JORDAN PROPERTY FOR LIE GROUPS AND AUTOMORPHISM GROUPS OF COMPLEX SPACES

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Abstract. We prove that all connected finite-dimensional real Lie groups are Jordan. This implies that all algebraic groups (not necessarily affine) over fields of characteristic zero and some transformation groups of complex spaces and Riemannian manifolds are Jordan.

1. Introduction. We recall the definition introduced in [Po 2011, Def. 2.1]:

Definition 1. Given a group G, put

$$J_G := \sup_F \min_A [F : A],$$

where F runs over all finite subgroups of G and A runs over all normal abelian subgroups of F. If $J_G \neq \infty$, then G is called a *Jordan group* and J_G is called the *Jordan constant* of G. In this case, we also say that G enjoys the *Jordan property.*

Informally, the Jordan property of G means that all finite subgroups of G are "almost abelian" in the sense that they are extensions of abelian groups by groups taken from only a finite list. Definition 1 is inspired by the classical theorem of Jordan [Jo 1878] claiming that $J_{\mathrm{GL}_n(\ell)} \neq \infty$ holds for every n and every field ℓ of characteristic zero. If ℓ is algebraically closed, then, for every fixed n, the constant $J_{\mathrm{GL}_n(\ell)}$ is independent of ℓ, so we denote it simply by $J(n)$. It has been computed in [Co2007]; in particular,

$$J(n) = (n+1)! \text{ for all } n \geqslant 71 \text{ and } n = 63, 65, 67, 69.$$

For more examples of Jordan groups see [Po 2014_1].

Below variety means algebraic variety over a fixed algebraically closed field k of characteristic zero; in particular, any algebraic group is defined over k. If G is either an algebraic group or a topological group, G^0 denotes the identity component of G.

After being posed seven years ago in [Po 2011, Sect. 2] (see also [Po 2014_1, Sect. 2]), the following problem was explored by a number of researchers (see the most recent brief survey and references in [PS 2017, Sect. 1]):

Problem. *Describe varieties X for which the group* $\mathrm{Aut}(X)$ *is Jordan.*

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At present (April 2018) it is still unknown whether there are varieties X such that the group $\mathrm{Aut}(X)$ is non-Jordan (note that complex manifolds whose automorphism groups are non-Jordan do exist, see below Remark 2). On the other hand, by now for many types of varieties X it is shown that the group $\mathrm{Aut}(X)$ is Jordan. In particular, S. Meng and D.-Q. Zhang recently proved the following

Theorem 1 ([MZ 2015, Thm. 1.6])**.** *For every projective variety X, the group $\mathrm{Aut}(X)$ is Jordan.*

Given a variety X, we denote by $\mathrm{Aut}(X)^0$ the identity component of $\mathrm{Aut}(X)$ in the sense of [Ra 1964]; see also [Po 2014_2]. By [Ra 1964, Cor. 1], if X is complete, then $\mathrm{Aut}(X)^0$ is a connected (not necessarily affine) algebraic group. Jordan's theorem cited above implies the claim that every affine algebraic group is Jordan; see [Po 2014_1, Thm. 2]. The key ingredient of the proof of Theorem 1 given in [MZ 2015] is the proof that the extension of this claim to all (i.e., not necessarily affine) algebraic groups holds true. The latter proof is rather involved.

In the present note we obtain, with a very short proof, a general result, from which the above-mentioned extension immediately follows (see Theorem 4 below). Namely, we prove that every finite-dimensional connected real Lie group is Jordan (the more precise and general statements are formulated in Theorems 2, 3, and Corollary 3 below). Then in Sections 5–7 we apply this to showing that some transformation groups of complex spaces and Riemannian manifolds are Jordan (see Theorems 5, 7, 8, 9 below).

The question of whether the Lie groups are Jordan was posed to me by A. M. Vershik (see [Po 2014_3, 95:20]) whom I thank. I am grateful to Yu. G. Zarhin for the valuable comments.

2. Lie groups. We now explore the Jordan property for finite-dimensional real Lie groups G. Note that non-Jordan groups of this type do exist, because every discrete group is a 0-dimensional real Lie group and there are non-Jordan discrete groups (see [Po 2014_1, 1.2.5]). Therefore, the Jordan property of G can be expected only under some constraint on the component group G/G^0.

To formulate this restriction we recall the following definition introduced in [Po 2011, Def. 2.9]:

Definition 2. Given a group H, put
$$b_H := \sup_F |F|,$$
where F runs over all finite subgroups of H. If $b_H \neq \infty$, then the group H is called *bounded*.

In particular, every finite group H is bounded and $b_H = |H|$.

In Theorems 2, 3 and Corollary 4 below, we consider the class of finite-dimensional real Lie groups G whose component group G/G^0 is bounded.

Note that every compact Lie group K belongs to this class, because K/K^0 is finite.

Theorem 2. *Let G be a finite-dimensional real Lie group whose component group G/G^0 is bounded. Then G is Jordan.*

Proof. By [Po 2011, Lem. 2.11] (or [Po 2014_1, Thm. 5]), we may (and shall) assume that G is connected. This assumption implies the existence of a compact Lie subgroup K of G such that every compact subgroup of G is conjugate to that of K (see, e.g., [Ho 1965, Chap. XV, Thm. 3.1(iii)]). In particular, every finite subgroup of G is conjugate to that of K. This and Definition 1 show that G is Jordan if and only if K is, and if they are, then

$$J_G = J_K. \tag{1}$$

Being compact, the group K admits a faithful finite-dimensional representation, i.e., is isomorphic to a subgroup of $\mathrm{GL}_m(\mathbb{R})$ for some m (see, e.g., [OV 1990, Chap. 5, §2, Thm. 10]). Since the latter group is Jordan, K is Jordan as well (see [Po 2014_1, Thm. 3(i)]. This completes the proof. $\square$

Corollary 1. *For every finite-dimensional real Lie group G whose component group G/G^0 is bounded, the set of isomorphism classes of all finite simple subgroups of G is finite.*

We now dwell on estimating the Jordan constants of Lie groups whose component group is finite, with a view of proving that the class of such groups enjoys a property stronger than that of all its members to be Jordan (see Corollary 3 below). Seeking only this goal, we did not seek to improve the estimates obtained.

Lemma 1. *Let S be a simply connected simple affine algebraic group. Then the minimum* $\operatorname{rdim} S$ *of dimensions of faithful linear algebraic representations of S is given by the following table:*

type of S	A_ℓ $\ell\geqslant 1$	B_ℓ $\ell\geqslant 2$	C_ℓ $\ell\geqslant 2$	D_ℓ $\ell\geqslant 3$, ℓ odd	D_ℓ $\ell\geqslant 4$, ℓ even	E_6	E_7	E_8	F_4	G_2
$\operatorname{rdim} S$	$\ell+1$	2^ℓ	2ℓ	$2^{\ell-1}$	$2\ell+2^{\ell-1}$	27	56	248	26	7

Remark 1. In the proof of Lemma 1 below, a faithful representation of S of dimension $\operatorname{rdim} S$ is explicitly specified for each type of S.

Proof of Lemma 1. By Lefschetz's principle (see, e.g., [Si 1968, VI.6]), we may (and shall) assume that k is $\mathbb{C}$. We fix a maximal torus T of S. Let $\alpha_1, \ldots, \alpha_\ell \in (\operatorname{Lie} T)^*$, $\varpi_1, \ldots, \varpi_\ell \in (\operatorname{Lie} T)^*$, and $\alpha_1^\vee, \ldots, \alpha_\ell^\vee \in \operatorname{Lie} T$ be respectively the system of simple roots, fundamental weights, and simple coroots of $\operatorname{Lie} T$ with respect to a fixed Borel subalgebra of $\operatorname{Lie} S$ containing $\operatorname{Lie} T$; we number them as in [OV 1990].

The center Z of S is a finite subgroup of T. Fix a subset $\widetilde{Z}$ of $\operatorname{Lie} T$ whose image under the exponential map $\operatorname{Lie} T \to T$ is the set of all nonidentity elements of Z.

For every dominant weight $\lambda \in (\operatorname{Lie} T)^*$, let $R(\lambda)$ be an irreducible representation of $\operatorname{Lie} S$ with the highest weight λ. The dimension of $R(\varpi_i)$ for every i is specified in [OV 1990, Ref. Chap., §2, Table 5, pp. 299–305]. Note that Weyl's dimension formula implies

$$\dim R\big(\textstyle\sum_{i=1}^{\ell} \lambda_i \varpi_i\big) \geqslant \dim R\big(\sum_{i=1}^{\ell} \mu_i \varpi_i\big) \quad \text{if } \lambda_i \geqslant \mu_i \text{ for every } i. \tag{2}$$

Since S is simply connected, $R(\lambda)$ is the differential of a finite-dimensional linear algebraic representation $\mathcal{R}(\lambda)$ of S. Since S is simple, for every finite set D of nonzero dominant weights and $\mathcal{R}(D) := \bigoplus_{\lambda \in D} \mathcal{R}(\lambda)$, we have $\ker \mathcal{R}(D) \subseteq Z$. Hence

$$\mathcal{R}(D) \text{ is faithful} \iff \text{for every } x \in \widetilde{Z} \text{ there is } \lambda \in D \text{ with } \lambda(x) \notin \mathbb{Z}. \tag{3}$$

As is well known, $\dim \mathcal{R}(\varpi_1)$ is the minimum of dimensions of nonzero finite-dimensional algebraic representations of S (see [OV 1990, pp. 299–305]).

If S is of type E_8, F_4, or G_2, then Z is trivial; hence in this case $\mathcal{R}(\varpi_1)$ is faithful and therefore we have the equality

$$\operatorname{rdim} S = \dim R(\varpi_1), \tag{4}$$

which proves the claim of Lemma 1 for these types.

If S is of type A_ℓ or C_ℓ, then S is respectively $\mathrm{SL}_{\ell+1}$ and $\mathrm{Sp}_{2\ell}$. Since for these groups $\mathcal{R}(\varpi_1)$ is the tautological faithful representation, in this case (4) holds as well, which proves the claim of Lemma 1 for these types.

For the other types, we apply (3) to the set $\widetilde{Z}$ taken from [OV 1990, Ref. Chap., §2, Table 3, p. 298]. Below is used that for any $\lambda_i, \mu_i \in k$,

$$\text{the value of } \textstyle\sum_{i=1}^{\ell} \lambda_i \varpi_i \in (\operatorname{Lie} T)^* \text{ in } \sum_{i=1}^{\ell} \mu_i \alpha_i^\vee \in \operatorname{Lie} T \text{ is } \sum_{i=1}^{\ell} \lambda_i \mu_i. \tag{5}$$

If S is of type E_7, then $\widetilde{Z}$ consists of only one element $\zeta := (\alpha_1^\vee + \alpha_3^\vee + \alpha_7^\vee)/2$. By (5), we have $\varpi_1(\zeta) = 1/2 \notin \mathbb{Z}$, so $\mathcal{R}(\varpi_1)$ is faithful. Therefore, in this case again (4) holds, which proves the claim of Lemma 1 for this type.

If S is of type E_6, then $\widetilde{Z}$ consists of two elements $\zeta := (\alpha_1^\vee - \alpha_2^\vee + \alpha_4^\vee - \alpha_5^\vee)/3$ and 2ζ. Since $\varpi_1(z) = 1/3 \notin \mathbb{Z}$, $\varpi_1(2z) = 2/3 \notin \mathbb{Z}$, in this case again $\mathcal{R}(\varpi_1)$ is faithful; whence (4) holds. This proves the claim of Lemma 1 for this type.

If S is of type B_ℓ, then $\widetilde{Z}$ consists of only one element $\alpha_\ell^\vee/2$. This and (3), (5) imply that $\mathcal{R}(D)$ is faithful if and only if D contains $\sum_{i=1}^{\ell} \lambda_i \varpi_i$ with odd λ_ℓ. Using (2), from this we infer that $\mathcal{R}(\varpi_\ell)$ is the faithful representation of minimal dimension. Hence $\operatorname{rdim} S = \dim R(\varpi_\ell)$. This proves the claim of Lemma 1 for this type.

If S is of type D_ℓ, $\ell \geqslant 3$, ℓ odd, then $\widetilde{Z}$ consists of three elements

$$\zeta := (\alpha_1^\vee + \alpha_3^\vee + \cdots + \alpha_{\ell-2}^\vee)/2 + (\alpha_{\ell-1}^\vee - \alpha_\ell^\vee)/4, \ \ 2\zeta, \ \ 3\zeta. \tag{6}$$

¿From (3), (5), (6) we infer that $\mathcal{R}(D)$ is faithful if and only if D contains $\sum_{i=1}^{\ell} \lambda_i \varpi_i$ such that 4 is coprime to either $\lambda_{\ell-1}$ or λ_ℓ. This and (2) show that

$\mathcal{R}(\varpi_\ell)$ is the faithful representation of minimal dimension. Hence $\operatorname{rdim} S = \dim R(\varpi_\ell)$, proving the claim of Lemma 1 for this type.

If S is of type D_ℓ, $\ell \geqslant 4$, ℓ even, then $\widetilde{Z}$ consists of three elements

$$\zeta_1 := (\alpha_1^\vee + \alpha_3^\vee + \cdots + \alpha_{\ell-1}^\vee)/2, \;\; \zeta_2 := (\alpha_{\ell-1}^\vee + \alpha_\ell^\vee)/2, \;\; \zeta_1 + \zeta_2. \tag{7}$$

Hence if $\mathcal{R}(D)$ is faithful, then D contains $\sum_{\imath=1}^{\ell} \lambda_i \varpi_i$ with odd λ_ℓ or $\lambda_{\ell-1}$ and $\sum_{\imath=1}^{\ell} \mu_i \varpi_i$ with odd μ_i for some odd $i \neq \ell - 1$. On the other hand, since in this case Z is not cyclic, Schur's lemma implies that $|D| \geqslant 2$. ¿From this it is not difficult to deduce that $\mathcal{R}(\varpi_1) \oplus \mathcal{R}(\varpi_\ell)$ is the faithful representation of minimal dimension. Hence $\operatorname{rdim} S = \dim R(\varpi_1) + \dim R(\varpi_\ell) = 2\ell + 2^{\ell-1}$. This completes the proof of Lemma 1. □

Corollary 2. *Every simply connected simple affine algebraic group of rank ℓ admits a faithful linear algebraic representation of dimension at most $2^\ell + 10$.*

Proof. Clearly if an algebraic group admits a faithful linear algebraic representation, then it admits a faithful linear algebraic representation of any bigger dimension. In view of this, the claim follows from the inequality $\operatorname{rdim} S \leqslant 2^\ell + 10$, which, in turn, follows from Lemma 1: indeed, the latter shows that $\operatorname{rdim} S \leqslant 2^\ell$ if the type of S differs from F_4 and G_2, and that $\operatorname{rdim} S = 2^\ell + 10$ and $2^\ell + 3$ respectively for the types F_4 and G_2. □

Theorem 3. *Let G be an n-dimensional real Lie group whose component group G/G^0 is bounded. Then*

$$J_G \leqslant b_{G/G^0} J\big(n(2^n + 10)\big)^{b_{G/G^0}}. \tag{8}$$

Proof. By [Po 2011, Lem. 2.11] (or [Po 2014_1, Thm. 5]), we may (and shall) assume that G is connected; in particular,

$$b_{G/G^0} = 1. \tag{9}$$

We use the notation of the proof of Theorem 2. Since G is connected, K is connected, too; see [Ho 1965, Chap. XV, Thm. 3.1(ii)]. Hence (see [Bo1982, §1, Prop. 4]) there are

(i) the compact simply connected simple Lie groups $K_1, \ldots, K_d$;
(ii) a compact torus S;
(iii) a group epimorphism with finite kernel

$$\pi\colon \widetilde{K} := K_1 \times \cdots \times K_d \times S \to K. \tag{10}$$

By [Po 2014_1, Thm. 3(ii)], from (iii) we infer that

$$J_K \leqslant J_{\widetilde{K}}. \tag{11}$$

Every K_i is a real form of the corresponding simply connected simple complex affine algebraic group. The rank ℓ_i of the latter is equal to that of K_i. By Corollary 2 we then conclude that K_i admits an embedding in $\mathrm{GL}_{2^{\ell_i}+10}(\mathbb{C})$. Since $\ell_i \leqslant \dim \widetilde{K} = \dim K \leqslant n$, this in turn implies that K_i admits an embedding in $\mathrm{GL}_{2^n+10}(\mathbb{C})$. Clearly, S admits an embedding in $\mathrm{GL}_{\dim S}(\mathbb{C})$, and therefore, in view of $\dim S \leqslant \dim \widetilde{K}$, also in $\mathrm{GL}_{2^n+10}(\mathbb{C})$.

This and the definition of $\widetilde{K}$ (see (10)) show that $\widetilde{K}$ admits an embedding in the direct product of $d+1$ copies of $\mathrm{GL}_{2^n+10}(\mathbb{C})$, hence in $\mathrm{GL}_{(d+1)(2^n+10)}(\mathbb{C})$. In turn, since, in view of (10), we have $d+1 \leqslant \dim \widetilde{K}$, from this we infer that $\widetilde{K}$ admits an embedding in $\mathrm{GL}_{n(2^n+10)}(\mathbb{C})$; whence,

$$J_{\widetilde{K}} \leqslant J\bigl(n(2^n+10)\bigr). \tag{12}$$

Now, putting (1), (11), (12), (9) together, we complete the proof. $\square$

Recall from [MZ 2015] the following

Definition 3. A family $\mathcal{F}$ of groups is called *uniformly Jordan* if every group in $\mathcal{F}$ is Jordan and there is an integer $J_{\mathcal{F}}$ such that $J_G \leqslant J_{\mathcal{F}}$ for every $G \in \mathcal{F}$.

Corollary 3. *Fix an integer $n \geqslant 0$. Let $\mathcal{L}_n$ be the family of all connected n-dimensional real Lie groups. Then*

(i) *the family $\mathcal{L}_n$ is uniformly Jordan;*
(ii) *one can take $J_{\mathcal{L}_n} = J\bigl(n(2^n+10)\bigr)$.*

Proof. This follows from (8) because $b_{G/G^0} = 1$ for every $G \in \mathcal{L}_n$. $\square$

Corollary 4. *For every integer $n \geqslant 0$, the set of isomorphism classes of finite simple groups embeddable in n-dimensional connected real Lie groups is finite.*

3. Algebraic groups. We now consider several applications of Theorems 2 and 3. First, we apply them to algebraic groups, answering Question 1.2 in [MZ 2015]:

Theorem 4. *Every (not necessarily affine) n-dimensional algebraic group G over an algebraically closed field k of characteristic 0 is Jordan. Moreover,*

$$J_G \leqslant [G:G^0]J\bigl(n(2^{2n+1}+20)\bigr)^{[G:G^0]}. \tag{13}$$

Proof. In this case, G/G^0 is finite. By Lefschetz's principle, we may (and shall) assume that k is $\mathbb{C}$. Then G has a structure of $2n$-dimensional real Lie group whose identity component is G^0. The claim then follows from Theorem 3. $\square$

Statement (i) of the next corollary is one of the main results of [MZ 2015]:

Corollary 5. *Fix an integer $n \geqslant 0$. Let $\mathcal{A}_n$ be the family of all (not necessarily affine) connected n-dimensional algebraic groups over an algebraically closed field k of characteristic 0. Then*

(i) ([MZ 2015, Thm. 1.3]) *the family $\mathcal{A}_n$ is uniformly Jordan;*
(ii) *one can take $J_{\mathcal{A}_n} = J\bigl(n(2^{2n+1}+20)\bigr)$.*

Proof. This follows from (13). $\square$

4. Automorphism groups of complex spaces. The next application is to automorphism groups of complex spaces.

Let C be a (not necessarily reduced) complex space. There exists a topology on $\mathrm{Aut}(C)$ with respect to which $\mathrm{Aut}(C)$ is a topological group (see [Ak 1995, 2.1]).

Theorem 5. *For every compact complex space C, the group $\mathrm{Aut}(C)^0$ is Jordan.*

Proof. By [Ka 1965], the compactness of C implies that $\mathrm{Aut}(C)$ is a complex Lie group. The claim then follows from Theorem 2. $\square$

We do not know whether the statement of Theorem 5 remains true if $\mathrm{Aut}(C)^0$ is replaced by $\mathrm{Aut}(C)$. By [PS 2017, Thm. 1.5], the answer is affirmative if C is a connected compact two-dimensional complex manifold. By Theorem 1, it is also affirmative if C is a projective variety. On the other hand, we recall that by [CPS 2014] there are connected smooth compact real manifolds whose diffeomorphism groups are non-Jordan (this disproves Ghys' conjecture).

Remark 2. There are connected noncompact complex manifolds, whose automorphism groups are non-Jordan. Indeed, by [Wi 2002], for any countable group Γ, there is a noncompact Riemann surface M such that $\mathrm{Aut}(M)$ is isomorphic to Γ; whence the claim because of the existence of countable non-Jordan groups (see [Po 2014_1, Sect. 1.2.5]).

In actual fact, using the idea exploited earlier in [Po 2015], one can prove more than is said in Remark 2, showing the existence of connected complex manifolds with monstrous automorphism groups, namely:

Theorem 6. *There is a 3-dimensional simply connected noncompact complex manifold M such that*

(i) *the group $\mathrm{Aut}(M)$ contains an isomorphic copy of every finitely presentable (in particular, every finite) group;*
(ii) *every such copy is a discrete transformation group of M acting freely.*

Proof. It follows (see, e.g., [Ro 1995, Thm. 12.29]) from Higman's embedding theorem [Hi 1961] that there is a universal finitely presented group, i.e., a finitely presented group $\mathcal{U}$ containing as a subgroup an isomorphic copy of every finitely presented group. In turn, by [ABCKT 1996, Cor. 1.66] the finite presentability of $\mathcal{U}$ implies the existence of a connected 3-dimensional compact complex manifold B whose fundamental group is isomorphic to $\mathcal{U}$. Consider the universal cover $\pi\colon \widetilde{B} \to B$. Then $\widetilde{B}$ is a simply connected noncompact 3-dimensional complex manifold and the deck transformation group of π is a subgroup of $\mathrm{Aut}\,\widetilde{B}$ isomorphic to $\mathcal{U}$, which acts on $\widetilde{B}$ freely. Hence one can take $M = \widetilde{B}$. $\square$

Remark 3. For M from Theorem 6, the group $\mathrm{Aut}(M)$ is non-Jordan, because for every integer n, there is a finite simple group of order $> n$ (cf. [Po 2014_1, Example 4].

5. Automorphism groups of hyperbolic complex manifolds. The next application is to complex manifolds hyperbolic in the sense of Kobayashi (in particular, to bounded domains in $\mathbb{C}^n$).

Theorem 7. *Fix an integer $n \geqslant 0$. Let $\mathcal{H}_n$ be the family of groups $\mathrm{Aut}(M)^0$, where M runs over all connected complex manifolds hyperbolic in the sense of Kobayashi and of complex dimension n. Then*

(i) *the family $\mathcal{H}_n$ is uniformly Jordan;*
(ii) *one can take $J_{\mathcal{H}_n} = J\big((2n+n^2)(2^{2n+n^2}+10)\big)$;*
(iii) *for every point $x \in M$, the $\mathrm{Aut}(M)$-stablizer $\mathrm{Aut}(M)_x$ of x is Jordan and $J_{\mathrm{Aut}(M)_x} \leqslant J(n)$.*

Proof. Let M be a connected complex manifolds hyperbolic in the sense of Kobayashi and of complex dimension n. By [Ko 2005, Thms. 2.1, 2.6], $\mathrm{Aut}(M)$ is a real Lie group of dimension $\leqslant 2n+n^2$; whence (i) and (ii) by Theorems 2 and 3. By [Ko 2005, Thm. 2.6], the isotropy representation of $\mathrm{Aut}(M)_x$ is faithful and its image is isomorphic to a subgroup of the unitary group $\mathrm{U}(n)$; whence (iii). $\square$

Remark 4. The group $\mathrm{Aut}(M)^0$ in the formulation of Theorem 7 cannot be replaced by $\mathrm{Aut}(M)$. Indeed, it follows from the construction in [Wi 2002] that the Riemann surface M in Remark 2 is hyperbolic in the sense of Kobayashi. Therefore there are connected hyperbolic complex manifolds M such that the group $\mathrm{Aut}(M)$ is not Jordan.

However, as the next theorem shows, for complex hyperbolic manifolds M of a special type, the Jordan property holds for the whole $\mathrm{Aut}(M)$ rather than only for $\mathrm{Aut}(M)^0$.

Theorem 8. *For every strongly pseudoconvex bounded domain M with smooth boundary in $\mathbb{C}^n$, the group $\mathrm{Aut}(M)$ of all biholomorphic transformations of M is Jordan.*

Proof. If the Lie group $\mathrm{Aut}(M)$ is compact, then the claim follows from Theorem 2. If the group $\mathrm{Aut}(M)$ is non-compact, then, by the Rosey–Wong theorem [Ro 1979], [Wo 1977], the domain M is biholomorphic to the unit ball B_n in $\mathbb{C}^n$. Since $\mathrm{Aut}(B_n)$ is $\mathrm{PU}(n,1)$ (see [Ak 1995, Sect. 2.7, Prop. 3]), and the latter Lie group is connected (see [He 1962, Chap. IX, Lem. 4.4]), the claim then follows from Theorem 2. $\square$

Corollary 6. *For every strongly pseudoconvex bounded domain M with smooth boundary in $\mathbb{C}^n$, the set of isomorphism classes of all finite simple groups of biholomorphic transformations of M is finite.*

6. Isometry groups of Riemannian manifolds. The last application is to isometry groups $\mathrm{Iso}(M)$ of Riemannian manifolds M. They are topological groups with respect to the compact-open topology [Ko 1995].

Theorem 9. *Fix an integer $n \geqslant 0$. Let $\mathcal{R}_n$ be the family of groups* $\mathrm{Iso}(M)^0$*, where M runs over all connected n-dimensional Riemannian manifolds. Then*

(i) *the family $\mathcal{R}_n$ is uniformly Jordan;*
(ii) *one can take $J_{\mathcal{R}_n} = J\big((n^2+n)(2^{(n^2+n-2)/2}+5)\big)$;*
(iii) *for every point $x \in M$, the* $\mathrm{Iso}(M)$*-stabilizer* $\mathrm{Iso}(M)_x$ *of x is Jordan;*
(iv) *if the manifold M is compact, then the group* $\mathrm{Iso}(M)$ *is Jordan.*

Proof. It is known (see, e.g., [Ko 1995, Chap. II, Thms. 1.2 and 3.1]) that $\mathrm{Iso}(M)$ is a real Lie group of dimension at most $n(n+1)/2$, the group $\mathrm{Iso}(M)_x$ is compact for every x, and the group $\mathrm{Iso}(M)$ is compact if the manifold M is compact. The claims then follows from combining these facts with Theorems 2 and 3. □

Remark 5. The group $\mathrm{Aut}(M)^0$ in the formulation of Theorem 9 cannot be replaced by $\mathrm{Aut}(M)$. Indeed, it follows from the construction in [Wi 2002] that the Riemann surface M in Remark 2 is a two-dimensional Riemannian manifold and $\mathrm{Aut}(M) = \mathrm{Iso}(M)$. Therefore there are connected Riemannian manifolds M such that the group $\mathrm{Iso}(M)$ is not Jordan.

7. Concluding remarks. In view of (1), computing the Jordan constants of connected real Lie groups is reduced to that of compact such groups. For instance, the results of [Co2007] may be interpreted as computing the Jordan constants of all unitary groups:

$$J_{\mathrm{U}_n} = J(n) \text{ for every } n.$$

This leads to the following natural

Problem. *Compute the Jordan constants of all simple compact connected real Lie groups.*

References

[Ak 1995] D. N. Akhiezer, *Lie Group Actions in Complex Analysis*, Aspects of Mathematics, Vol. E 27, Vieweg, Braunschweig, 1995.

[ABCKT 1996] J. Amorós, M. Burger, K. Corlette, D. Kotschick, D. Toledo, *Fundamental Groups of Compact Kähler Manifolds*, Mathematical Surveys and Monographs, Vol. 44, American Mathematical Society, Providence, RI, 1996.

[Bo1982] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. 9, *Groupes de Lie Réels Compacts*, Masson, Paris, 1982.

[Co2007] M. J. Collins, *On Jordan's theorem for complex linear groups*, J. Group Theory **10** (2007), 411–423.

[CPS 2014] B. Csikós, L. Pyber, E. Szabó, *Diffeomorphism groups of compact* 4*-manifolds are not always Jordan*, `arXiv:1411.7524` (2014).

[He 1962] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

[Hi 1961] G. Higman, *Subgroups of finitely presented groups*, Proc. R. Soc. London A **262** (1961), 455–475.

[Ho 1965] G. Hochschild, *The Structure of Lie Groups*, Holden-Day, San Francisco, 1965.

[Jo 1878] C. Jordan, *Mémoire sur les equations différentielle linéaire à intégrale algébrique*, J. Reine Angew. Math. **84** (1878), 89–215.

[Ka 1965] W. Kaup, Infinitesimale Transformationsgruppen komplexer Räume, Math. Ann. **160** (1965), 72–92.

[Ko 1995] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer, Berlin, 1995.

[Ko 2005] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings. An Introduction*, 2nd ed., World Scientific, New Jersey, 2005.

[MZ 2015] S. Meng, D.-Q. Zhang, *Jordan property for non-linear algebraic groups and projective varieties*, to appear in Amer. J. Math., available at `https://preprint.press.jhu.edu/ajm/article/jordan-property-non-linear-algebraic-groups-and-projective-varieties`.

[OV 1990] A. L. Onishchik, È. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990.

[Po 2011] V. L. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*, in: *Affine Algebraic Geometry: The Russell Festschrift*, CRM Proceedings and Lecture Notes, Vol. 54, American Mathematical Society, Providence, RI, 2011, pp. 289–311.

[Po 2014_1] V. L. Popov, *Jordan groups and automorphism groups of algebraic varieties*, in: *Automorphisms in Birational and Affine Geometry* (Levico Terme, Italy, October 2012), Springer Proceedings in Mathematics & Statistics, Vol. 79, Springer, Heidelberg, 2014, pp. 185–213.

[Po 2014_2] V. L. Popov, *On infinite dimensional algebraic transformation groups*, Transform. Groups **19** (2014), no. 2, 549–568.

[Po 2014_3] V. L. Popov, *Jordan groups*, talk at the St. Petersburg Division of Steklov Institute of Mathematics, Russian Academy of Sciences, December 18, 2014, videorecord available at `http://www.mathnet.ru/php/seminars.phtml?&presentid=10616&option_lang=eng`.

[Po 2015] V. L. Popov, *Finite subgroups of diffeomorphism groups*, Proc. Steklov Inst. Math. **289** (2015), 221–226.

[PS 2017] Yu. Prokhorov, C. Shramov, *Automorphism groups of compact complex surfaces*, `arXiv:1708.03566` (2017).

[Ra 1964] C. P. Ramanujam, *A note on automorphism groups of algebraic varieties*, Math. Annalen **156** (1964), 25–33.

[Ro 1979] J.-P. Rosay, *Sur une caractérisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe dautomorphismes*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 4, 91–97.

[Ro 1995] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th ed., Graduate Texts in Mathematics, Vol. 148, Springer, New York, 1995.

[Si 1968] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Graduate Texts in Mathematics, Vol. 106, Springer, Dordrecht, 2009.

[Wi 2002] J. Winkelmann, *Realizing countable groups as automorphism groups of Riemann surfaces*, Documenta Math. **7** (2002), 413–417.

[Wo 1977] B. Wong, *Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group*, Invent. math. **41** (1977), 253–257.

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