

THE RATIONALITY PROBLEM FOR FORMS OF $\overline{M}_{0,n}$

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ABSTRACT. Let X be a del Pezzo surface of degree 5 defined over a field F . A theorem of Yu. I. Manin and P. Swinnerton-Dyer asserts that every Del Pezzo surface of degree 5 is rational. In this paper we generalize this result as follows. Recall that del Pezzo surfaces of degree 5 over a field F are precisely the F -forms of the moduli space $\overline{M}_{0,5}$ of stable curves of genus 0 with 5 marked points. Suppose $n \geq 5$ is an integer, and F is an infinite field of characteristic $\neq 2$. It is easy to see that every twisted F -form of $\overline{M}_{0,n}$ is unirational over F . We show that

- (a) If n is odd, then every twisted F -form of $\overline{M}_{0,n}$ is rational over F .
- (b) If n is even, there exists a field extension F/k and a twisted F -form X of $\overline{M}_{0,n}$ such that X is not retract rational over F .

1. INTRODUCTION

Let X be a del Pezzo surface of degree 5 defined over a field F . Yu. I. Manin [Man63, Theorem 3.15] showed that if X has an F -point, then X is rational over F . P. Swinnerton-Dyer [SD72] then proved that X always has an F -point; for alternative proofs of this assertion, see [SB92] and [Sko93]. In summary, one obtains the following result, published earlier by F. Enriques [E1897] (with an incomplete proof).

Theorem 1.1. (*Enriques, Manin, Swinnerton-Dyer*) *Every del Pezzo surface of degree 5 defined over a field F is F -rational. Equivalently, every F -form of $\overline{M}_{0,5}$ is F -rational.*

The purpose of this paper is to generalize this celebrated theorem as follows. As usual, we will denote the moduli space of smooth (respectively, stable) curves of genus g with n marked points by $M_{g,n}$ (respectively, $\overline{M}_{g,n}$). Recall that these moduli spaces are defined over the prime field. A *form* of a scheme X defined over a field F is an F -scheme Y , such that X and Y become isomorphic over the separable closure F^{sep} . We will use the terms “form”, “ F -form” and “twisted form” interchangeably throughout this paper. For a discussion of this notion and further references, see Section 2.

We now recall that $\overline{M}_{0,5}$ is a split del Pezzo surface of degree 5, and F -forms of $\overline{M}_{0,5}$ are precisely the del Pezzo surfaces of degree 5 defined over F . The main result of this paper is Theorem 1.2 below.

Theorem 1.2. *Let $n \geq 5$ be an integer and F be an infinite field of characteristic $\neq 2$.*

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(a) Assume n is odd. Then every F -form of $\overline{M}_{0,n}$ is rational over F .

(b) Assume n is even. If $\mathrm{Br}_2(F) \neq 0$, then there exists an F -form X of $\overline{M}_{0,n}$ such that X is not retract rational over F .

If F is assumed to be an infinite field, Theorem 1.1 is recovered from Theorem 1.2(a) by setting $n = 5$. In part (b), Br_2 denotes the 2-torsion of the Brauer group. The condition that $\mathrm{Br}_2(F) \neq 0$ is fairly mild; it is equivalent to the existence of a non-split quaternion algebra over F . In particular, $\mathrm{Br}_2(F) \neq 0$ if $F = k(t_1, t_2)$ where k is an arbitrary field of characteristic $\neq 2$ and t_1, t_2 are independent variables. Note also that for $n \geq 5$, all F -forms of $\overline{M}_{0,n}$ are unirational over F ; see [DR15, Theorem 6.1] or Proposition 4.4(a) below.

It is natural to ask if similar rationality results hold for forms of $\overline{M}_{g,n}$ for $g \geq 1$. Theorems of J. Harris, D. Mumford, D. Eisenbud and G. Farkas [HM82, EH87, Fa00, Fa11], assert that $M_{g,0}$ is not unirational for any $g \geq 23$, and hence, neither is $M_{g,n}$ for any $n \geq 0$. Moreover, A. Logan [Lo03] exhibited an explicit integer $f(g)$, for each $1 \leq g \leq 22$ such that $M_{g,n}$ is not unirational as long as $n \geq f(g)$. Deciding for which of the finitely many remaining pairs (g, n) the moduli space $M_{g,n}$ is rational, stably rational or unirational, over \mathbb{C} is a problem of ongoing interest; see, e.g., [CF07]. We have shown that in some cases (for small $n, g \geq 1$), every form of $\overline{M}_{g,n}$ is stably rational. We plan to publish these results in a forthcoming paper.

2. PRELIMINARIES ON MODULI SPACES OF CURVES AND THEIR TWISTED FORMS

The F -forms of a quasi-projective variety X are in a natural bijective correspondence with $H^1(F, \mathrm{Aut}(X))$; see [Se97, II.1.3]. Here $\mathrm{Aut}(X)$ is a functor which associates to the scheme S/F the abstract group $\mathrm{Aut}(X_S)$. This functor is not representable by an algebraic group defined over F in general. If it is, one usually says that $\mathrm{Aut}(X)$ is an algebraic group. In the case, where $\mathrm{Aut}(X)$ is an algebraic group, the bijective correspondence between $H^1(F, \mathrm{Aut}(X))$ and the set of F -forms of X (up to F -isomorphism) can be described explicitly by using the twisting operation. That is, to an $\mathrm{Aut}(X)$ -torsor $\tau: Y \rightarrow \mathrm{Spec}(F)$, we associate the F -variety ${}^\tau X := (X \times Y)/\mathrm{Aut}(X)$, which is a twisted form of X . Up to F -isomorphism, ${}^\tau X$ depends only on the class α of τ in $H^1(F, \mathrm{Aut}(X))$; see [Se97, Section III.1.3]. By abuse of notation, we will sometimes write ${}^\alpha X$ in place of ${}^\tau X$. For the definition and basic properties of the twisting operation we refer the reader to [Flo08, Section 2] or [DR15, Section 3]. Conversely, to a twisted form X' of X defined over F , we associate the $\mathrm{Aut}(X)$ -torsor $\mathrm{Isom}_F(X, X') \rightarrow \mathrm{Spec}(F)$.

The following recent result is the starting point for our investigation.

Theorem 2.1. *Let F be a field of characteristic $\neq 2$. If $2g + n \geq 5$, then the natural embedding $S_n \rightarrow \mathrm{Aut}_F(\overline{M}_{g,n})$ is an isomorphism.*

In the case $g = 0$ and $F = \mathbb{C}$, Theorem 2.1 was proved by A. Bruno and M. Mella [BM13]. In the more general situation, where $F = \mathbb{C}$ but $g \geq 0$ is arbitrary, it is due to A. Massarenti [Mas14], and in full generality to B. Fantechi and A. Massarenti [FM14, Theorem A.2 and Remark A.4]. As an immediate consequence, we obtain the following.

Corollary 2.2. *Let F be a field of characteristic $\neq 2$, and g, n be non-negative integers such that $2g + n \geq 5$. Then every F -form of $\overline{M}_{g,n}$ is isomorphic to ${}^\alpha\overline{M}_{g,n}$ for some $\alpha \in H^1(F, S_n)$. \square*

Remark 2.3. If S_n is the full automorphism group of the moduli space $M_{0,n}$ of smooth marked curves, then $\overline{M}_{0,n}$ can be replaced by $M_{0,n}$ in the statement of Theorems 1.2. The proof remains unchanged. In particular, by [Lin11, Section 4.10, Corollary 7], this is the case if $F = \mathbb{C}$.

Remark 2.4. Recall that $\overline{M}_{g,n}$ is, by definition, the coarse moduli space of the functor which assigns to a scheme X , defined over F , the set of isomorphism classes of pairs (C, s) , where $C \rightarrow X$ is a stable curve of genus g over X and $s = (s_1, \dots, s_n)$ is an n -tuple of disjoint sections $s_i: X \rightarrow C$. Equivalently, we may view s as a single closed embedding $s: X^{(n)} \hookrightarrow C$ (over X), where $X^{(n)} = X \times_{\text{Spec}(F)} \text{Spec}(F^n)$ is the disjoint union of n copies of X . To place our results into the context of moduli theory, we remark that if $2g + n \geq 5$, then every form of $\overline{M}_{g,n}$ admits a similar functorial interpretation. Suppose $\alpha: Y \rightarrow \text{Spec}(F)$ is an S_n -torsor represented by an n -dimensional étale algebra E/F . Then ${}^\alpha\overline{M}_{g,n}$ is the coarse moduli space for the functor

$$X \mapsto \{\text{isomorphism classes of pairs } (C, s)\},$$

where $C \rightarrow X$ is a stable curve of genus g , and s is an embedding $X \times_{\text{Spec}(F)} \text{Spec}(E) \rightarrow C$ (over X). We will not use this functorial description of ${}^\alpha\overline{M}_{g,n}$ in the sequel.

3. PRELIMINARIES ON THE NOETHER PROBLEM

Let G be a linear algebraic group, and $G \rightarrow \text{GL}(V)$ be a finite-dimensional representation of G , both defined over a field F . We will assume that this representation is generically free, i.e., there is a dense open subset $U \subset V$ such that the scheme-theoretic stabilizer of every point of U is trivial.

The following questions originated in the work of E. Noether. Here (R) stands for rationality, (SR) for stable rationality and (RR) for retract rationality.

Noether's problem (R): Is $F(V)^G$ rational over F ?

Noether's problem (SR): Is $F(V)^G$ stably rational over F ? That is, is there a field $E/F(V)^G$ such that E is rational over both $F(V)^G$ and F ?

Noether's Problem (RR): Is $F(V)^G$ retract rational over F ?

Recall that an irreducible variety Y defined over F is called *retract rational* if the identity map $Y \rightarrow Y$ factors through the affine space \mathbb{A}_F^n for some $n \geq 1$:

$$(3.1) \quad \begin{array}{ccc} Y & \xrightarrow{\text{id}} & Y \\ & \searrow i & \nearrow j \\ & & \mathbb{A}_F^n \end{array}$$

Here i and j are composable rational maps, i.e., the image of i and the domain of j intersect non-trivially. A finitely generated field extension L/F is called *retract rational* if some (and thus any) model Y of L/F is retract rational. Here by a model of L/F we mean an irreducible variety Y defined over F such that $F(Y) = L$.

Noether's original paper [Noe13] only considered problem (R) (and only in the case, where G is a finite group and V is the regular representation of G). Subsequent attempts to solve problem (R) naturally led to problems (SR) and (RR). Note, in particular, that the answers to problems (SR) and (RR) depend only on the group G and not on the choice of generically free representation V . For this reason we will refer to these problems as *Noether's problems (SR) and (RR) for G* in the sequel. The answer to problem (R) may a priori depend on the choice of V .

Remark 3.2. (see [CTS08, Section 4.2]) Suppose G be a special group defined over F , i.e., $H^1(K, G) = \{1\}$ for every field extension K/F . Recall that a special group is always linear and connected; see [Se58, Theorem 4.4.1.1].

Let $\pi: V \dashrightarrow V/G$ be the rational quotient map. That is, V/G is any variety defined over F whose function field is $F(V)^G$, and π is induced by the inclusion of fields $F(V)^G \hookrightarrow F(V)$. If G is special, π has a rational section and thus V is birationally isomorphic to $V/G \times G$ over F . Consequently, Noether's problem (SR) has a positive solution for G if and only if G is itself stably rational over F , and similarly for Noether's problem (RR).

Definition 3.3. We will say that a G -torsor α over a field K is r -trivial if it can be connected to the trivial torsor by a rational curve. In other words, α is r -trivial if there exists an open subset $C \subset \mathbb{A}^1$ defined over K , a G -torsor $Y \rightarrow C$, and K -points $p_1, p_2: \text{Spec}(K) \rightarrow C$ such that $p_1^*(Y) \simeq \alpha$ and $p_2^*(Y)$ is split.

Note that our notion of r -triviality is a minor variant of the more commonly used notion of R -triviality, introduced by Manin [Man72]. A G -torsor α is called R -trivial if it can be connected to the trivial torsor by a chain of rational curves.

Lemma 3.4. *Suppose Noether's problem (RR) has a positive solution for an affine algebraic group G/F . Then every G -torsor $\alpha: X \rightarrow \text{Spec}(K)$ is r -trivial, for every field extension K/F .*

Proof. There is a dense G -invariant open subset $V_0 \subset V$ which is the total space of a G -torsor $\pi: V_0 \rightarrow Y$; see [Se03, Section 5]. Here $\pi^*F(Y) = F(V)^G$. Recall that we are assuming Y is retract rational. After replacing Y by a dense open subset, we may further assume that Y is a locally closed subvariety of \mathbb{A}^n , $i: Y \dashrightarrow \mathbb{A}^n$ in (3.1) is the inclusion map, and $j: \mathbb{A}^n \dashrightarrow Y$ is regular on some dense open subset U of \mathbb{A}^n containing Y .

It is well known that π is a versal torsor; once again, see [Se03, Section 5] or [DR15]. In particular, there is a K -point $p_1: \text{Spec}(K) \rightarrow Y$ such that π restricts to α over p_1 , i.e., $p_1^*(\pi) = \alpha$. Similarly, there is a point $p_2: \text{Spec}(K) \rightarrow Y$ such that π splits over p_2 . It now suffices to connect p_1 and p_2 by an affine rational curve $C \subset Y$, defined over K , which is smooth at p_1 and p_2 . After removing a closed subset from C away from p_1 and p_2 , we may assume that C is isomorphic to an open subset of \mathbb{A}_K^1 . Then we obtain a torsor $T \rightarrow C$ with desired properties by pulling back π to C .

To construct C , we first connect p_1 and p_2 by a rational curve C_0 in \mathbb{A}^n , then set $C := j(C_0)$. Note that since $j: U \rightarrow Y$ is the identity map on Y , the differential dj_p is surjective for every $p \in Y$. Hence, we can choose C_0 so that C is smooth at p_1 and p_2 . \square

4. THE NOETHER PROBLEM FOR A CLASS OF TWISED GROUPS

Let $G_0 := G(F^n/F) = (\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m$, where \mathbb{G}_m is centrally embedded into $\mathrm{GL}_2 \times \mathbb{G}_m^n$ by $t \mapsto (t^{-1} \mathrm{Id}, t, \dots, t)$. The group G_0 and its twisted forms,

$$(4.1) \quad G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m,$$

where E/F is an étale algebra of degree n , will play a prominent role in this paper.

Recall that $\overline{M}_{0,n}$ is S_n -equivariantly birationally isomorphic to $(\mathbb{P}^1)^n/\mathrm{PGL}_2$. In turn, $(\mathbb{P}^1)^n/\mathrm{PGL}_2$ is S_n -equivariantly birationally isomorphic to

$$(\mathbb{A}^2)^n/(\mathrm{GL}_2 \times (\mathbb{G}_m)^n).$$

Here we identify \mathbb{G}_m^n with the diagonal maximal torus in GL_n , and $(\mathbb{A}^2)^n$ with the affine space $\mathrm{Mat}_{2,n}$ of $2 \times n$ matrices. The group GL_2 acts on $\mathrm{Mat}_{2,n}$ via multiplication on the left, and the torus \mathbb{G}_m^n acts via multiplication on the right. These two commuting linear actions give rise to a linear representation

$$\mathrm{GL}_2 \times \mathbb{G}_m^n \rightarrow \mathrm{GL}(\mathrm{Mat}_{2,n}).$$

One readily checks that the kernel of this representation is

$$H = \{(t^{-1} \mathrm{Id}, t, \dots, t) \in \mathrm{GL}_2 \times \mathbb{G}_m^n \mid t \in \mathbb{G}_m\} \simeq \mathbb{G}_m$$

and that the induced representation

$$\phi: G_0 = (\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m \rightarrow \mathrm{GL}(\mathrm{Mat}_{2,n})$$

is generically free. (Recall that we are assuming that $n \geq 5$ throughout.) Now identify S_n with the subgroup of permutation matrices in GL_n , and let this group act on $\mathrm{Mat}_{2,n}$ linearly, via multiplication on the right. In summary,

$$(4.2) \quad \overline{M}_{0,n} \simeq (\mathbb{P}^1)^n/\mathrm{PGL}_2 \simeq \mathrm{Mat}_{2,n}/G_0,$$

where \simeq denotes an S_n -equivariant birational isomorphism.

Let τ be a S_n -torsor over $\mathrm{Spec}(F)$. Since S_n normalizes \mathbb{G}_m^n in GL_n , we can twist the group G_0 and the representation ϕ by τ and obtain a new group

$$(4.3) \quad {}^\tau G_0 := {}^\tau(\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/{}^\tau H := G(E/F)$$

and a new representation ${}^\tau \phi: {}^\tau G_0 \rightarrow \mathrm{GL}({}^\alpha \mathrm{Mat}_{2,n})$ defined over F . Note that S_n acts trivially on H , and thus ${}^\tau H \simeq H \simeq \mathbb{G}_m$ over F . Moreover, by Hilbert's Theorem 90, ${}^\tau \mathrm{Mat}_{2,n}$ is isomorphic to $\mathrm{Mat}_{2,n}$ as an F -vector space. Explicitly,

$${}^\tau G_0 \simeq G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m,$$

${}^\tau \mathrm{Mat}_{2,n}$ is the affine space $\mathbb{A}(F^2 \otimes_F E)$, where GL_2 acts F -linearly on $F^2 \otimes_F E$ via multiplication on F^2 and $R_{E/F}(\mathbb{G}_m)$ acts via multiplication on E . We have thus proved the following:

Proposition 4.4. *Let F be an infinite field, τ be an S_n -torsor over $\mathrm{Spec}(F)$, and E/F be the étale algebra associated to τ .*

(a) (cf. [DR15, Theorem 6.1]) ${}^\tau \overline{M}_{0,n}$ is unirational.

(b) ${}^\tau \overline{M}_{0,n}$ is rational over F if and only if Noether's problem (R) for the representation ${}^\tau \phi$ of the group $G(E/F)$ has a positive solution.

(c) ${}^\tau\overline{M}_{0,n}$ is stably rational over F if and only if Noether's problem (SR) for the group $G(E/F)$ has a positive solution.

(d) ${}^\tau\overline{M}_{0,n}$ is retract rational over F if and only if and only if the Noether's problem (RR) for the group $G(E/F)$ has a positive solution. \square

5. THE GALOIS COHOMOLOGY OF $G(E/F)$

Let E/F be an finite-dimensional étale algebra and $G := G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$ be the algebraic group we considered in the previous section; see (4.1).

Lemma 5.1. *Let $\mathcal{F}: \mathbf{Fields}_F \rightarrow \mathbf{Sets}$ be the functor from the category of field extensions of F to the category of sets, defined as follows:*

$$\mathcal{F}(K) := \{ \text{isomorphism classes of quaternion } K\text{-algebras } A \\ \text{such that } A \text{ is split by } E \otimes_F K \}.$$

Then the functors \mathcal{F} and $H^1(*, G)$ are isomorphic.

Proof. Consider the the short exact sequence

$$(5.2) \quad 1 \rightarrow R_{E/F}(\mathbb{G}_m) \rightarrow G \rightarrow \mathrm{PGL}_2 \rightarrow 1$$

of algebraic groups and the associated long exact sequence

$$(5.3) \quad H^1(K, R_{E/F}(\mathbb{G}_m)) \rightarrow H^1(K, G) \xrightarrow{\alpha} H^1(K, \mathrm{PGL}_2) \xrightarrow{\delta} H^2(K, R_{E/F}(\mathbb{G}_m))$$

of Galois cohomology sets. By Shapiro's Lemma,

$$H^1(K, R_{E/F}(\mathbb{G}_m)) \simeq H^1(K \otimes_F E, \mathbb{G}_m) = \{1\},$$

and $H^2(K, R_{E/F}(\mathbb{G}_m)) \simeq H^2(K \otimes_F E, \mathbb{G}_m)$ is in a natural bijective correspondence with the Brauer group $\mathrm{Br}(K \otimes_F E)$. Thus the long exact sequence (5.3) simplifies to

$$(5.4) \quad \{1\} \rightarrow H^1(K, G) \xrightarrow{\alpha} H^1(K, \mathrm{PGL}_2) \xrightarrow{\delta} \mathrm{Br}(K \otimes_F E).$$

Here $H^1(K, \mathrm{PGL}_2)$ is the set of isomorphism classes of quaternion algebras A/K . The connecting map δ takes an algebra A/K to $A \otimes_K (K \otimes_F E)$. By [Se97, Proposition 42], α is injective.¹ Hence, we can identify $H^1(K, G)$ with the kernel of δ , and the lemma follows. \square

Remark 5.5. When n is odd, Lemma 5.1 tells us that $H^1(K, G) = \{1\}$ for every field K/F . In other words, $G(E/F)$ is a special group. Using the short exact sequence (5.2) one readily checks that $G(E/F)$ is rational over F . By Remark 3.2, we conclude that the Noether problem (SR) for this group has a positive solution. In other words, every F -form of $\overline{M}_{0,n}$ is stably rational over F . This is a bit weaker than Theorem 1.2(a), which will be proved in the next section.

¹Note that a priori the exact sequence (5.4) only tells us that α has trivial kernel. Injectivity is not automatic, since $H^1(K, R_{E/F}(\mathbb{G}_m))$ and $H^1(K, G)$ are pointed sets with no group structure.

6. PROOF OF THEOREM 1.2(A)

Suppose $n = 2s + 1 \geq 5$ is odd. Our goal is to show that ${}^\tau \overline{M}_{0,n}$ is rational over F for every field F and every $\tau \in H^1(F, S_n)$. Let E/F be the étale algebra representing τ . In view of Proposition 4.4(b), it suffices to show that Noether's problem (R) for the representation ${}^\tau \phi$ of the group $G(E/F)$ has a positive solution.

Recall that ${}^\tau \phi$ is the natural representation of $G(E/F)$ on $F^2 \otimes_F E$ of $G(E/F)$. The quotient $\mathbb{A}(F^2 \otimes_F E)/\mathrm{GL}_2$ is the Grassmannian $\mathrm{Gr}(2, E)$ (up to birational equivalence). Thus the quotient $\mathbb{A}(F^2 \otimes_F E)/G(E/F)$ is birational to the quotient $\mathrm{Gr}(2, E)/R_{E/F}(\mathbb{G}_m)$.

Note that the diagonal subgroup $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^n$ acts trivially on $\mathrm{Gr}(2, n)$. Hence,

$$\mathbb{G}_m = {}^\alpha(\mathbb{G}_m) \hookrightarrow {}^\alpha(\mathbb{G}_m^n) = R_{E/F}(\mathbb{G}_m)$$

acts trivially on $\mathrm{Gr}(2, E)$, and $T(E) := R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$ acts faithfully on $\mathrm{Gr}(2, E)$.

Our proof of the rationality of the quotient variety $\mathrm{Gr}(2, E)/T(E)$ below is inspired by the arguments in [Flo13]. Here E is an arbitrary n -dimensional étale F -algebra and $n = 2s + 1 \geq 5$. Fix an F -vector subspace W of E of dimension s , and define the rational map

$$\begin{aligned} \phi_W: \mathrm{Gr}(2, E) &\dashrightarrow \check{\mathbb{P}}(E) \\ &V \rightarrow V \cdot W, \end{aligned}$$

Here $V \cdot W$ is the F -linear span of elements of the form $v \cdot w$ in E , as v ranges over V and w ranges over W , $v \cdot w$ stands for the product of v and w in E , and $\check{\mathbb{P}}(E)$ denotes the dual projective space to $\mathbb{P}(E)$, where E is viewed as an n -dimensional vector space over F . In other words, points of $\check{\mathbb{P}}(E)$ are $2s$ -dimensional linear subspaces of E .

Lemma 6.1. *Suppose $W \in \mathrm{Gr}(2, E)$ is chosen in general position. Then*

(a) ϕ_W is well-defined. In other words, $\dim(V \cdot W) = 2s$ for $V \in \mathrm{Gr}(2, E)$ in general position.

(b) ϕ_W is equivariant with respect to the natural action of $T(E) := R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$ on $\mathrm{Gr}(2, E)$ and $\check{\mathbb{P}}(E)$.

Proof. For the purpose of proving part (a), it suffices to work over the algebraic closure \overline{F} . In other words, we may assume that F is algebraically closed and thus $E = F^n$, as an F -algebra. Then $\mathrm{Gr}(2, E)$ becomes $\mathrm{Gr}(2, 2s + 1)$ and $\mathrm{Gr}(s, E)$ becomes $\mathrm{Gr}(s, 2s + 1)$. Recall that multiplication in $E = F^n$ is given by the formula

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n).$$

From the definition of $V \cdot W$ it is clear that $\dim(V \cdot W) \leq 2s$, and that equality holds for (V, W) in a Zariski open subset of $\mathrm{Gr}(2, 2s + 1) \times \mathrm{Gr}(s, 2s + 1)$. It thus suffices to show that this open subset is non-empty, i.e., to exhibit one particular example of $V \in \mathrm{Gr}(2, 2s + 1)$ and $W \in \mathrm{Gr}(s, 2s + 1)$ such that $\dim(V \cdot W) = 2s$.

Let e_1, \dots, e_{2s+1} be the standard basis of $E = F^{2s+1}$. We define V as the 2-dimensional subspace of F^{2s+1} spanned by

$$v_1 := e_1 + \dots + e_s = (1, \dots, 1, 0, \dots, 0, 0)$$

and

$$v_2 := e_{s+1} + \dots + e_{2s} = (0, \dots, 0, 1, \dots, 1, 0)$$

and W as the s -dimensional F -vector space spanned by

$$\begin{aligned} w_1 &:= e_1 + e_{s+1} = (1, 0, \dots, 0, 1, 0, \dots, 0, 0) \\ w_2 &:= e_2 + e_{s+2} = (0, 1, \dots, 0, 0, 1, \dots, 0, 0) \\ &\dots\dots\dots \\ w_s &:= e_s + e_{2s} = (0, 0, \dots, 1, 0, 0, \dots, 1, 0) \end{aligned}$$

It is easy to see that $V \cdot W$ contains the $2s$ linearly independent vectors

$$e_1 = v_1 \cdot w_1, \quad e_2 = v_1 \cdot w_2, \dots, \quad e_{2s} = v_2 \cdot w_s$$

in F^{n+1} . Hence, $\dim(V \cdot W) = 2s$, as claimed. This completes the proof of part (a).

To prove part (b), we may once again pass to the algebraic closure of F and thus assume that $E = F^n$ as an F -algebra and $T(E) = \mathbb{G}_m^n / \mathbb{G}_m$. Given $t = (t_1, \dots, t_n) \in \mathbb{G}_m^n$, we see that

$$(tv) \cdot w = (t_1 a_1 b_1, \dots, t_n a_n b_n) = t(v \cdot w).$$

for any $v = (a_1, \dots, a_n) \in V$ and $w = (b_1, \dots, b_n) \in W$. Hence, $(tV) \cdot W = t(V \cdot W)$, as desired. \square

We are now ready to complete the proof of Theorem 1.2(a). Note that the $T(E)$ -action on $\mathrm{Gr}(2, E)$ is generically free (to see this, we may once again pass to the algebraic closure of F). By Lemma 6.1(b), ϕ_W is $T(E)$ -equivariant. Moreover, $\check{\mathbb{P}}(E)$ contains a dense and open $T(E)$ -orbit with trivial stabilizer. Hence, $\check{\mathbb{P}}(E)$ is birationally isomorphic to $T(E)$, as a $T(E)$ -variety (over F), and we obtain the following diagram of $T(E)$ -equivariant dominant rational maps:

$$\begin{array}{ccc} \phi_W: \mathrm{Gr}(2, E) & \dashrightarrow & T(E) \simeq \check{\mathbb{P}}(E) \\ & \downarrow \pi & \\ & \mathrm{Gr}(2, E)/T & \end{array}$$

Here π is a torsor over the generic point of $\mathrm{Gr}(2, E)/T$. From this diagram we see that for $H \in \check{\mathbb{P}}(E)$ in general position, $\phi_W^{-1}(H)$ is a rational section of π . In particular, $\mathrm{Gr}(2, E)/T(E)$ is birationally isomorphic to $\phi_W^{-1}(H)$ over F , where $H \in \check{\mathbb{P}}(E)$ is a hyperplane in general position in $\mathbb{P}(E)$. It remains to show that for $H \in \check{\mathbb{P}}(E)$ in general position, $\phi_W^{-1}(H)$ is rational over F .

Here by $\phi_W^{-1}(H)$ we mean the preimage of H under the regular map $\phi_W: U \rightarrow \check{\mathbb{P}}(E)$, where U is the domain of ϕ_W , i.e., the largest dense open subset of $\mathrm{Gr}(2, E)$, where ϕ_W is regular. (Thus dense open subset is necessarily $T(E)$ -invariant.) Let Z be the F -vector subspace of E given by

$$Z := \{a \in E \mid a \cdot W \subset H\}.$$

Clearly $V \in \mathrm{Gr}(2, E)$ belongs to $\phi^{-1}(H)$ if and only if (i) ϕ_W is defined at V , and (ii) $V \subset Z$ or equivalently, $V \in \mathrm{Gr}(2, Z)$. Since H is in general position, we may assume that Z does not lie entirely in the indeterminacy locus of ϕ_W . We conclude that $\phi_W^{-1}(H)$ is a dense open subset of $\mathrm{Gr}(2, Z)$. In particular, this implies that $\phi_W^{-1}(H)$ is rational over F , and hence, so is $\mathrm{Gr}(2, E)/T(E)$, as desired. This completes the proof of Theorem 1.2(a). \square

7. PROOF OF THEOREM 1.2(B)

We will deduce Theorem 1.2(b) from the following proposition.

Proposition 7.1. *Suppose F is a field of characteristic $\neq 2$ and $A := (a_1, a_2)$ is a quaternion division algebra over F , for some $a_1, a_2 \in F^*$. Set $a_3 := a_1 a_2$ and $E_i := F(\sqrt{a_i})$, for $i = 1, 2, 3$. Consider the étale F -algebra*

$$E := E_1^{n_1} \times E_2^{n_2} \times E_3^{n_3},$$

for some $n_1, n_2, n_3 \geq 1$. Then Noether's problem (RR) has a negative solution for the group $G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$.

Assuming that Proposition 7.1 is established, we can complete the proof of Theorem 1.2 as follows. By a theorem of A. S. Merkurjev [Mer81], $\mathrm{Br}_2(F)$ is generated, as an abelian group by classes of quaternion algebras. Since we are assuming that $\mathrm{Br}_2(F) \neq 0$, one of these classes, say, (a_1, a_2) is non-split, i.e., is a division algebra. Since we are assuming that $n \geq 6$ is even, we can choose $n_1, n_2, n_3 \geq 1$ so that $n_1 + n_2 + n_3 = n$. For example, we can take $n_1 = \frac{n}{2} - 2$, $n_2 = 1$ and $n_3 = 1$. By Proposition 7.1, Noether's problem (RR) has a negative solution for the group $G(E/F) := (\mathrm{GL}_2 \times R_{E/F}(\mathbb{G}_m))/\mathbb{G}_m$. By Proposition 4.4(d), the F -form ${}^\tau \overline{M}_{0,n}$ of $\overline{M}_{0,n}$ is not retract rational over F , where $\tau \in H^1(K, \mathbb{S}_n)$ is the class of the étale algebra E/F . This completes the proof of Theorem 1.2(b). \square

Proof of Proposition 7.1. Since E_1, E_2, E_3 are maximal subfields of A ,

$$A \otimes_F E_i \simeq \mathrm{Mat}_2(E_i)$$

for $i = 1, 2, 3$. In other words, A is split by E/F . Thus by Lemma 5.1, A corresponds to a class in $H^1(F, G)$, where $G := G(E/F)$. Denote this class by α .

By Lemma 3.4 it suffices to show that α is not r -trivial. Assume the contrary. Using Lemma 5.1 once again, we see that this means the following: there exists a quaternion algebra $A(t)$ over $F(t)$ such that

(a) $A(t)$ is split by $F(t) \otimes_F E$, and

(b) $A(t)$ is unramified at $t = 0$ and $t = 1$, $A(0)$ is split over F , and $A(1)$ is isomorphic to A .

Here $A(0)$ and $A(1)$ denote $A(t)$ specialized to the points $t = 0$ and $t = 1$. We now recall the Faddeev exact sequence

$$(7.2) \quad 0 \rightarrow \mathrm{Br}(F) \rightarrow \mathrm{Br}(F(t)) \rightarrow \bigoplus_{\eta \in \mathbb{P}_F^1} H^1(F_\eta, \mathbb{Q}/\mathbb{Z});$$

see e.g., [GS06, Corollary 6.4.6]. For $\eta \in \mathbb{P}_F^1$ denote the image of the Brauer class $[A(t)] \in \mathrm{Br}(F(t))$ in $H^1(F_\eta, \mathbb{Q}/\mathbb{Z})$ by α_η .

By property (a) above, $A(t)$ is split by $E_i(t) := F(t) \otimes_F E_i = F(t)(\sqrt{a_i})$ for $i = 1, 2, 3$. Note that $E_i(t)$ is a field extension of $F(t)$ of degree 2. Thus $A(t)$ is a central simple algebra over $F(t)$ of index 1 or 2. Consequently, $A(t)^{\otimes 2}$ is split over $F(t)$ and hence, $2\alpha_\eta = 0$ for every $\eta \in \mathbb{P}^1$. In particular, every α_η lies in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^1(F_\eta, \mathbb{Q}/\mathbb{Z})$.

We claim that α_η is the trivial class in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$ for every $\eta \in \mathbb{P}^1$. If we can prove this claim, then the Faddeev exact sequence (7.2) will tell us that $A(t)$ is constant, i.e., that $A(t)$ is isomorphic to $B \otimes_F F(t)$ over $F(t)$, for some quaternion

algebra B defined over F . Consequently, $A(0)$ and $A(1)$ are both isomorphic to B over F and hence, are isomorphic to each other. Since $A(0)$ is split over F , and $A(1) \simeq A$ is a quaternion division algebra, this is a contradiction, and the proof of Proposition 7.1 will be complete.

It remains to prove the claim. Assume the contrary. Suppose $\alpha_\eta = (b)$ for $i = 1, 2, 3$, where (b) denotes the class of $b \in F_\eta^*$ in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$. Since we are assuming $\alpha_\eta \neq (0)$, b is not a square in F_η^* . On the other hand, since $A(t)$ splits over $F(t)(\sqrt{a_i})$, b becomes a square in $E_i = F_\eta(\sqrt{a_i})$ for $i = 1, 2, 3$. This is only possible if $F_\eta(\sqrt{a_i})$ is a field extension of F_η of degree 2 and $\sqrt{b} = f_i\sqrt{a_i}$ for some $f_i \in F_\eta^*$, where $i = 1, 2, 3$. Equivalently, $b = f_i^2 a_i$ or $(b) = (a_i)$ in $H^1(F_\eta, \mathbb{Z}/p\mathbb{Z})$. Since $a_1 a_2 a_3$ is a complete square in F^* , we conclude that

$$\alpha_\eta = (b) = (b) + (b) + (b) = (a_1) + (a_2) + (a_3) = (a_1 a_2 a_3) = 0$$

is the trivial class in $H^1(F_\eta, \mathbb{Z}/2\mathbb{Z}) = F_\eta^*/(F_\eta^*)^2$, as desired. \square

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