TRIPLE LINKAGE

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Abstract. For a field the condition is studied that any triple of (bilinear) Pfister forms of a given dimension are linked. This is a strengthening of the condition of linkage investigated by Elman and Lam, which asks the same for pairs of Pfister forms. In characteristic different from two this condition for triples of 2-fold Pfister forms is related to the Hasse number.

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1. Introduction

Milnor's seminal article [9] on K-theory of fields had an enormous impact on quadratic form theory. In a series of articles Elman and Lam explored the correspondence between Pfister forms and symbols (canoncial generators) in the K-theory modulo 2 of a field. The notion of *linkage* for Pfister forms was introduced in [4]. With the definition from [4, Sect. 4] one can consider linkage of a finite number of Pfister forms. However, the study of linkage has mostly been limited to pairs of Pfister forms. Initially, this study was restricted to fields of characteristic different from 2, where quadratic forms are characterised by their associated (symmetric bilinear) polar forms.

When trying to extend notions and statements to cover the case of characteristic 2, one has to choose between quadratic forms or symmetric bilinear forms. In this article we work mainly in the setup of Milnor K-theory over a field of arbitrary characteristic. We study linkage of symbols in the Milnor K-groups modulo 2, or equivalently, of symmetric bilinear Pfister forms. In particular we study the condition that a certain Milnor K-group modulo 2 has *triple linkage*, i.e. that any three symbols have a common linkage. This condition turns out to have stronger consequences than usual linkage, in particular on the vanishing of higher K-groups. In the last section we focus on fields of characteristic different from 2 and relate the condition of triple linkage to quadratic forms and the Hasse number $\widetilde{u}$ (the u-invariant if the field is nonreal).

For a recent study of triple linkage of quadratic Pfister forms covering fields of characteristic 2 we refer the reader to [1].

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2. Symbols and Pfister forms

We refer to [3] for standard results used from quadratic form theory. Let E be a field. By a *form over* E we mean a pair (V, b) of a finite-dimensional E-vector space V and a non-degenerate symmetric bilinear form b on V. We use an equality sign to indicate that two forms are isometric.

Let n always denote a nonnegative integer. Given $a_1, \dots, a_n \in E^\times$ we denote the bilinear n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ over E by $\langle\langle a_1, \dots, a_n\rangle\rangle$. In the sequel we refer to bilinear Pfister forms simply as *Pfister forms*. Given a Pfister form π, the orthogonal complement of the subform $\langle 1\rangle$ in π is called the *pure part of* π.

Theorem 2.1 (Elman-Lam). *Let $r \in \mathbb{N}$. Let ρ be an anisotropic r-fold Pfister form over E and let ρ' denote its pure part. Let π be a Pfister form over E such that $\pi \otimes \rho$ is anisotropic and let $c_1 \in E^\times$ be such that $-c_1$ is represented by $\pi \otimes \rho'$. Then there exist $c_2, \dots, c_r \in E^\times$ such that $\pi \otimes \rho \simeq \pi \otimes \langle\langle c_1, \dots, c_r\rangle\rangle$.*

Proof: See [3, Proposition 6.15] or [4, Theorem 2.6]. □

We denote by $\mathsf{k}_n E$ the nth Milnor K-group of E modulo 2; this is the abelian group generated by symbols $\{a_1, \dots, a_n\}$ with $a_1, \dots, a_n \in E^\times$ which are subject to the defining relations that the map $(E^\times)^n \longrightarrow \mathsf{k}_n E$ given by $(a_1, \dots, a_n) \mapsto \{a_1, \dots, a_n\}$ is multilinear and further that $\{a_1, \dots, a_n\} = 0$ whenever $a_i \in E^{\times 2}$ for some $i \leqslant n$ or $a_i + a_{i+1} = 1$ for some $i < n$. The direct sum $\bigoplus_{n\in\mathbb{N}} \mathsf{k}_n E$ is a graded ring with the multiplication induced by concatenation of symbols.

We recall some results from [3] on the relation of symbols and Pfister forms, which for fields of characteristic different from 2 go back to [4]. We begin with the one-to-one correspondence between symbols and Pfister forms.

Theorem 2.2 (Elman-Lam). *For $a_1 \dots, a_n, b_1, \dots, b_n \in E^\times$, we have*

$$\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\} \quad \textit{if and only if} \quad \langle\langle a_1, \dots, a_n\rangle\rangle = \langle\langle b_1, \dots, b_n\rangle\rangle\,.$$

Proof: See [3, Theorem 6.20]. □

We denote by ΣE^2 the subgroup of $E^\times$ consisting of the non-zero sums of squares in E. Recall that the field E is *real* if $-1 \notin \Sigma E^2$, *nonreal* otherwise. For $m \in \mathbb{N}$ we denote by $\mathsf{D}_E(m)$ the subset of ΣE^2 consisting of the elements that are sums of m squares in E.

Corollary 2.3. *For $a \in E^\times$ and the symbol $\tau = \{-1, \dots, -1\}$ in $\mathsf{k}_n E$ the following hold:*

(*a*) $a \in \mathsf{D}_E(2^n)$ *if and only if* $\tau \cdot \{a\} = 0$ *in* $\mathsf{k}_{n+1} E$.
(*b*) $a \in \mathsf{D}_E(2^n - 1)$ *if and only if* $\tau = \{a, a_2, \dots, a_n\}$ *for certain* $a_2, \dots, a_n \in E^\times$.

Proof: This follows from Theorem 2.1 and Theorem 2.2. □

Lemma 2.4. *Let τ and τ' be symbols in $\mathsf{k}_n E$ and let $a, a' \in E^\times$ be such that $\tau \cdot \{a\} = \tau' \cdot \{a'\}$. Then $\tau \cdot \{a\} = \tau \cdot \{c\} = \tau' \cdot \{c\} = \tau' \cdot \{a'\}$ for some $c \in E^\times$.*

Proof: This follows from [3, Corollary 6.16 and Theorem 6.20]. □

3. Linkage

Assume in the sequel that $n \geqslant 2$. Given two symbols $\sigma_1, \sigma_2 \in \mathsf{k}_n E$, the sum $\sigma_1 + \sigma_2 \in \mathsf{k}_n E$ is equal to a symbol if and only if there exists a symbol $\sigma' \in \mathsf{k}_{n-1} E$ and $b_1, b_2 \in E^\times$ such that $\sigma_i = \sigma' \cdot \{b_i\}$ for $i = 1, 2$ (see [4, Lemma 5.4]); in this case we say that σ_1 and σ_2 are *linked*.

We say that $\mathsf{k}_n E$ *is linked* if any two symbols in $\mathsf{k}_n E$ are linked (which in the terminology of [6] corresponds to saying that $\mathsf{I}^n E$ is linked.) Obviously, if $\mathsf{k}_n E$ is linked, then so is $\mathsf{k}_m E$ for any integer $m \geqslant n$.

The following statement was obtained in [6, Corollary 2.8 and Corollary 2.9]. For convenience of the reader we include a compact proof, whose first lines follow [5, Sect. 3, Example 3]. The statement should be compared with Theorem 5.1.

Theorem 3.1 (Elman-Lam)**.** *Assume that $\mathsf{k}_n E$ is linked. Then we have that $\Sigma E^2 = \mathsf{D}_E(2^{n+1})$ and $\mathsf{k}_{n+2} E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$. In particular, if E is nonreal then $\mathsf{k}_{n+2} E = 0$.*

Proof: Consider an arbitrary symbol $\tau \in \mathsf{k}_{n-2} E$ and $a_1, a_2, b_1, b_2 \in E^\times$. Since $\mathsf{k}_n E$ is linked and by Lemma 2.4, there exist $c_1, c_2 \in E^\times$ and a symbol $\sigma \in \mathsf{k}_{n-1} E$ such that $\tau \cdot \{a_i, b_i\} = \tau \cdot \{a_i, c_i\} = \sigma \cdot \{c_i\}$ for $i = 1, 2$. It follows that $\tau \cdot \{a_1, c_1, c_2\} = \sigma \cdot \{c_1, c_2\} = \tau \cdot \{a_2, c_1, c_2\}$, whereby

$$\tau \cdot \{a_1, b_1, a_2, b_2\} = \tau \cdot \{a_1, c_1, a_2, c_2\} = \tau \cdot \{a_2, c_1, a_2, c_2\} = \{-1, c_1\} \cdot \tau \cdot \{a_2, b_2\}\,.$$

This argument shows that for any $a_1, \dots, a_{n+2} \in E^\times$ there exists $c \in E^\times$ such that

$$\{a_1, \dots, a_{n+2}\} = \{a_2, \dots, a_{n+1}, -1, c\}\,.$$

Applying this rule $n+1$ times, we conclude that every symbol in $\mathsf{k}_{n+2} E$ is of the form $\{-1, \dots, -1, c\}$ with $c \in E^\times$. Hence $\mathsf{k}_{n+2} E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$.

Moreover, if $a_1, \dots, a_{n+2} \in E^\times$ are such that $a_2 \in \mathsf{D}_E(2)$, then $\{-1, a_2\} = 0$ and we obtain from the above rule that $\{a_1, \dots, a_{n+2}\} = 0$. Hence we have $\{a\} \cdot \mathsf{k}_{n+1} E = 0$ for any $a \in \mathsf{D}_E(2)$.

Consider an element $c \in \mathsf{D}_E(2^{n+1} + 1)$. We write $c = a + b$ with $a \in \mathsf{D}_E(2)$ and $b \in \mathsf{D}_E(2^{n+1} - 1)$. In $\mathsf{k}_{n+1} E$ we obtain that $\{-1, \dots, -1\} = \{-b\} \cdot \tau$ for a symbol τ in $\mathsf{k}_n E$, by Corollary 2.3. Since $c - b = a$ we have $\{-b, c\} = \{a, bc\}$. As $\{a\} \cdot \mathsf{k}_{n+1} E = 0$ we obtain in $\mathsf{k}_{n+2} E$ that

$$\{-1, \dots, -1, c\} = \{-b, c\} \cdot \tau = \{a, bc\} \cdot \tau = 0\,,$$

which shows that $c \in \mathsf{D}_E(2^{n+1})$. This argument shows that $\Sigma E^2 = \mathsf{D}_E(2^{n+1})$.

Assume finally that E is nonreal. If $-1 \in E^{\times 2}$ then $\{-1\} = \{1\} = 0$ in $\mathsf{k}_1 E$. If $\mathsf{char}(E) \neq 2$ then $E^\times = \Sigma E^2 = \mathsf{D}_E(2^{n+1})$. Hence in any case we obtain that $\mathsf{k}_{n+2} E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E = 0$. $\square$

If E is nonreal and $\mathsf{k}_n E$ is linked where $n \geqslant 2$, then $\mathsf{k}_{n+2} E$ vanishes by Theorem 3.1, but we may have that $\mathsf{k}_{n+1} E \neq 0$, as the following well-known example shows.

Example 3.2. For $E = \mathbb{C}(\!(t_1)\!) \dots (\!(t_{n+1})\!)$, $\mathsf{k}_n E$ is linked and $\mathsf{k}_{n+1} E \simeq \mathbb{Z}/2\mathbb{Z}$.

4. The linkage pairing

We are going to investigate an operation on linked symbols. Let $n \geqslant 2$. To any pair of linked symbols in $\mathsf{k}_n E$ we associate a symbol in $\mathsf{k}_{n+1}E$.

Proposition 4.1. *Let $\sigma_1, \sigma_2 \in \mathsf{k}_n E$ be two linked symbols. There is a unique symbol $\rho \in \mathsf{k}_{n+1}E$ such that, for any symbol $\tau \in \mathsf{k}_{n-1}E$ and any $a_1, a_2 \in E^\times$ with $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2$, we have $\rho = \tau \cdot \{a_1, a_2\}$.*

Proof: By the hypothesis there exist a symbol $\tau \in \mathsf{k}_{n-1}E$ and two elements $a_1, a_2 \in E^\times$ with $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2$. Suppose we have another symbol $\tau' \in \mathsf{k}_{n-1}E$ and $a'_1, a'_2 \in E^\times$ with $\sigma_i = \tau' \cdot \{a'_i\}$ for $i = 1, 2$. By Lemma 2.4 there exist $c_1, c_2 \in E^\times$ such that $\tau \cdot \{c_i\} = \sigma_i = \tau' \cdot \{c_i\}$ holds for $i = 1, 2$. We obtain that $\tau \cdot \{a_1, a_2\} = \tau \cdot \{c_1, c_2\} = \tau' \cdot \{c_1, c_2\} = \tau' \cdot \{a'_1, a'_2\}$. □

Corollary 4.2. *Let $a, b, c \in E^\times$ and let τ be a symbol in $\mathsf{k}_{n-1}E$ such that $\tau \cdot \{-1, a\} = 0$. Assume that there exist a symbol ρ in $\mathsf{k}_n E$ and $x, y, z \in E^\times$ such that $\tau \cdot \{a, b\} = \rho \cdot \{x\}$, $\tau \cdot \{a, c\} = \rho \cdot \{y\}$ and $\tau \cdot \{b, c\} = \rho \cdot \{z\}$. Then $\tau \cdot \{a, b, c\} = 0$.*

Proof: By Proposition 4.1 we obtain that $\rho \cdot \{x, z\} = \tau \cdot \{a, b, c\} = \rho \cdot \{y, z\}$, whereby $\rho \cdot \{xy, z\} = 0$. We have $\tau \cdot \{b, -bc\} = \tau \cdot \{b, c\} = \rho \cdot \{z\}$. Since $\tau \cdot \{-1, a\} = 0$ we further have $\tau \cdot \{a, -bc\} = \tau \cdot \{a, bc\} = \rho \cdot \{xy\}$. We conclude with Proposition 4.1 that $\tau \cdot \{a, b, c\} = \tau \cdot \{a, b, -bc\} = \rho \cdot \{xy, z\} = 0$. □

Corollary 4.3. *Assume that $\mathsf{k}_n E$ is linked. We obtain a surjective pairing*

$$\langle \cdot, \cdot \rangle : \mathsf{k}_n E \times \mathsf{k}_n E \longrightarrow \mathsf{k}_{n+1}E$$

by letting $\langle \tau \cdot \{a_1\}, \tau \cdot \{a_2\} \rangle = \tau \cdot \{a_1, a_2\}$ for any symbol $\tau \in \mathsf{k}_{n-1}E$ and any $a_1, a_2 \in E^\times$.

Proof: Let $\sigma_1, \sigma_2 \in \mathsf{k}_n E$ be given. As $\mathsf{k}_n E$ is linked, there exist $a_1, a_2 \in E^\times$ and a symbol $\tau \in \mathsf{k}_{n-1}E$ such that $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2$. By Proposition 4.1 the symbol $\rho = \tau \cdot \{a_1, a_2\} \in \mathsf{k}_{n+1}E$ only depends on σ_1 and σ_2 but not on the choice of τ and $a_1, a_2 \in E^\times$. Hence, the pairing is well-defined. As $\mathsf{k}_n E$ is linked, so is $\mathsf{k}_{n+1}E$, and it follows that the pairing is surjective. □

If $\mathsf{k}_n E$ is linked then we call the pairing in Corollary 4.3 the *linkage pairing on* $\mathsf{k}_n E$.

Theorem 4.4. *Assume that $\mathsf{k}_n E$ is linked. Then the following are equivalent:*
- *(i) The linkage pairing on $\mathsf{k}_n E$ is bilinear.*
- *(ii) $\Sigma E^2 = \mathsf{D}_E(2^n)$ and $\mathsf{k}_{n+1}E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$.*
- *(iii) Either $\mathsf{k}_{n+1}E = 0$, or E is real and the rule $c \mapsto \{-1, \dots, -1, c\}$ determines an isomorphism $E^\times / \Sigma E^2 \longrightarrow \mathsf{k}_{n+1}E$.*

Proof: As a consequence of the definition of the linkage pairing

$$\langle \cdot, \cdot \rangle : \mathsf{k}_n E \times \mathsf{k}_n E \longrightarrow \mathsf{k}_{n+1}E$$

we have for any $\rho, \rho' \in \mathsf{k}_n E$ that $\langle \rho, \rho' \rangle = \rho \cdot \{d\}$ for some $d \in E^\times$.

($i \Rightarrow ii$) Consider an arbitrary symbol $\tau \in \mathsf{k}_{n-2}E$ and $a, b, c \in E^\times$. Set $\sigma_1 = \{a, b\}$, $\sigma_2 = \{a, c\}$, $\sigma_3 = \{b, c\}$ and $\sigma_4 = \{-ab, c\}$. We obtain that

$\langle \tau\cdot\sigma_i, \tau\cdot\sigma_1\rangle = \tau\cdot\{a,b,c\}$ for $i=2,3,4$. Assuming that the pairing is bilinear, we get that

$$\langle \tau\cdot(\sigma_2+\sigma_3+\sigma_4), \tau\cdot\sigma_1\rangle = \tau\cdot\{a,b,c\}\,.$$

Since $\sigma_2+\sigma_3+\sigma_4 = \{-1,c\}$, we conclude that

$$\tau\cdot\{a,b,c\} = \langle \tau\cdot\{-1,c\}, \tau\cdot\{a,b\}\rangle = \tau\cdot\{-1,c,d\}$$

for some $d\in E^\times$. This argument shows that, for any $c_1,\dots,c_n,c_{n+1}\in E^\times$, there exists $d\in E^\times$ such that $\{c_1,\dots,c_n,c_{n+1}\} = \{-1,c_1,\dots,c_{n-1},d\}$ in $\mathsf{k}_{n+1}E$. Using this rule n times we obtain for any $c_1,\dots,c_n,c_{n+1}\in E^\times$ that

$$\{c_1,\dots,c_{n+1}\} = \{-1,\dots,-1,c_1,d\} = \{-1,\dots,-1,d'\}$$

for some $d,d'\in E^\times$. This shows that $\mathsf{k}_{n+1}E = \{-1,\dots,-1\}\cdot\mathsf{k}_1E$ and that $\{a\}\cdot\mathsf{k}_nE = 0$ for any $a\in\mathsf{D}_E(2^{n-1})$.

Consider now an element $c\in\mathsf{D}_E(2^n+1)$. We choose $a,b\in\mathsf{D}_E(2^{n-1})$ such that $c-a-b$ is a square in E. Then we have $\{-a,-b,c\}=0$ in k_3E, whereby $\{a,-b,c\} = \{-1,-b,c\}$. For the symbol $\epsilon = \{-1,\dots,-1\}$ in $\mathsf{k}_{n-2}E$ we obtain that $\epsilon\cdot\{-1,b\}=0$ and conclude that

$$\epsilon\cdot\{-1,-1,c\} = \epsilon\cdot\{-1,-b,c\} = \epsilon\cdot\{a,-b,c\} = 0\,,$$

for $a\in\mathsf{D}_E(2^{n-1})$. Hence $\{-1,\dots,-1,c\}=0$ in $\mathsf{k}_{n+1}E$, whereby $c\in\mathsf{D}_E(2^n)$. This shows that $\Sigma E^2 = \mathsf{D}_E(2^n)$.

$(ii\Rightarrow iii)$ This implication is obvious.

$(iii\Rightarrow i)$ Let $\varepsilon = \{-1,\dots,-1\}$ in $\mathsf{k}_{n-1}E$. For any symbol $\tau\in\mathsf{k}_{n-1}E$ and any $a,b\in E^\times$ we have $\tau\cdot\{a\}\cdot\tau\cdot\{b\} = \varepsilon\cdot\tau\cdot\{a,b\} = \varepsilon\cdot\langle\tau\cdot\{a\},\tau\cdot\{b\}\rangle$. Since k_nE is linked, this means that

$$\rho\cdot\rho' = \varepsilon\cdot\langle\rho,\rho'\rangle \text{ in } \mathsf{k}_{2n}E \text{ for any } \rho,\rho'\in\mathsf{k}_nE\,.$$

Hence, the pairing $\mathsf{k}_nE\times\mathsf{k}_nE\longrightarrow\mathsf{k}_{2n}E, (\rho,\rho')\mapsto\varepsilon\cdot\langle\rho,\rho'\rangle$ is bilinear. On the other hand, (iii) implies that $\mathsf{k}_{n+1}E\longrightarrow\mathsf{k}_{2n}E, \xi\mapsto\varepsilon\cdot\xi$ is an isomorphism. Therefore the pairing $\langle\cdot,\cdot\rangle$ is bilinear. □

5. Triple linkage

Let $n\geqslant 2$. We say that k_nE has *triple linkage* if for any three symbols $\sigma_1,\sigma_2,\sigma_3\in\mathsf{k}_nE$ there exist a symbol $\tau\in\mathsf{k}_{n-1}E$ and $a_1,a_2,a_3\in E^\times$ such that $\sigma_i = \tau\cdot\{a_i\}$ for $i=1,2,3$. Note that this implies that k_nE is linked.

Theorem 5.1. *Assume that k_nE has triple linkage. Then $\Sigma E^2 = \mathsf{D}_E(2^n)$ and $\mathsf{k}_{n+1}E = \{-1,\dots,-1\}\cdot\mathsf{k}_1E$. In particular, if E is nonreal then $\mathsf{k}_{n+1}E=0$.*

Proof: Let $\langle\cdot,\cdot\rangle : \mathsf{k}_nE\times\mathsf{k}_nE\longrightarrow\mathsf{k}_{n+1}E$ be the linkage pairing. Consider three symbols $\sigma_1,\sigma_2,\sigma_3\in\mathsf{k}_nE$. By the hypothesis there exist a symbol $\tau\in\mathsf{k}_{n-1}E$ and $a_1,a_2,a_3\in E^\times$ such that $\sigma_i=\tau\cdot\{a_i\}$ for $i=1,2,3$. As $\sigma_1+\sigma_2=\tau\cdot\{a_1a_2\}$ we obtain that

$$\langle\sigma_1+\sigma_2,\sigma_3\rangle = \tau\cdot\{a_1a_2,a_3\} = \tau\cdot\{a_1,a_3\}+\tau\cdot\{a_2,a_3\} = \langle\sigma_1,\sigma_3\rangle+\langle\sigma_2,\sigma_3\rangle\,.$$

Hence the linkage pairing is bilinear and Theorem 4.4 yields the statement. □

Question 5.2. *If k_nE has triple linkage, does then any finite number of symbols in k_nE have a common linkage (by a symbol in $\mathsf{k}_{n-1}E$)?*

Triple linkage holds for $\mathsf{k}_n E$ if E is a $\mathcal{C}_n$-field, in the terms of Tsen-Lang Theory (see [11, Chap. 5]). This is a direct consequence of the next statement. For $n = 1$ and $|\mathcal{S}| = 3$ the statement corresponds to [13, Proposition 9].

Proposition 5.3. *Assume that there exists a finite system $\mathcal{S}$ of nonzero symbols in $\mathsf{k}_n E$ that do not have a common linkage. Then there exists an anisotropic system of $|\mathcal{S}| - 1$ quadratic forms in $|\mathcal{S}| \cdot 3 \cdot 2^{n-2}$ variables over E.*

Proof: Let $m \in \mathbb{N}$ be as large as possible such that there exist $a_1, \ldots, a_m \in E^\times$ for which the symbol $\{a_1, \ldots, a_m\}$ factors every element of $\mathcal{S}$. By the hypothesis $m \leqslant n-2$. We set $\pi = \langle\langle a_1, \ldots, a_m\rangle\rangle$. Using Theorem 2.1 and the one-to-one correspondence between Pfister forms and symbols, we choose for $\sigma \in \mathcal{S}$ an $(n-m)$-fold Pfister form ρ_σ over E such that σ corresponds to the n-fold Pfister form $\pi \otimes \rho_\sigma$ over E and we denote by ρ'_σ the pure part of ρ_σ. Note that $\dim(\pi \otimes \rho'_\sigma) = 2^n - 2^m$ for any $\sigma \in \mathcal{S}$. By Theorem 2.1 and by the maximality of m, there exists no element $c \in E^\times$ such that $-c$ is represented by all the forms $\pi \otimes \rho'_\sigma$ with $\sigma \in \mathcal{S}$. We fix $\sigma_0 \in \mathcal{S}$ and set $\mathcal{S}' = \mathcal{S} \smallsetminus \{\sigma_0\}$. Considering each of the forms $\pi \otimes \rho'_\sigma$ for $\sigma \in \mathcal{S}$ with its own variables, we obtain an anisotropic system of quadratic forms $(\pi \otimes \rho'_{\sigma_0} - \pi \otimes \rho'_\sigma)_{\sigma \in \mathcal{S}'}$ in $|\mathcal{S}| \cdot (2^n - 2^m)$ variables over E. If $m < n-2$ we may substitute zero for some of these variables. So in any case we obtain an anisotropic system of $|\mathcal{S}| - 1$ quadratic forms over E in exactly $|\mathcal{S}| \cdot 3 \cdot 2^{n-2}$ variables. $\square$

Let $\widetilde{u}(E)$ denote the *Hasse number of E*, which is defined as the supremum in $\mathbb{N} \cup \{\infty\}$ on the dimension of anisotropic totally indefinite quadratic forms over E (cf. [11, Chap. 8, Sect. 3]). The study of this invariant was initiated in [7], the notation was introduced in [2]. The definition of the Hasse number captures one of several possibilities to study bounds on the dimension of anisotropic quadratic forms in a meaningful way without restriction to nonreal fields. The results below have their main interest in the case where E is nonreal, and in this case $\widetilde{u}(E)$ is the usual u-invariant (cf. [11, Chap. 8] and [3, Chap. VI]).

Corollary 5.4. *If $\widetilde{u}(E(t)) \leqslant 2^{n+1}$ then $\mathsf{k}_n E$ has triple linkage.*

Proof: Suppose first that E is real and $\widetilde{u}(E(t)) < \infty$. It follows by [7, Theorem I] that E is hereditarily euclidean. Hence $\mathsf{k}_n E \simeq \mathbb{Z}/2\mathbb{Z}$, whereby $\mathsf{k}_n E$ has triple linkage.

Assume now that E is nonreal and $\widetilde{u}(E(t)) \leqslant 2^{n+1} < 9 \cdot 2^{n-2}$. By the Amer-Brumer Theorem [11, Chap. 9, Prop. 1.10], it follows that every pair of quadratic forms in $9 \cdot 2^{n-2}$ variables over E is isotropic. Hence $\mathsf{k}_n E$ has triple linkage, by Proposition 5.3. $\square$

The next example shows that the converse to the statement in Corollary 5.4 does not hold.

Example 5.5. Let E_0 be a quadratically closed field of characteristic not 2 having a finite field extension of even degree. (One can for example take E_0 as the quadratic closure of $\mathbb{Q}$: any polynomial over $\mathbb{Q}$ having as Galois group a dihedral group of order $2m$ for an odd positive integer m will have as splitting field over E_0 an extension of order $2m$. See also the discussion of finite extensions of quadratically closed fields [8, Chapter VII, §7].) It follows from this choice of E_0 that, $\widetilde{u}(E_0) = 1$ while $\widetilde{u}(F_0) \geqslant 2$ for some finite separable extension F_0/E_0. Thus $\mathsf{k}_1 F_0 \neq 0$. Consider the fields of iterated power series in n variables

$$E = E_0(\!(u_1)\!) \dots (\!(u_n)\!) \text{ and } F = F_0(\!(u_1)\!) \cdots (\!(u_n)\!)\,.$$

We obtain that $\widetilde{u}(E) = 2^n$ and $\mathsf{k}_{n+1} F \neq 0$. Since F/E is a finite separable extension, F is the residue field of a discrete valuation on $E(t)$. We conclude that $\mathsf{k}_{n+2} E(t) \neq 0$. After translation to Pfister forms via [3, Theorem 6.20], the Arason-Pfister Hauptsatz [3, Theorem 6.18] yields that $\widetilde{u}(E(t)) \geqslant 2^{n+2}$.

Note that $\{u_1, \dots, u_n\}$ is the only nonzero symbol in $\mathsf{k}_n E$. In particular $\mathsf{k}_n E$ has triple linkage. Finally, $\mathsf{k}_n F$ does not have triple linkage, for $\mathsf{k}_{n+1} F \neq 0$.

The following reformulates and enhances [7, Theorem G]. The notation $\mathsf{I}^n E$ refers to the nth power of the fundamental ideal $\mathsf{I} E$ in the Witt ring (of symmetric bilinear forms) of E; recall that $\mathsf{I}^n E$ is additively generated by the classes of the n-fold Pfister forms over E.

Theorem 5.6 (Elman-Lam-Prestel)**.** *Assume that* $\mathsf{char}(E) \neq 2$. *The following are equivalent:*

(*i*) $\widetilde{u}(E) \leqslant 4$.
(*ii*) *E is linked and $\mathsf{I}^3 E$ is torsion-free.*
(*iii*) *E is linked and the linkage pairing $\mathsf{k}_2 E \times \mathsf{k}_2 E \longrightarrow \mathsf{k}_3 E$ is bilinear.*
(*iv*) $\Sigma E^2 = \mathsf{D}_E(4)$ *and* $\mathsf{k}_3 E = \{-1,-1\} \cdot \mathsf{k}_1 E$.
(*v*) *$\mathsf{k}_2 E$ is linked and either $\mathsf{k}_3 E = 0$, or E is real and $c \mapsto \{-1,-1,c\}$ defines an isomorphism $E^\times / \Sigma E^2 \longrightarrow \mathsf{k}_3 E$.*

Proof: Conditions $(iii) - (v)$ are equivalent by Theorem 4.4. The equivalence of (i) and (ii) is shown in [7, Theorem G] and in [2, Theorem 4.7]. By [6, Corollary 2.9] (ii) implies that $\mathsf{I}^3 E = 4 \cdot \mathsf{I} E$. Therefore the equivalence of (ii) and (iv) follows using Theorem 2.2. □

Corollary 5.7. *Assume that* $\mathsf{char}(E) \neq 2$ *and that* $\mathsf{k}_2 E$ *has triple linkage. Then* $\widetilde{u}(E) \leqslant 4$.

Proof: This follows from Theorem 5.1 together with Theorem 5.6. □

In the case where E is nonreal, one can show the converse of Corollary 5.7 by using the following statement, which is a direct consequence of a deep result of Peyre [10, Proposition 6.1] combined with an observation by Sivatski [13, Corollary 11].

Proposition 5.8 (Peyre-Sivatski)**.** *Assume that* $\mathsf{char}(E) \neq 2$. *Let $\mathcal{H}$ be a subgroup of $\mathsf{k}_2 E$ with $|\mathcal{H}| \leqslant 8$. Assume that every element of $\mathcal{H}$ is a symbol and that $\mathsf{I}^3 E = 0$. Then there exists $a \in E^\times$ such that for every $\sigma \in \mathcal{H}$ one has $\sigma = \{a, b_\sigma\}$ for some $b_\sigma \in E^\times$.*

Proof: By the hypothesis every $\sigma \in \mathcal{H}$ corresponds to an E-quaternion algebra. Let $\sigma_1, \sigma_2, \sigma_3 \in \mathsf{k}_2 E$ be three symbols that generate $\mathcal{H}$ and let Q_1, Q_2, Q_3 denote the corresponding quaternion algebras. Since $\mathsf{I}^3 E = 0$ we have that $H^3(E, \mathbb{Z}/\mathbb{Q}(2)) = 0$. It follows by [10, Proposition 6.1] that there exists a field extension F/E with $[F : E] = 2m$ for an odd integer m and such that $(Q_i)_F$ is split for $i = 1, 2, 3$. Then, by [13, Corollary 11], Q_1, Q_2, Q_3 have a common slot $a \in E^\times$. It follows that any $\sigma \in \mathcal{H}$ is of the form $\sigma = \{a, b_\sigma\}$ with $b_\sigma \in E^\times$. $\square$

In [13] Sivatski seems to be unaware that he provides a proof of Proposition 5.8. In [13, Corollary 12] he comes to a closely related conclusion, but at the end of his article he asks whether fields of cohomological 2-dimension 2 satisfy the conclusion stated in Proposition 5.8. This was pointed out to the author of the present article by Adam Chapman and David Leep.

Corollary 5.9. *Assume that E is nonreal with $\mathsf{char}(E) \neq 2$. Then $\widetilde{u}(E) \leqslant 4$ if and only if k_2E has triple linkage.*

Proof: One implication is Corollary 5.7, the converse follows from Proposition 5.8. $\square$

The hypothesis in Proposition 5.8 on E can be weakened. Instead of assuming $\mathsf{I}^3E = 0$, which requires E to be nonreal, it is sufficient to assume that I^3E is torsion-free and that E has the so-called *ED-property* about field orderings introduced in [12]. This can be proven by using algebras with involutions and skew-hermitian forms over quaternion algebras. In this way Proposition 5.8 is recovered with a different proof, which in particular is independent of the algebraic geometry behind Peyre's result [10, Proposition 6.1].

Since $\widetilde{u}(E) < \infty$ implies that E satisfies the *ED*-property, the generalisation carries over to Corollary 5.9 and makes the condition that E is nonreal superfluous: If $\mathsf{char}(E) \neq 2$, then $\widetilde{u}(E) \leqslant 4$ if and only if k_2E has triple linkage.

The author is planning to give details on this argument in a future article.

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