $\alpha$ is not $r$-trivial. Assume the contrary. Using Lemma 5.1 once again, we see that this means the following: there exists a quaternion algebra $A(t)$ over $F(t)$ such that

(a) $A(t)$ is split by $F(t) \otimes_F E$, and

(b) $A(t)$ is unramified at $t = 0$ and $t = 1$, $A(0)$ is split over $F$, and $A(1)$ is isomorphic to $A$.

Here $A(0)$ and $A(1)$ denote $A(t)$ specialized to the points $t = 0$ and $t = 1$. We now recall the Faddeev exact sequence

$$0 \to \text{Br}(F) \to \text{Br}(F(t)) \to \bigoplus_{\eta \in \mathbb{P}^1} H^1(F_\eta, \mathbb{Q}/\mathbb{Z});$$

see e.g., [GS06, Corollary 6.4.6]. For $\eta \in \mathbb{P}^1_F$ denote the image of the Brauer class $[A(t)] \in \text{Br}(F(t))$ in $H^1(F_\eta, \mathbb{Q}/\mathbb{Z})$ by $\alpha_\eta$.

By property (a) above, $A(t)$ is split by $E_i(t) := F(t) \otimes_F E_i = F(t)(\sqrt{a_i})$ for $i = 1, 2, 3$. Note that $E_i(t)$ is a field extension of $F(t)$ of degree 2. Since $A(t)$ is a quaternion algebra over $F(t)$, $A(t)^{\otimes 2}$ is split over $F(t)$ and hence, $2\alpha_\eta = 0$ for every $\eta \in \mathbb{P}^1$. In particular, every $\alpha_\eta$ lies in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^1(F_\eta, \mathbb{Q}/\mathbb{Z})$.

We claim that $\alpha_\eta$ is the trivial class in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$ for every $\eta \in \mathbb{P}^1$. If we can prove this claim, then the Faddeev exact sequence (7.2) will tell us that $A(t)$ is constant, i.e., that $A(t)$ is isomorphic to $B \otimes_F F(t)$ over $F(t)$, for some quaternion algebra $B$ defined over $F$. Consequently, $A(0)$ and $A(1)$ are both isomorphic to $B$ over $F$ and hence, are isomorphic to each other. Since $A(0)$ is split over $F$, and $A(1) \simeq A$ is a quaternion division algebra, this is a contradiction, and the proof of Proposition 7.1 will be complete.

It remains to prove the claim. Assume the contrary. Suppose $\alpha_\eta = (b)$, where $(b)$ denotes the class of $b \in F_\eta^*$ in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$. Since we are assuming $\alpha_\eta \neq (0)$, $b$ is not a square in $F_\eta^*$. On the other hand, since $A(t)$ splits over $F(t)(\sqrt{a_i})$, $b$ becomes
a square in $F_\eta(\sqrt{a_i})^*$ for $i = 1, 2, 3$. This is only possible if $F_\eta(\sqrt{a_i})$ is a field extension of $F_\eta$ of degree 2 and $\sqrt{b} = f_i \sqrt{a_i}$ for some $f_i \in F_\eta^*$, where $i = 1, 2, 3$. Equivalently, $b = f_i^2 a_i$ or $(b) = (a_i)$ in $H^1(F_\eta, \mathbb{Z}/p\mathbb{Z})$. Since $a_1 a_2 a_3$ is a complete square in $F^*$, we conclude that

$$\alpha_\eta = (b) = (b) + (b) + (b) = (a_1) + (a_2) + (a_3) = (a_1 a_2 a_3) = 0$$

is the trivial class in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$, a contradiction. This completes the proof of the claim and thus of Proposition 7.1 and of Theorem 1.2(b).

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