TRIANGULAR WITT GROUPS.
PART I: THE 12-TERM LOCALIZATION EXACT SEQUENCE.

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March 1999

ABSTRACT. To a short exact sequence of triangulated categories with duality, we associate a long exact sequence of Witt groups. For this, we introduce *higher Witt groups* in a very algebraic and pretty explicit way. Since those Witt groups are 4-periodic, this long exact sequence reduces to a cyclic 12-term one. Of course, in addition to higher Witt groups, we need to construct connecting homomorphisms, hereafter called *residue homomorphisms*.

INTRODUCTION.

The usual Witt group of a scheme is obtained by considering symmetric vector bundles (up to isometry) modulo the bundles possessing a maximal totally isotropic sub-bundle (called a *lagrangian*, see Knebusch [5, definition p. 133]). This is an invariant whose behavior with respect to localization is pretty tough.

Several attempts have been made to define higher Witt groups. For instance, one might mention the famous articles of Karoubi (see [3] and [4]), Pardon (see [6]) and Ranicki (see [7]). However, the existence of a localization sequence for Witt groups over arbitrary schemes and open subschemes is unknown, even in the affine case. My opinion on the question is that there is no natural kernel of the localization map and therefore no obvious result to conjecture, beyond the naive idea that some role has to be attributed to the closed complement of the open subscheme.

In part II of this series, I shall establish that the Witt group of a scheme X can be computed in terms of the derived category of bounded complexes of locally free $\mathcal{O}_X$-modules of finite type. The affine case has already been treated in [1, theorem 4.1] and [2, *théorème* 3.29, p. 89]. This result leads us to rephrase the localization problem in terms of triangulated categories.

It has been a real pleasure for me to see how everything can be expressed in this framework and how the axioms of triangulated categories (including the octahedron axiom!) are sufficient to establish the 12-term exact sequence. Nevertheless, the results are not always easier to prove. On the contrary, some of the questions are harder to tackle in the triangulated case than in the usual one (the better example being the *sub-lagrangian construction* treated in § 3). As an illustration of the beauty of this new approach, the experts will hopefully appreciate theorem 2.5 asserting that stably metabolic forms are metabolic.

One great advantage of the triangulated point of view is the existence of a *kernel category*. Let me make this more precise.

First of all, recall that a triangulated category K possesses a *duality* if it is endowed with a contravariant exact functor:

$$\# : K \longrightarrow K$$

such that $\#^2 \cong \mathrm{Id}$. Exactness means that $\#$ sends exact triangles of K to exact triangles. The Witt group of K is defined to be the quotient of the monoid of isometry classes of symmetric spaces by the sub-monoid formed by those spaces possessing a *lagrangian*. The notion of lagrangian used here generalizes the classical one in a natural way.

Now, consider a localization of triangulated categories:

$$q : \ K \longrightarrow S^{-1}K$$

1991 *Mathematics Subject Classification.* 11E81, 18E30, 19G12.

and suppose that both are equipped with a duality in a compatible way ($\# \circ q = q \circ \#$). Then this localization has a kernel (compare with the classical case!):

$$\text{(LOC)} \qquad 0 \longrightarrow J \longrightarrow K \longrightarrow S^{-1}K \longrightarrow 0$$

namely the full subcategory J of K on the objects M such that $q(M)$ is isomorphic to zero in $S^{-1}K$ (sounds really like a kernel, isn't it?). Using the notion of *épaisse* subcategory (*confer* [8]), one can start from J and see $S^{-1}K$ as K/J. A kernel category like above is always épaisse. I prefer to stick to the presentation of (LOC) because I have localization of schemes in mind, that is, in Nature, K and $S^{-1}K$ are given first and then J is introduced as the kernel category.

If we have an exact sequence like in (LOC) and if the dualities on K and $S^{-1}K$ are compatible, it is obvious that the duality $\#$ of K restricts to a duality on J. Now the conjecture is clear:

> *To such a localization exact sequence of triangulated categories with duality, we can associate a long exact sequence of Witt groups (introducing higher and lower Witt groups).*

This is the result we are going to establish (see theorem 5.1). There is no assumption to be made on the triangulated category except a very light *noetherianity* hypothesis (see definition 3.3).

The *higher* Witt groups are defined as follows. They are the Witt groups of the same category with *shifted* dualities ($T^n \circ \#$, $n \in \mathbb{Z}$). In other words, these higher Witt groups can be defined without topology and without metaphysical considerations on the category of triangulated categories (e.g., no use of *flasque* triangulated categories). This means that the theory of Witt groups over triangulated categories is in some sense *complete with respect to localization*. Moreover the simplicity of their definition allows the computation of some of these higher Witt groups.

Forgetting about the higher groups, this result implies in particular that the Witt-group functor is *exact in the middle*, in the sense that

$$\mathrm{W}(J) \longrightarrow \mathrm{W}(K) \longrightarrow \mathrm{W}(S^{-1}K)$$

is exact at $\mathrm{W}(K)$.

It will be our next duty to interpret this 12-term exact sequence in the special case of the derived category of a scheme (or the derived category of any exact category with duality of course). This requires additional work and is not presented here.

One word about the proofs. The main result depends upon two facts: first, stably neutral spaces are neutral (*confer* theorem 2.5) and, second, we can more or less generalize the classical sub-lagrangian construction to the derived framework (*confer* theorem 3.17). The exposition is such that the reader can follow the reasoning step by step. The only facts needed about triangulated categories are those recalled in § 0 and the four basic axioms. Especially the octahedron is used with no mercy.

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0. Background on triangulated categories.

Here is a little *baise-en-ville* of the triangulated mathematician. Details can be found in [8], for instance. Let K denote a triangulated category. Recall that K is additive and endowed with a *translation* additive functor $T : K \to K$ which is an automorphism. We have exact triangles:

$$(\Delta) \qquad A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

which replace the exact sequences of the abelian (or exact) framework. We do not want to write down the axioms but we would like to recall some very elementary and useful techniques. First, if Δ is exact, so is any isomorphic triangle (axiom) and in particular any triangle obtained from Δ by changing the sign of two of the three morphisms.

To explain how triangles replace exact sequences, recall that necessarily the composition of two consecutive morphisms in Δ is trivial: $v\,u = 0$, $w\,v = 0$ but also $T(u)\,w = 0$, $u\,T^{-1}(w) = 0$ and so on. Let us focus on the middle morphism v. Then (A, u) is a *weak kernel* of v in the sense that $v\,u = 0$ and that any morphism $f : X \to B$ such that $v\,f = 0$ factors (non uniquely) through A:

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & T(A). \\ & X \arrow[ul, dashed, "\exists\,\tilde f"] \arrow[u, "f"] \arrow[ur, "v\,f = 0"'] & & \end{tikzcd}$$

As well, $\big(T(A), w\big)$ is a *weak cokernel* of v: for $g : C \to Y$, we have $g\,v = 0$ if and only if g factors (non uniquely) through $T(A)$:

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] \arrow[dr, "g\,v = 0"'] & C \arrow[r, "w"] \arrow[d, "g"] & T(A) \arrow[dl, dashed, "\exists\,\bar g"] \\ & & Y. & \end{tikzcd}$$

Another very common trick is the following. Given a morphism of exact triangles:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & T(A') \end{tikzcd}$$

if two among f, g and h are isomorphisms so is the third (analogue of the five lemma). This is an easy consequence of the following also useful remark: Given any endomorphism of an exact triangle of the form:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "0"] & B \arrow[r, "v"] \arrow[d, "0"] & C \arrow[r, "w"] \arrow[d, "k"] & T(A) \arrow[d, "0"] \\ A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & T(A), \end{tikzcd}$$

we have necessarily $k^2 = 0$. This, in turn, is an easy consequence of the above remarks about weak kernels and cokernels.

This implies that the triples (C, v, w) such that Δ is an exact triangle (for $u : A \to B$ fixed) are all isomorphic. Any such triple (or sometimes only the object C) is called *the cone* of u and will be written as $\mathrm{Cone}(u)$. The cone simultaneously plays the role of weak kernel and cokernel. It contains essentially all the *homological* information. A very important question is then to know how to compare the cone of a composition $u' \circ u$ with the cones of u' and u. The answer is the famous axiom of the octahedron, which says: there is a *good* exact triangle linking the cone of the composition of two morphisms and the cones of these morphisms. Any more precise formulation should be close to the classical one due to Verdier.

1. The four Witt groups of a triangulated category with duality.

1.1. Once and for all. Let K denote a triangulated category and T be its *translation automorphism.* Here, we will always suppose that $\frac{1}{2} \in K$. This means that the abelian group $\mathrm{Hom}_K(A, B)$ is uniquely 2-divisible for all objects A and B in K.

1.2. Definition. Let $\delta = \pm 1$. An additive contravariant functor $\# : K \to K$ is said to be *δ-exact* if $T \circ \# = \# \circ T^{-1}$ and if, for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A),$$

the following triangle is exact:

$$C^{\#} \xrightarrow{v^{\#}} B^{\#} \xrightarrow{u^{\#}} A^{\#} \xrightarrow{\delta \cdot T(w^{\#})} T(C^{\#})$$

where $(-)^{\#}$ trickily denotes $\#(-)$. Suppose, moreover, that there exists an isomorphism of functors $\varpi : \mathrm{Id} \xrightarrow{\sim} \# \circ \#$ such that $\varpi_{T(M)} = T(\varpi_M)$ and $(\varpi_M)^{\#} \circ \varpi_{M^{\#}} = \mathrm{Id}_{M^{\#}}$ for any object M of K. Then the triple $(K, \#, \varpi)$ is called a *triangulated category with δ-duality.* In case $\delta = 1$, we shall simply talk about a *duality* and in case $\delta = -1$, we shall use the word *skew-duality.*

1.3. Very easy and very important exercise. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality for $\delta = \pm 1$.

(1) Let $n \in \mathbb{Z}$. Prove that $(K, T^n \circ \#, \varpi)$ is again a triangulated category with $\big((-1)^n \cdot \delta\big)$-duality.
(2) Prove that $(K, \#, -\varpi)$ is again a triangulated category with δ-duality.

1.4. Notions using only the additive structure. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$). In particular, K is an additive category with duality. A *symmetric space* is a pair (P, φ) such that P is an object in K and $\varphi : P \xrightarrow{\sim} P^{\#}$ is a *symmetric form*, meaning that $\varphi^{\#} \circ \varpi_P = \varphi$. Orthogonal sum and isometries are defined as usual. A *skew-symmetric form* $\varphi : P \xrightarrow{\sim} P^{\#}$ verifies $\varphi^{\#} \circ \varpi_P = -\varphi$ or, in other words, is a symmetric form in $(K, \#, -\varpi)$. Pay attention: the sign $\delta = \pm 1$ has nothing to do *a priori* with symmetry and skew-symmetry.

1.5. Remark. Our K is more than an additive category: we have the cone data. If u is symmetric morphism (*i.e.* $u^{\#} = u$ **without** assuming u to be an isomorphism), then the cone of u is more than the homological information about u, it also carries a *symmetric structure.* Let us make this more precise.

1.6. Cone of symmetric morphisms.
This is a central point of the study of Witt groups over triangulated categories with duality and we recommend the reader not to skip this part. It will be useful both for the definition of *neutral* spaces (and then Witt groups) and also for the definition of the connecting homomorphism in the long exact sequence of localization.

Suppose that $(K, \#, \varpi)$ is a triangulated category with duality; the case of skew-duality is rapidly treated in the end of this paragraph.

Let $\epsilon = \pm 1$. Consider a morphism $u : A \to A^{\#}$ such that u is ϵ-symmetric, *i.e.*

$$u^{\#} \circ \varpi_A = \epsilon \cdot u.$$

For instance, you can take $u = 0$. Choose an exact triangle containing u:

$$A \xrightarrow{u} A^{\#} \xrightarrow{u_1} C \xrightarrow{u_2} T(A).$$

Then it is natural to ask if $C = \mathrm{Cone}(u)$ is *symmetric* in some sense. To answer this question, we start by considering the dual triangle:

$$A^{\#\#} \xrightarrow{u^\#} A^\# \xrightarrow{-T(u_2^\#)} T(C^\#) \xrightarrow{T(u_1^\#)} T(A^{\#\#}).$$

This triangle is obtained from the previous one by using the duality (definition 1.2), the rotation axiom and by changing two signs. Now, compare these two exact triangles, using Id and ϖ, to get:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal, "\epsilon\cdot\varpi_A"', "\cong"] & A^\# \arrow[r, "u_1"] \arrow[d, equal] & C \arrow[r, "u_2"] \arrow[d, dashed, "\exists\,\psi"] & T(A) \arrow[d, "\epsilon\cdot T(\varpi_A)", "\cong"'] \\
A^{\#\#} \arrow[r, "u^\#"'] & A^\# \arrow[r, "-T(u_2^\#)"'] & T(C^\#) \arrow[r, "T(u_1^\#)"'] & T(A^{\#\#}).
\end{tikzcd}$$

Of course, we complete the above diagram with a morphism $\psi : C \to T(C^\#)$ such that it commutes. It is an axiom of triangulated categories that we can do so. Now, by dualizing the above diagram, we observe that

$$-\epsilon \cdot T(\psi^\#) \circ \varpi_C$$

also makes the above diagram commute. Since $\frac{1}{2} \in K$, you can replace ψ in the above diagram by $\frac{1}{2}\big(\psi - \epsilon \cdot T(\psi^\#) \circ \varpi_C\big)$ and get a symmetric form over C in the triangulated category with skew-duality $(K, T \circ \#, -\epsilon\,\varpi)$.

For now on, in order to focus on the spirit of the construction, we consider ϖ to be an identity, meaning that we identify A with $A^{\#\#}$ and of course u with $u^{\#\#}$. With this convention, we get the following diagrams:

If u is symmetric ($\epsilon = 1$):

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & A^\# \arrow[r, "u_1"] \arrow[d, equal] & C \arrow[r, "u_2"] \arrow[d, "\psi = -T(\psi^\#)", "\simeq"'] & T(A) \arrow[d, equal] \\
A \arrow[r, "u^\#"'] & A^\# \arrow[r, "-T(u_2^\#)"'] & T(C^\#) \arrow[r, "T(u_1^\#)"'] & T(A).
\end{tikzcd} \qquad (\Gamma_1)$$

If u is skew-symmetric ($\epsilon = -1$):

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & A^\# \arrow[r, "u_1"] \arrow[d, equal] & C \arrow[r, "u_2"] \arrow[d, "\psi = T(\psi^\#)", "\simeq"'] & T(A) \arrow[d, equal] \\
A \arrow[r, "-u^\#"'] & A^\# \arrow[r, "T(u_2^\#)"'] & T(C^\#) \arrow[r, "T(u_1^\#)"'] & T(A).
\end{tikzcd} \qquad (\Gamma_2)$$

We know from § 0 that ψ has to be an isomorphism. We proved in [1, lemma 3.4] or in [2, *lemme* 2.47, p. 39], still under the assumption that $\frac{1}{2} \in K$, that the space (C, ψ) is uniquely determined up to isometry by u, *i.e.* the space (C, ψ) obtained from any other choice of C, u_1, u_2 and ψ, would be isometric.

Now, consider the case of a skew-duality. Suppose that $(K, *, \varpi)$ is a triangulated category with a skew-duality $*$. Then diagrams Γ_1 and Γ_2 have to be replaced by:

If u is symmetric ($\epsilon = 1$):

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & A^* \arrow[r, "u_1"] \arrow[d, equal] & C \arrow[r, "u_2"] \arrow[d, "\psi = T(\psi^*)", "\simeq"'] & T(A) \arrow[d, equal] \\
A \arrow[r, "u^*"'] & A^* \arrow[r, "T(u_2^*)"'] & T(C^*) \arrow[r, "T(u_1^*)"'] & T(A).
\end{tikzcd} \qquad (\Gamma_3)$$

If u is skew-symmetric ($\epsilon = -1$):

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] & A^* \arrow[r, "u_1"] \arrow[d, equal] & C \arrow[r, "u_2"] \arrow[d, "\simeq"', "\psi = -T(\psi^*)"] & T(A) \arrow[d, equal] \\ A \arrow[r, "-u^*"'] & A^* \arrow[r, "-T(u_2^*)"'] & T(C^*) \arrow[r, "T(u_1^*)"'] & T(A). \end{tikzcd} \qquad (\Gamma_4)$$

Observe that in the skew-dual case, the cone of a symmetric morphism carries a symmetric form. There is no longer a change of sign as there was for a duality.

1.7. Definition. Let $(K, \#)$ be a triangulated category with δ-duality (for $\delta = \pm 1$). Let A be an object in K and $u : A \to A^{\#}$ be an ϵ-symmetric morphism ($\epsilon = \pm 1$). The *cone* of (A, u) is defined to be the $(-\delta \cdot \epsilon)$-symmetric space

$$\mathrm{Cone}(A, u) := (C, \psi)$$

fitting in the diagram Γ corresponding to δ and ϵ in the previous paragraph. This is a space for the shifted duality $T \circ \#$. It is well defined up to (non-unique) isometry.

1.8. Remark. In other words, there is a way, given any symmetric morphism u for a specified duality $\#$, to construct over the cone of this morphism a symmetric form for the translated duality $T \circ \#$. If u is symmetric, then this cone-form is skew-symmetric and if u is skew-symmetric, the cone-form is symmetric. We would like to emphasize the fact that if $\#$ was a skew-duality instead of a duality, then this change of sign would disappear. A symmetric u will have a symmetric cone-form and a skew-symmetric u will have a skew-symmetric cone-form.

Another observation is that one can change the point of view. Consider $\#$ to be the translated duality of $T^{-1} \circ \#$. The above construction allows us to construct a lot of *trivial* forms starting with symmetric morphisms u for the *previous* skew-duality $T^{-1} \circ \#$. These are the forms for the duality $\#$ which we are going to ignore in the Witt group. In [1] and [2], we chose to call these forms "neutral" because we also had to deal with classical metabolic forms and we wanted to avoid the confusion.

1.9. Definition. Let $(K, \#)$ be a triangulated category with δ-duality ($\delta = \pm 1$). An ϵ-symmetric space (P, φ) is *neutral* (or *metabolic*) if there exist an object A and a morphism $u : A \to T^{-1}(A^{\#})$ such that

$$T^{-1}(u^{\#}) = (\epsilon\,\delta) \cdot u \qquad \text{and} \qquad (P, \varphi) = \mathrm{Cone}(A, u).$$

It is easy to convince oneself that this is the natural generalization of Knebusch's definition of *metabolic forms* (see [5]). Compare with [2, *definition* 2.18, p. 32] and the subsequent remarks or with lemma 2.1 (2) below for the optical approach to the analogy.

1.10. Definition. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$). We define the *Witt monoid* of K to be the monoid of isometry classes of symmetric spaces endowed with the orthogonal sum. We write it as

$$\mathrm{MW}(K, \#, \varpi).$$

Obviously, since this cone-construction is additive and since neutrality is preserved by isometry, the set of isometry classes of neutral spaces form a well defined sub-monoid of $\mathrm{MW}(K, \#, \varpi)$ that we shall call

$$\mathrm{NW}(K, \#, \varpi).$$

The quotient monoid is a group (for quotient of abelian monoids, see [2, *remarque* 2.27, p. 35] if necessary), called the *Witt group* of $(K, \#, \varpi)$ and written as

$$\mathrm{W}(K, \#, \varpi) = \frac{\mathrm{MW}(K, \#, \varpi)}{\mathrm{NW}(K, \#, \varpi)}.$$

Referring to exercise 1.3, we define

$$\mathrm{W}^{\epsilon}_n(K, \#, \varpi) := \mathrm{W}(K, T^n \circ \#, \epsilon \cdot \varpi)$$

for all $n \in \mathbb{Z}$ and $\epsilon = \pm 1$. These are the *shifted* Witt groups of $(K, \#, \varpi)$.

If (P, φ) is a symmetric space, we write $[P, \varphi]$ for its class in the corresponding Witt group. We say that two symmetric spaces are *Witt-equivalent* if their classes in the Witt groups are the same.

1.11. Proposition. *The translation functor $T : K \to K$ induces isomorphisms:*

$$\mathrm{W}^{\epsilon}_{n}(K,\#,\varpi) \xrightarrow{\sim} \mathrm{W}^{\epsilon}_{n+2}(K,\#,\varpi).$$

1.12. Proof. See [1, proposition 1.20] or [2, *proposition* 2.38, p. 37] or do it as an easy exercise. ♯

1.13. Remark. The reader should keep in mind the following interpretation of this proposition: on a given triangulated category with a duality $\#$, there is essentially one *associated* skew-duality, namely $T\circ\#$. In terms of Witt groups, it means that we have two Witt groups for the given duality and two Witt groups for *the* associated skew-duality, since in both cases one can consider symmetric or skew-symmetric forms. Then we keep four Witt groups. We introduce the following notation:

$\mathrm{W}(K)$ $:= \mathrm{W}^{1}_{0}(K,\#,\varpi)$ the *natural* Witt group of symmetric forms.

$\mathrm{W}^{-}(K)$ $:= \mathrm{W}^{-1}_{0}(K,\#,\varpi)$ the *natural* Witt group of skew-symmetric forms.

$\mathrm{W}_1(K)$ $:= \mathrm{W}^{1}_{1}(K,\#,\varpi)$ the *associated* Witt group of symmetric forms, using *the* skew-duality associated to $\#$.

$\mathrm{W}^{-}_1(K)$ $:= \mathrm{W}^{-1}_{1}(K,\#,\varpi)$ the *associated* Witt group of skew-symmetric forms.

The reader might, out of nostalgia for existing notations, prefer something like $\mathrm{W}_0, \mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3$ or might want to replace "W" by "L" or whatever. He should feel free to modify the subsequent notations. This might have the advantage to see the 12-term exact sequence as a *homological* sequence but this notation does not make sense in the degree of generality adopted here. One can make a simple story complicated by introducing signs in the properties of the duality just to fit with the uses in some previous approaches. This may help to interpret some of our Witt groups in terms of classical ones, over rings for instance. In this case, it turns out that they should be numbered in the following order:

$$\mathrm{W},\ \mathrm{W}_1,\ \mathrm{W}^{-},\ \mathrm{W}^{-}_1.$$

Let us now return to the localization problem.

2. Neutrality and stable neutrality.

2.1. Lemma. *Let $(K,\#)$ be a triangulated category with duality. Let (P,φ) be a symmetric space. The following conditions are equivalent:*

(1) *(P,φ) is neutral;*
(2) *there exist L, α and w such that the triangle:*

$$T^{-1}(L^{\#}) \xrightarrow{\ w\ } L \xrightarrow{\ \alpha\ } P \xrightarrow{\ \alpha^{\#}\varphi\ } L^{\#}$$

is exact and such that $T^{-1}(w^{\#}) = w$;
(3) *there exists an exact triangle:*

$$T^{-1}(M^{\#}) \xrightarrow{\ \nu_0\ } L \xrightarrow{\ \nu_1\ } P \xrightarrow{\ \nu_2\ } M^{\#}$$

and an isomorphism $h : L \xrightarrow{\sim} M$ such that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(h^{\#})"', "\simeq"] & L \arrow[r, "\nu_1"] \arrow[d, "\simeq"', "h"] & P \arrow[r, "\nu_2"] \arrow[d, "\simeq"', "\varphi"] & M^{\#} \arrow[d, "\simeq"', "h^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}.
\end{tikzcd}$$

2.2. Proof. Using definitions 1.7, 1.9 and diagram Γ_3 of construction 1.6 (recall that $T^{-1}\#$ is a skew-duality), it is obvious that condition (1) and (2) are equivalent. Clearly (2) implies (3) by taking $h = \mathrm{Id}$.

Suppose now that there exists a diagram like in (3). The triangle

$$T^{-1}(M^\#) \xrightarrow{\quad h \circ \nu_0 \quad} M \xrightarrow{\quad \nu_1 \circ h^{-1} \quad} P \xrightarrow{\quad \nu_2 \quad} M^\#$$

is isomorphic to the exact triangle of (3) and is therefore exact. Let $w = h \circ \nu_0$ and $\alpha = \nu_1 \circ h^{-1}$. Then the triple (L, α, w) satisfies (2). $\sharp$

2.3. Definition. Given a neutral symmetric space (P, φ), a triple (L, α, w) satisfying condition (2) of the above lemma is called a *lagrangian* of (P, φ). In particular, $\alpha^\# \varphi\, \alpha = 0$.

Given any symmetric space (P, φ), a pair (L, α) is called a *sub-lagrangian* of (P, φ) if $\alpha^\# \varphi\, \alpha = 0$. We shall deal with sub-lagrangians in paragraph 3.

2.4. Remark. A symmetric space (P, φ) is said to be *stably neutral* if there exists a neutral space (R, ψ) such that

$$(P, \varphi) \perp (Q, \chi) \simeq (R, \psi) \perp (Q, \chi)$$

for some symmetric space (Q, χ). This is the same as saying that $(P, \varphi) \perp$ some neutral space is neutral (by adding $(Q, -\chi)$ on both side). This is also the same as saying that $[P, \varphi] = 0$ in $\mathrm{W}(K)$. Obviously neutral spaces are stably neutral. The converse is wrong in the *usual* framework (there exist non-metabolic stably metabolic spaces) but is true in the *triangulated* one.

2.5. Theorem. *Let K be a triangulated category with duality. Suppose that $\frac{1}{2} \in K$. Any stably neutral symmetric space is neutral.*

2.6. Proof. Let (P, φ) be a stably neutral space. By the above remark, there exists a space (Q, χ) such that the space

$$(P, \varphi) \perp (Q, \chi) \perp (Q, -\chi)$$

is neutral. Since $(Q, \chi) \perp (Q, -\chi)$ is isometric to $(Q \oplus Q^\#, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, this means that the space

$$(P \oplus Q \oplus Q^\#, \begin{pmatrix} \varphi & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix})$$

is neutral. By condition (2) of lemma 2.1 applied to the above symmetric space, there exists an exact triangle:

$$T^{-1}(L^\#) \xrightarrow{w} L \xrightarrow{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} P \oplus Q \oplus Q^\# \xrightarrow{\quad (a^\# \varphi \quad c^\# \quad b^\#) \quad} L^\#$$

with $w = T^{-1}(w^\#)$ for some morphisms a, b and c as above.

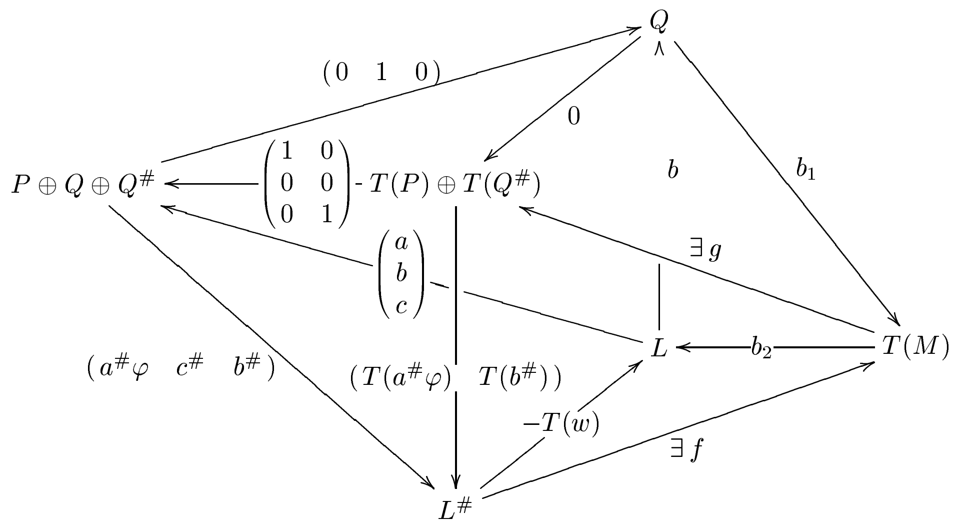
Consider an exact triangle containing b:

$$L \xrightarrow{b} Q \xrightarrow{b_1} T(M) \xrightarrow{b_2} T(L)$$

(we choose to call $T(M)$ the cone of b). Now, apply the octahedron axiom to the relation:

$$(0 \quad 1 \quad 0) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = b.$$

This gives the following:



for some morphisms $f : L^{\#} \to T(M)$ and $g : T(M) \to T(P) \oplus T(Q^{\#})$ such that the above diagram is an octahedron (meaning that faces are alternatively commutative or exact triangles and that the two ways from the left to the right (respectively the two ways from the right to the left) coincide). Confer [8] for the octahedron axiom.

In particular, the condition

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot g = T(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) \cdot b_2$$

forces g to be $\begin{pmatrix} T(a)b_2 \\ T(c)b_2 \end{pmatrix}$. Thus the above octahedron reduces to the existence of a morphism $f : L^{\#} \to T(M)$ such that:

(1) the following triangle is exact:

$$L^{\#} \xrightarrow{\ f\ } T(M) \xrightarrow{\begin{pmatrix} T(a)b_2 \\ T(c)b_2 \end{pmatrix}} T(P) \oplus T(Q^{\#}) \xrightarrow{(T(a^{\#}\varphi) \quad T(b^{\#}))} T(L^{\#})$$

(2) $b_2\, f = -T(w)$;
(3) $f\, c^{\#} = b_1$.

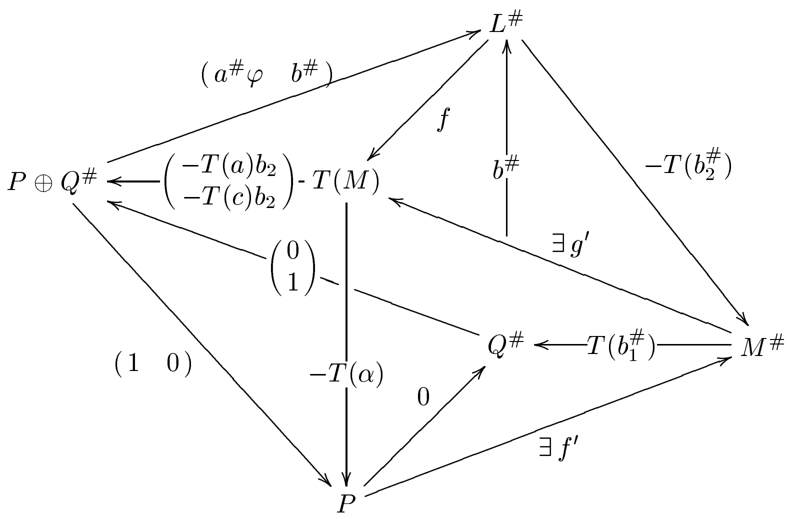
It will appear that $(M, a \circ T^{-1}(b_2), m)$ is a lagrangian of (P, φ) for a suitable morphism $m : T^{-1}(M^{\#}) \to M$ (more or less). Therefore we baptize:

$$\alpha := a \circ T^{-1}(b_2) \,:\, M \longrightarrow P.$$

The reluctant reader can motivate himself by checking that $\alpha^{\#}\varphi\,\alpha = 0$. The octahedron axiom will help us to establish an exact triangle over α. Let us consider the following identity:

$$(\,a^{\#}\varphi \quad b^{\#}\,) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = b^{\#}$$

and apply to this the octahedron axiom to get:



The exact triangles over $(\, a^\# \varphi \quad b^\# \,)$ and $b^\#$ are obtained respectively by rotating (1) and by dualizing the exact triangle over b from the beginning.

Here, the relation

$$f' \circ (\,1 \quad 0\,) = -T(b_2^\#) \circ (\, a^\# \varphi \quad b^\# \,)$$

forces $f' = -T(b_2^\#)\, a^\# \varphi = -\big(a\, T^{-1}(b_2)\big)^\# \varphi = -\alpha^\# \varphi$. Thus the information of the octahedron reduces to the existence of some $g' : M^\# \to T(M)$ that we astutely re-baptize $g' = T(m)$ for $m : T^{-1}(M^\#) \to M$ such that:

(4) the following triangle is exact:

$$T^{-1}(M^\#) \xrightarrow{\ m\ } M \xrightarrow{\ \alpha\ } P \xrightarrow{\ \alpha^\# \varphi\ } M^\#$$

(5) $m\, b_2^\# = -T^{-1} f$

(6) $c\, T^{-1}(b_2)\, m = -b_1^\#$.

To make (M, α, m) a lagrangian of (P, φ) we still need $T^{-1}(m^\#) = m$. So let us consider the triangles over m and over $T^{-1}(m^\#)$.

To lighten a little bit the notations, we shall write

$$* = T^{-1} \circ \#.$$

Dualizing the exact triangle (4), we get the second line of the following diagram, whose first line is simply exact triangle (4):

$$\begin{tikzcd} M^* \arrow[r, "m"] \arrow[d, equal] & M \arrow[r, "\alpha"] \arrow[d, dashed, "\exists\, 1+x"', "\simeq"] & P \arrow[r, "\alpha^\# \varphi"] \arrow[d, "\varphi"', "\simeq"] & M^\# \arrow[d, equal] \\ M^* \arrow[r, "m^*"'] & M \arrow[r, "\varphi\alpha"'] & P^\# \arrow[r, "\alpha^\#"'] & M^\# \end{tikzcd} \tag{7}$$

and it is an axiom of triangulated categories that the above morphism of exact triangles can be completed. The third morphism is called $1 + x$ because we are going to prove that x is nilpotent. This is quite a technical computation. Let us do it step by step, very carefully. Directly from (7), we get the two following equalities:

(8) $x\, m = m^* - m$

(9) $\alpha\, x = 0$.

¿From (8), we immediately have $m^* x^* = (x\,m)^* = (m^* - m)^* = m - m^* = -x\,m$. Call this

$$m^* x^* = -x\,m. \tag{10}$$

Now, compose (5) on the right with c^* and use (3) to obtain

$$m\,b_2^{\#} c^* = -T^{-1}(b_1). \tag{11}$$

Applying $*$ to this last equality and using (6) gives

$$c\,T^{-1}(b_2)\,m^* = -b_1^{\#} = c\,T^{-1}(b_2)\,m.$$

Replacing in this equation m^* by $(1+x)\,m$, which is allowed by (7), we obtain:

$$c\,T^{-1}(b_2)\,x\,m = 0. \tag{12}$$

Dualizing this last relation and using (10) gives:

$$x\,m\,b_2^{\#} c^* = 0.$$

But (11) allows us to replace $m\,b_2^{\#} c^*$ to get:

$$x\;T^{-1}(b_1) = 0. \tag{13}$$

Now we are going to use the symmetry of the w given at the beginning to deduce some information about x. Compose relation (5) on the left by $T^{-1}(b_2)$ and use (2) to obtain that

$$T^{-1}(b_2)\,m\,b_2^{\#} = -T^{-1}(b_2)\,T^{-1}(f) = w.$$

Since $w = w^*$, the left hand side of the above equation is also $*$-symmetric. This gives

$$T^{-1}(b_2)\,m\,b_2^{\#} = T^{-1}(b_2)\,m^* b_2^{\#}.$$

Again replacing m^* by $(1+x)\,m$, we obtain

$$T^{-1}(b_2)\,x\,m\,b_2^{\#} = 0.$$

But then (5) allows us to replace $m\,b_2^{\#}$ and to have

$$T^{-1}(b_2)\,x\,T^{-1}(f) = 0. \tag{14}$$

We are almost done. Recalling the definition of $\alpha = a\,T^{-1}(b_2)$, composing (9) with m on the right gives

$$a\,T^{-1}(b_2)\,x\,m = 0.$$

Since, on the other hand (12) insures us that $c\,T^{-1}(b_2)\,x\,m = 0$, we have

$$\begin{pmatrix} a\,T^{-1}(b_2) \\ c\,T^{-1}(b_2) \end{pmatrix} x\,m = 0.$$

Since triangle (1) is exact (rotate it a little bit, if you need), there exists a morphism $y : M^* \to L^*$ such that

$$xm = T^{-1}(f)\,y. \tag{15}$$

¿From relation (13) and the exact triangle over b that we chose at the very beginning of the proof, there exists a morphism $z : L \to M$ such that

$$x = z\,T^{-1}(b_2). \tag{16}$$

Now we are going to compute $x^3 m$ using the above relations. By (15) and (16), we have

$$x^3 m = x \cdot x \cdot (xm) = z\, T^{-1}(b_2) \cdot x \cdot T^{-1}(f)\, y.$$

But (14) insures that the composition of the 3 morphisms in the middle is zero. Thus we have established

$$x^3 m = 0.$$

Of course, from (9), we have that $\alpha\, x^3 = 0$. In other words, x^3 makes the following diagram commute:

$$\begin{tikzcd} M^* \arrow[r, "m"] \arrow[d, "0"'] & M \arrow[r, "\alpha"] \arrow[d, "x^3"] & P \arrow[r, "\alpha^{\#}\varphi"] \arrow[d, "0"] & M^{\#} \arrow[d, "0"] \\ M^* \arrow[r, "m"'] & M \arrow[r, "\alpha"'] & P \arrow[r, "\alpha^{\#}\varphi"'] & M^{\#}. \end{tikzcd}$$

It is an easy exercise (*confer* § 0.) to show that such endomorphisms have trivial square:

$$x^6 = 0.$$

Therefore, $h := 1 + \frac{1}{2}\, x$ is an automorphism of M. Relation (10) says that $m^* x^* = -x\, m$. This and the fact that $m^* = (1+x)\, m$ insures that

$$m^*\, h^* = m^*(1 + \frac{1}{2} x)^* = m^* + \frac{m^* x^*}{2} = (1+x)\, m - \frac{x\, m}{2} = (1 + \frac{1}{2}\, x)\, m = h\, m.$$

In other words, the following diagram commutes (since, of course, we still have $\alpha\, h = \alpha$ by (9)):

$$\begin{tikzcd} M^* \arrow[r, "m"] \arrow[d, "h^*"', "\simeq"] & M \arrow[r, "\alpha"] \arrow[d, "h"', "\simeq"] & P \arrow[r, "\alpha^{\#}\varphi"] \arrow[d, "\varphi"', "\simeq"] & M^{\#} \arrow[d, "h^{\#}"', "\simeq"] \\ M^* \arrow[r, "m^*"'] & M \arrow[r, "\varphi\alpha"'] & P^{\#} \arrow[r, "\alpha^{\#}"'] & M^{\#}. \end{tikzcd}$$

Lemma 2.1 insures that (P, φ) is neutral. $\sharp$

2.7. Remark. The same proof goes through for a skew-duality.

2.8. Exercise. Establish how a stably metabolic space in the usual sense becomes neutral in the derived category. The above proof gives an explicit way to find the lagrangian. The latter will not be (of course) a complex concentrated in degree 0.

3. The sub-lagrangian construction.

In this paragraph, we fix the triangulated category K and its duality $\#$. The case of a skew-duality can be treated as well and is left to the reader.

3.1. Presentation of the problem.
The classical theory of symmetric spaces over a scheme as well as some technical problems we shall encounter in the subsequent paragraphs lead us to consider the following questions.

Consider a symmetric space (P,φ). Suppose you have a sub-lagrangian (L,ν_1), *i.e.* L is an object and $\nu_1 : L \to P$ is a morphism such that

$$\nu_1^{\#}\varphi\,\nu_1 = 0.$$

Here are natural questions:
Can we *correctly* define $L^{\perp}$, the orthogonal of L?
Can we map $L \to L^{\perp}$?
Can we endow $\mathrm{Cone}(L \to L^{\perp})$ with a structure of symmetric space possibly Witt-equivalent to (P,φ)?
The first mental step is to renounce to the constraint for L or for $L^{\perp}$ to be subobjects of P, as well as L to be a subobject of $L^{\perp}$. In triangulated categories, it is definitely a too strong hypothesis since any monomorphism (*i.e.* any morphism α such that $\alpha\,\beta = 0$ forces $\beta = 0$) is a split inclusion as a direct summand. In other words, this would be the same as forgetting the triangulated structure and focusing only on the additive one!
In the classical case, $L^{\perp}$ is defined to be the kernel of $\nu_1^{\#} \circ \varphi$. Since we know that $T^{-1}\big(\mathrm{Cone}(u)\big)$ is a weak kernel of u, the natural analogue of $L^{\perp}$ would be here

$$T^{-1}\big(\mathrm{Cone}(\nu_1^{\#} \circ \varphi)\big)$$

which is the same as $\big(\mathrm{Cone}(\nu_1)\big)^{\#}$ as can be deduced from exactness of $\#$ (*confer* definition 1.2). In other words, if we choose an exact triangle containing ν_1:

$$L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} C \xrightarrow{\nu_3} T(L)$$

then $L^{\perp} = C^{\#}$. Therefore, we prefer to introduce directly the following exact triangle (setting, if you prefer $\nu_0 = -T^{-1}(\nu_3)$ and $M = C^{\#}$):

$$(\Delta) \qquad\qquad T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}$$

in which M is already the orthogonal of L. Then, dualizing the above triangle, we get:

$$(\Delta^{\#}) \qquad\qquad T^{-1}(L^{\#}) \xrightarrow{T^{-1}(\nu_0^{\#})} M \xrightarrow{\nu_2^{\#}} P^{\#} \xrightarrow{\nu_1^{\#}} L^{\#}.$$

This allows us to understand the more precise statement $(L,\nu_1)^{\perp} = (M,\varphi^{-1}\nu_2^{\#})$.
We are going to construct a morphism of triangles between Δ and $\Delta^{\#}$. Its first part is given by φ:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] & L \arrow[r, "\nu_1"] & P \arrow[r, "\nu_2"] \arrow[d, "\simeq"', "\varphi"] & M^{\#} \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}.
\end{tikzcd}$$

Since $\nu_1^{\#} \circ (\varphi\, \nu_1) = 0$ and since $\Delta^{\#}$ is exact, there exists a morphism $\alpha : L \to M$ (*i.e.* we map L in its orthogonal) such that $\varphi\, \nu_1 = \nu_2^{\#}\, \alpha$. The third axiom of triangulated categories insures you of the existence of a third morphism $\beta : M^{\#} \to L^{\#}$ such that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, dashed, "T^{-1}(\beta)"'] & L \arrow[r, "\nu_1"] \arrow[d, "\alpha"'] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, dashed, "\exists\beta"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}.
\end{tikzcd}$$

Of course, we would prefer to have $\beta = \alpha^{\#}$. This is easy to obtain. It suffices to define

$$\eta_0 := \frac{1}{2}(\alpha + \beta^{\#})$$

and to check, using the above diagram, that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\eta_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"'] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\eta_0^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}.
\end{tikzcd}$$

In other words, we have mapped L reasonably in its orthogonal, M. Now, we introduce the cone of this map. That is: we consider an exact triangle over η_0:

$$L \xrightarrow{\eta_0} M \xrightarrow{\eta_1} Q \xrightarrow{\eta_2} T(L)$$

and we use it (and its dual) to extend the last diagram to the following key-diagram, which will be used several times in the subsequent pages:

$$(\Omega) \qquad \begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\eta_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\eta_0^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] \arrow[d, "\eta_2^{\#}"'] & M \arrow[r, "\nu_2^{\#}"'] \arrow[d, "\eta_1"] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#} \\
Q^{\#} \arrow[d, "\eta_1^{\#}"'] & Q \arrow[d, "\eta_2"] & & \\
M^{\#} \arrow[r, "T(\nu_0)"'] & T(L). & &
\end{tikzcd}$$

3.2. The question.

Of course, if you consider a diagram like Ω, (L,ν_1) is necessarily a sub-lagrangian of (P,φ). Therefore, the sub-lagrangian question is now:

Given a commutative diagram Ω with exact lines and columns, can we construct a symmetric form $\psi : Q \xrightarrow{\sim} Q^{\#}$ such that (P,φ) and (Q,ψ) are Witt-equivalent?

More precisely, we also could ask ψ to fit in the above diagram. As an exercise, the reader should try to solve the problem for $L=0$. Even in this very case, it is immediate that we cannot expect any form making the above diagram commute to be Witt-equivalent to (P,φ). It appears also from this example that one cannot simply complete the above diagram using the third axiom. The induced morphism $Q^{\#}\to Q$ might very well not be an isomorphism (can even be zero). For these reasons and also for technical ones appearing in the proof, it seems very important to consider the morphism:

$$s := \nu_2\,\varphi^{-1}\nu_2^{\#} \;:\; M \to M^{\#}.$$

This is, in some sense, the symmetric map induced by φ on M (if you send M in P through $\varphi^{-1}\nu_2^{\#}$). This morphism will play a central role hereafter. For some reasons, the form ψ we shall be able to construct and that will fit in Ω will actually give the opposite of $[P,\varphi]$ in the Witt group. Of course, this answers also the question since you can replace ψ by $-\psi$.

Observe that the choice of a diagram Ω is more than the sub-lagrangian hypothesis, because there might be several such morphisms η_0. The question is actually quite complicated because of that precise point. In the application we shall make of this sub-lagrangian construction to the 12-term sequence, we will have the diagram Ω imposed by the opponent. The result goes as follows.

Given a sub-lagrangian (L,ν_1) of a symmetric space (P,φ), there exists always a way to construct a *good* η_0, that is an η_0 such that the corresponding diagram Ω can be completed and such that the form on Q is Witt-equivalent to the opposite of (P,φ). If, moreover, the morphism η_0 is imposed, then there exists a way to modify it into a *good* one (in the above sense) without modifying its cone too much. Since we proved in 3.1, that there always exists an η_0 (not necessarily good), the first assertion is a consequence of the second. This explains the weird presentation of the results (theorem 3.17 and corollary 3.24).

As mentioned in the introduction, we were not able to establish the sub-lagrangian construction without some finiteness hypothesis, called here *noetherianity*. Let us introduce it.

3.3. Definition. We say that a triangulated category K is *noetherian* if it satisfies the following two properties:

(1) If A and B are objects in K such that there exists an isomorphism $A\oplus B\simeq A$ then $B\simeq 0$.
(2) For any pair of fixed objects $A,\ B$ in K and for any sequence of exact triangles

$$\Big(A \xrightarrow{u_i} B \xrightarrow{v_i} C_i \xrightarrow{w_i} T(A) \Big)_{i\in\mathbb{N}}$$

over $A,\ B$ such that there exists split injections

$$\Big(f_i : C_i \hookrightarrow C_{i+1}\Big)_{i\in\mathbb{N}}$$

verifying $w_{i+1}\,f_i = w_i$ for all i, this sequence of inclusions as direct summands

$$C_1 \underset{f_1}{\hookrightarrow} C_2 \underset{f_2}{\hookrightarrow} \cdots C_i \underset{f_i}{\hookrightarrow} C_{i+1} \hookrightarrow \cdots$$

is stationary. In other words, there exists $n\in\mathbb{N}$ such that f_i is an isomorphism for all $i\geq n$.

3.4. Remark. It is an open question to give a good definition of *noetherian* triangulated categories. The one given here has the following property: Let X be a noetherian scheme and consider the derived category of bounded complexes of coherent $\mathcal{O}_X$-modules (resp. locally free $\mathcal{O}_X$-modules of finite type). Then a simple argument using long exact homology sequences allows us to prove that this category is noetherian in the sense of definition 3.3. The definition would be "good" if the converse was true. Anyway, for the future applications to schemes and open subschemes, it will be sufficient to restrict ourselves to noetherian schemes to be able to use the results of this paragraph.

Under this noetherianity assumption, we can explain how the completion of diagram Ω is sufficient to have a Witt-equivalent form on the cone of $L \longrightarrow L^{\perp}$.

3.5. Theorem. *Let $(K, \#)$ be a noetherian triangulated category with duality. Let (P, φ) be a symmetric space and (L, ν_1) be a sub-lagrangian of this space. Suppose that you have a commutative diagram with exact rows and columns:*

$$(\bar\Omega)\qquad \begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\eta_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\eta_0^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] \arrow[d, "\eta_2^{\#}"'] & M \arrow[r, "\nu_2^{\#}"'] \arrow[d, "\eta_1"] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#} \\
Q^{\#} \arrow[d, "\eta_1^{\#}"'] & Q \arrow[l, "\psi = \psi^{\#}", "\simeq"'] \arrow[d, "\eta_2"] & & \\
M^{\#} \arrow[r, "T(\nu_0)"'] & T(L) & &
\end{tikzcd}$$

such that

$$\eta_1^{\#}\psi\,\eta_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#}.$$

Then, $[P, \varphi] = [Q, -\psi]$ *in* $\mathrm{W}(K)$.

3.6. Proof. The proof is quite a long way and we shall establish partial results first. As already asserted above, the symmetric morphism $\nu_2\,\varphi^{-1}\nu_2^{\#}$ plays a important role and we give it a name:

$$s := \nu_2\,\varphi^{-1}\nu_2^{\#} \ : \ M \longrightarrow M^{\#}$$

once for all. We shall also consider the morphism $\eta_0\,\nu_0 : T^{-1}(M^{\#}) \to M$ which is symmetric for the duality $T^{-1} \circ \#$. We adopt the following notations:

$$* = T^{-1} \circ \#$$

and $w := \eta_0\,\nu_0 = w^* : M^* \to M$. It is clear that $\left(M, \begin{pmatrix} \varphi^{-1}\nu_2^{\#} \\ \eta_1 \end{pmatrix}\right)$ is a sub-lagrangian of $(P, \varphi) \perp (Q, \psi)$ but it is rather hard to remove the prefix.

The first lemmas ignore the lower part of $\bar\Omega$ (*i.e.* the form ψ) and focus on Ω.

3.7. Lemma. *Given a diagram Ω like in 3.1, there exists a morphism $y : Q \to M^{\#}$ such that*

(1) $T(\nu_0)\,y = \eta_2$;
(2) *the following triangle is exact where $w = \eta_0\,\nu_0$:*

$$T^{-1}(M^{\#}) \xrightarrow{\ w\ } M \xrightarrow{\begin{pmatrix} \varphi^{-1}\nu_2^{\#} \\ \eta_1 \end{pmatrix}} P \oplus Q \xrightarrow{(\nu_2 \quad y)} M^{\#}.$$

3.8. Proof. ¿From the commutativity of Ω, we have $\varphi\,\nu_1\,T^{-1}(\eta_2) = \nu_2^{\#}\,\eta_0\,T^{-1}(\eta_2) = 0$ and since φ is an isomorphism, we have

$$\nu_1 \circ \big(-T^{-1}(\eta_2)\big) = 0.$$

Apply the forth axiom of triangulated categories to get the following octahedron, in which we immediately used the commutativity of the faces containing $\begin{pmatrix}1\\0\end{pmatrix}$ and $(\,0 \quad 1\,)$:

$$\begin{tikzcd}
& & & P \arrow[dll, "\nu_2"'] \arrow[ddrr, "\begin{pmatrix}1\\0\end{pmatrix}"] \\
L \arrow[urrr, "\nu_1"] \arrow[ddr, "\eta_0"'] & M^{\#} \arrow[l, "-T(\nu_0)"'] \arrow[dd, "-T(\eta_0)\,T(\nu_0)"] \\
& & & T^{-1}(Q) \arrow[uu, "0"] \arrow[ulll, "-T^{-1}(\eta_2)"] & & P\oplus Q \arrow[ll, "(\,0\quad 1\,)"'] \arrow[ullll, "\exists\,(\,\nu_2\quad y\,)"'] \\
& M \arrow[urr, "\eta_1"] \arrow[urrrr, "\exists\begin{pmatrix}x\\ \eta_1\end{pmatrix}"']
\end{tikzcd}$$

for some $x\,:M\to P$ and $y\,:Q\to M^{\#}$ such that the triangle

$$T^{-1}(M^{\#}) \xrightarrow{\ w\ } M \xrightarrow{\begin{pmatrix}x\\ \eta_1\end{pmatrix}} P\oplus Q \xrightarrow{(\,\nu_2\quad y\,)} M^{\#}$$

is exact (recall that $w=\eta_0\,\nu_0$) and such that

$$\begin{aligned} x\,\eta_0 &= \nu_1\\ T(\nu_0)\,y &= \eta_2\,. \end{aligned}$$

Observe that $x\,\eta_0=\nu_1=\varphi^{-1}\nu_2^{\#}\eta_0$ by Ω. In other words, $(x-\varphi^{-1}\nu_2^{\#})\,\eta_0=0$. Using the exact triangle containing η_0, we find some $a\,:Q\to P$ such that

$$x-\varphi^{-1}\nu_2^{\#} = a\,\eta_1\,.$$

This can also be expressed by saying that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "w"] \arrow[d, equal] & M \arrow[r, "\begin{pmatrix}x\\ \eta_1\end{pmatrix}"] \arrow[d, equal] & P\oplus Q \arrow[r, "(\,\nu_2\quad y\,)"] \arrow[d, "\simeq"', "\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}"] & M^{\#} \arrow[d, equal] \\
T^{-1}(M^{\#}) \arrow[r, "w"'] & M \arrow[r, "\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix}"'] & P\oplus Q \arrow[r, "(\,\nu_2\quad y+\nu_2\,a\,)"'] & M^{\#}.
\end{tikzcd}$$

But then, since the first triangle is exact, so is the second one. It suffices now to replace y by $\bar{y} := y+\nu_2\,a$ which still satisfies

$$T(\nu_0)\circ\bar{y} = \underbrace{T(\nu_0)\,y}_{=\,\eta_2} + \underbrace{T(\nu_0)\,\nu_2}_{=\,0}\,a = \eta_2\,.$$

$\sharp$

3.9. Lemma. *Consider the diagram Ω and suppose moreover that*

$$\nu_2\,\varphi^{-1}\nu_2^{\#}=0.$$

Then Ω can be completed in an $\bar{\Omega}$ like in theorem 3.5 in such a way that the following triangle is exact:

$$T^{-1}(M^{\#})\xrightarrow{\;w\;}M\xrightarrow{\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix}}P\oplus Q\xrightarrow{\begin{pmatrix}\nu_2 & \eta_1^{\#}\psi\end{pmatrix}}M^{\#}$$

where $w=\eta_0\,\nu_0$.

In particular (P,φ) and $(Q,-\psi)$ are Witt-equivalent.

3.10. Proof. By the previous lemma, we have an exact triangle

$$(0)\qquad M^{*}\xrightarrow{\;w\;}M\xrightarrow{\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix}}P\oplus Q\xrightarrow{\begin{pmatrix}\nu_2 & y\end{pmatrix}}M^{\#}$$

for some morphism $y:Q\to M^{\#}$ such that $T(\nu_0)\,y=\eta_2$. Since $w^{*}=w$, we can consider $\chi:=\mathrm{Cone}(M^{*},w)$ as defined in 1.7, *i.e.* a neutral symmetric form

$$\chi=\begin{pmatrix}a & b\\ b^{\#} & c\end{pmatrix}=\chi^{\#}$$

such that the following diagram commutes (in which the second line is the dual of the first):

$$(1)\qquad \begin{array}{ccccccc}
M^{*} & \xrightarrow{\;w\;} & M & \xrightarrow{\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix}} & P\oplus Q & \xrightarrow{\begin{pmatrix}\nu_2 & y\end{pmatrix}} & M^{\#}\\
\Big\| & & \Big\| & & \simeq\Big\downarrow\begin{pmatrix}a & b\\ b^{\#} & c\end{pmatrix} & & \Big\|\\
M^{*} & \xrightarrow[\;w^{*}\;]{} & M & \xrightarrow[\begin{pmatrix}\nu_2^{\#}\\ y^{\#}\end{pmatrix}]{} & P^{\#}\oplus Q^{\#} & \xrightarrow[\begin{pmatrix}\nu_2\,\varphi^{-1} & \eta_1^{\#}\end{pmatrix}]{} & M^{\#}.
\end{array}$$

We have $a=a^{\#}$, $c=c^{\#}$ and

$$(2)\qquad a\,\varphi^{-1}\nu_2^{\#}+b\,\eta_1=\nu_2^{\#}$$

$$(3)\qquad b^{\#}\varphi^{-1}\nu_2^{\#}+c\,\eta_1=y^{\#}.$$

Composing (2) on the right with η_0 and using $\eta_1\,\eta_0=0$, we have

$$a\,\underbrace{\varphi^{-1}\nu_2^{\#}\eta_0}_{=\nu_1}=\nu_2^{\#}\eta_0=\varphi\nu_1$$

by Ω. In other words, $(a-\varphi)\,\nu_1=0$ which means that there exists a morphism $d:M^{\#}\to P^{\#}$ such that

$$a=\varphi+d\,\nu_2.$$

Therefore, using our extra hypothesis that $\nu_2\,\varphi^{-1}\nu_2^{\#}=0$, we have:

$$a\,\varphi^{-1}\nu_2^{\#}=(\varphi+d\,\nu_2)\,\varphi^{-1}\nu_2^{\#}=\nu_2^{\#}+d\,\underbrace{\nu_2\,\varphi^{-1}\nu_2^{\#}}_{=0}=\nu_2^{\#}.$$

This last equality compared with (2) forces

$$(4) \qquad b\,\eta_1 = 0.$$

Putting (3) and (4) together, we obtain the commutativity of the following diagram (compare with (1)) :

$$(5) \qquad \begin{tikzcd} M^* \arrow[r, "w"] \arrow[d, equal] & M \arrow[r, "\begin{pmatrix} \varphi^{-1}\nu_2^\# \\ \eta_1 \end{pmatrix}"] \arrow[d, equal] & P \oplus Q \arrow[r, "(\,\nu_2 \quad y\,)"] \arrow[d, "\simeq"', "\begin{pmatrix} \varphi & b \\ b^\# & c \end{pmatrix}"] & M^\# \arrow[d, equal] \\ M^* \arrow[r, "w^*"'] & M \arrow[r, "\begin{pmatrix} \nu_2^\# \\ y^\# \end{pmatrix}"'] & P^\# \oplus Q^\# \arrow[r, "(\,\nu_2\,\varphi^{-1} \quad \eta_1^\#\,)"'] & M^\#. \end{tikzcd}$$

In other words, we can replace a by φ in (1). By definition, the following form over $P \oplus Q$ is neutral :

$$\begin{pmatrix} \varphi & b \\ b^\# & c \end{pmatrix}.$$

But, of course, we can get rid of the b through an isometry of the type :

$$h := \begin{pmatrix} 1 & -\varphi^{-1}b \\ 0 & 1 \end{pmatrix} \ :\ P \oplus Q \xrightarrow{\sim} P \oplus Q.$$

This gives the following. Define $\psi : Q \to Q^\#$ to be

$$\psi := c - b^\#\varphi^{-1}b$$

and check that we have the following isometry :

$$(6) \qquad \begin{pmatrix} 1 & 0 \\ -b^\#\varphi^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi & b \\ b^\# & c \end{pmatrix} \cdot \begin{pmatrix} 1 & -\varphi^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}.$$

Use this automorphism h of $P \oplus Q$ to modify the triangle (0) :

$$\begin{tikzcd} M^* \arrow[r, "w"] \arrow[d, equal] & M \arrow[r, "\begin{pmatrix} \varphi^{-1}\nu_2^\# \\ \eta_1 \end{pmatrix}"] \arrow[d, equal] & P \oplus Q \arrow[r, "(\,\nu_2 \quad y\,)"] \arrow[d, "\simeq"', "h^{-1} = \begin{pmatrix} 1 & \varphi^{-1}b \\ 0 & 1 \end{pmatrix}"] & M^\# \arrow[d, equal] \\ M^* \arrow[r, "w"'] & M \arrow[r, "w_1"'] & P \oplus Q \arrow[r, "w_2"'] & M^\#. \end{tikzcd}$$

The first line is exact triangle (0) and therefore the second is exact, defining w_1 and w_2 such that the diagram commutes. By (4), a direct computation gives :

$$w_1 = \begin{pmatrix} 1 & \varphi^{-1}b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi^{-1}\nu_2^\# \\ \eta_1 \end{pmatrix} = \begin{pmatrix} \varphi^{-1}\nu_2^\# \\ \eta_1 \end{pmatrix}.$$

Now, the easiest way to compute w_2 is by using the fact that the induced form is $\varphi\perp\psi$, *i.e.* concretely :

$$\begin{aligned} w_2 = (\,\nu_2 \quad y\,) \circ h &\overset{(5)}{=} (\,\nu_2\,\varphi^{-1} \quad \eta_1^\#\,) \cdot \begin{pmatrix} \varphi & b \\ b^\# & c \end{pmatrix} \circ h = \\ &\overset{(4)}{=} (\,\nu_2\,\varphi^{-1} \quad \eta_1^\#\,) \cdot \begin{pmatrix} 1 & 0 \\ -b^\#\varphi^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi & b \\ b^\# & c \end{pmatrix} \circ h = \\ &\overset{(6)}{=} (\,\nu_2\,\varphi^{-1} \quad \eta_1^\#\,) \cdot \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} = (\,\nu_2 \quad \eta_1^\#\psi\,). \end{aligned}$$

This gives the announced triangle.

Use this triangle and condition (2) of lemma 2.1 to understand that $\varphi\perp\psi$ is neutral. Then $[P,\varphi] = -[Q,\psi] = [Q,-\psi]$ which is the result. $\sharp$

3.11. Lemma. *Let (P,φ) be a symmetric space in K such that there exists an exact triangle*

$$T^{-1}(L^{\#}) \xrightarrow{z} L \xrightarrow{\alpha} P \xrightarrow{\beta} L^{\#}$$

satisfying all the following properties:
(1) $\alpha^{\#}\varphi\,\alpha = 0$
(2) $\beta\,\varphi^{-1}\,\beta^{\#} = 0$
(3) $T^{-1}(z^{\#}) = z$.

Then, there exists a morphism $\xi : L \to L$, a morphism $\eta : L \to Q$, where $Q = \mathrm{Cone}(\xi)$ and a symmetric form ψ over Q such that the following triangle is exact:

$$T^{-1}(L^{\#}) \xrightarrow{w} L \xrightarrow{\begin{pmatrix}\varphi^{-1}\beta^{\#}\\ \eta\end{pmatrix}} P \oplus Q \xrightarrow{\begin{pmatrix}\beta & \eta^{\#}\psi\end{pmatrix}} L^{\#}$$

where $w = \xi\,z$ is moreover symmetric: $T^{-1}(w^{\#}) = w$.

In particular (P,φ) and $(Q,-\psi)$ are Witt-equivalent.

3.12. Proof. This is an immediate corollary of the previous lemma and of the fact that there exists a diagram Ω by discussion 3.1. This was only stated to make the proof of the next proposition easier. $\sharp$

3.13. Proposition. *Let K be noetherian. Let (P,φ) be a symmetric space in K. Suppose that there exists an exact triangle*

$$T^{-1}(L^{\#}) \xrightarrow{z} L \xrightarrow{\alpha} P \xrightarrow{\beta} L^{\#}$$

such that all the following properties are satisfied:
(1) $\alpha^{\#}\varphi\,\alpha = 0$
(2) $\beta\,\varphi^{-1}\,\beta^{\#} = 0$
(3) $T^{-1}(z^{\#}) = z$.

Then, (P,φ) is neutral.

3.14. Proof. We are going to apply several times the previous lemma. It is not necessary but more convenient to proceed *ab absurdo*. So, let us suppose that (P,φ) is *not* neutral. By theorem 2.5, this is the same as saying that (P,φ) is not stably neutral.

Let us first of all introduce a neutral form on P that will be useful in the proof. Recall that, since $z^* = z$, there exists a neutral symmetric structure on P defined by:

$$(4)\qquad \begin{array}{ccccccc}
T^{-1}(L^{\#}) & \xrightarrow{z} & L & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & L^{\#}\\
\| & & \| & & \simeq\big\downarrow \chi = \chi^{\#} & & \|\\
T^{-1}(L^{\#}) & \xrightarrow[z^*]{} & L & \xrightarrow[\beta^{\#}]{} & P^{\#} & \xrightarrow[\alpha^{\#}]{} & L^{\#}.
\end{array}$$

Keep this in mind.

By lemma 3.11, there exists a morphism $\xi_1 : L \to L$, an object Q_1 (the cone of ξ_1), a symmetric form ψ_1 on Q_1 and a morphism $\eta_1 : L \to Q_1$ such that the following triangle is exact:

$$(5)\qquad T^{-1}(L^{\#}) \xrightarrow{z_1} L \xrightarrow{\begin{pmatrix}\varphi^{-1}\beta^{\#}\\ \eta_1\end{pmatrix}} P \oplus Q_1 \xrightarrow{\begin{pmatrix}\beta & \eta_1^{\#}\psi_1\end{pmatrix}} L^{\#}$$

where $z_1 = \xi_1\,z = z_1^*$. Moreover, (Q_1,ψ_1) is Witt-equivalent to $(P,-\varphi)$ and therefore is not neutral by hypothesis. Call this

$$(6)\qquad (Q_1,\psi_1)\quad \text{is not neutral.}$$

In particular, Q_1 is not zero. Note that $0 \overset{(5)}{=} \begin{pmatrix} \beta & \eta_1^{\#}\psi_1 \end{pmatrix} \cdot \begin{pmatrix} \varphi^{-1}\beta^{\#} \\ \eta_1 \end{pmatrix} = \beta\,\varphi^{-1}\beta^{\#} + \eta_1^{\#}\psi_1\,\eta_1 \overset{(2)}{=} \eta_1^{\#}\psi_1\,\eta_1$. Call this

$$\eta_1^{\#}\psi_1\,\eta_1 = 0. \tag{7}$$

Define $P_1 := P \oplus Q_1$ and endow it with the form

$$\varphi_1 := \chi \perp \psi_1$$

(pay attention: χ replaces φ on P!). Keep an eye on triangle (5) while defining

$$\alpha_1 := \begin{pmatrix} \varphi^{-1}\beta^{\#} \\ \eta_1 \end{pmatrix} \qquad \text{and} \qquad \beta_1 := \begin{pmatrix} \beta & \eta_1^{\#}\psi_1 \end{pmatrix}.$$

We want to check that $P_1, \varphi_1, L, z_1, \alpha_1, \beta_1$ satisfy the hypothesis of the proposition, in place of P, φ, L, z, α and β. First, the exact triangle is simply (5). We already have property (3) and it suffices to check that $\alpha_1^{\#}\varphi_1\,\alpha_1 = 0$ and that $\beta_1\,\varphi_1^{-1}\,\beta_1^{\#} = 0$. This is easy. Recall that (2) says that $\beta \circ (\varphi^{-1}\,\beta^{\#}) = 0$. Then, exactness of the triangle of the proposition insures the existence of a morphism $\mu : L \to L$ such that $\varphi^{-1}\,\beta^{\#} = \alpha\,\mu$. Then

$$\alpha_1^{\#}\varphi_1\,\alpha_1 = \underbrace{(\beta\,\varphi^{-1})}_{=(\alpha\,\mu)^{\#}}\,\chi\,\underbrace{(\varphi^{-1}\beta^{\#})}_{=\alpha\,\mu} + \underbrace{\eta_1^{\#}\psi_1\,\eta_1}_{\overset{(7)}{=}\,0} = \mu^{\#}\,\underbrace{\alpha^{\#}\chi\,\alpha}_{\overset{(4)}{=}\,0}\,\mu = 0 \tag{8}$$

and

$$\beta_1\,\varphi_1^{-1}\,\beta_1^{\#} = \underbrace{\beta\,\chi^{-1}\beta^{\#}}_{\overset{(4)}{=}\,0} + \underbrace{\eta_1^{\#}\psi_1\eta_1}_{\overset{(7)}{=}\,0} = 0. \tag{9}$$

Now we can apply the previous lemma to this new space and this new triangle.

So there exists a morphism $\xi_2 : L \to L$, an object Q_2 (the cone of ξ_2), a morphism $\eta_2 : L \to Q_2$ and a symmetric form ψ_2 over Q_2 such that the following triangle is exact:

$$T^{-1}(L^{\#}) \xrightarrow{z_2} L \xrightarrow{\begin{pmatrix} \varphi_1^{-1}\beta_1^{\#} \\ \eta_2 \end{pmatrix}} P_1 \oplus Q_2 \xrightarrow{\begin{pmatrix} \beta_1 & \eta_2^{\#}\psi_2 \end{pmatrix}} L^{\#} \tag{10}$$

where $z_2 = \xi_2\,z_1 = z_2^*$. Note that (P_1, φ_1) cannot be neutral, otherwise (since χ is) so would be (Q_1, ψ_1), in contradiction with (6). Moreover, (Q_2, ψ_2) is Witt-equivalent to $(P_1, -\varphi_1)$ and therefore isn't neutral neither. Call this

$$(Q_2, \psi_2) \quad \text{is not neutral.} \tag{11}$$

In particular, Q_2 is not zero. But we can deduce something else from the fact that (Q_2, ψ_2) is Witt-equivalent to $(P_1, -\varphi_1)$, namely that

$$\underbrace{\chi \perp \psi_1}_{=\varphi_1} \perp \psi_2 \qquad \text{is neutral}$$

and therefore, since χ is neutral, we have that

$$\psi_1 \perp \psi_2 \qquad \text{is neutral.} \tag{12}$$

A last property that we can get from (10) is that

$$0 = \begin{pmatrix} \beta_1 & \eta_2^{\#}\psi_2 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1^{-1}\beta_1^{\#} \\ \eta_2 \end{pmatrix} = \underbrace{\beta_1\,\varphi_1^{-1}\beta_1^{\#}}_{\overset{(9)}{=}\,0} + \eta_2^{\#}\psi_2\,\eta_2 = \eta_2^{\#}\psi_2\,\eta_2. \tag{13}$$

Finally, a direct computation gives that $\varphi_1^{-1}\beta_1^{\#} = \begin{pmatrix} \chi^{-1}\beta^{\#} \\ \eta_1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \eta_1 \end{pmatrix}$, by (4). In other words, replacing this in (10) gives the following exact triangle:

$$(14) \qquad T^{-1}(L^{\#}) \xrightarrow{z_2} L \xrightarrow{\begin{pmatrix} \alpha \\ \eta_1 \\ \eta_2 \end{pmatrix}} P \oplus Q_1 \oplus Q_2 \xrightarrow{\begin{pmatrix} \beta & \eta_1^{\#}\psi_1 & \eta_2^{\#}\psi_2 \end{pmatrix}} L^{\#}.$$

Using (12), $\varphi\perp\psi_1\perp\psi_2$ is not neutral, otherwise so would be φ. But this new form $\varphi\perp\psi_1\perp\psi_2$ satisfies again the hypothesis of the proposition. In fact, the exact triangle is (14) and since $z_2 = z_2^*$, we only have to check conditions (1) and (2). These are all immediate consequences of (1), (2), (7) and (13). The good reader checks this carefully.

Consider this as the first step of the proof. To a (non-neutral) symmetric space like in the proposition, we associate these two spaces (Q_1, ψ_1) and (Q_2, ψ_2) and an exact triangle

$$T^{-1}(L^{\#}) \xrightarrow{z_2} L \xrightarrow{\alpha_2 := \begin{pmatrix} \alpha \\ \eta_1 \\ \eta_2 \end{pmatrix}} P \oplus Q_1 \oplus Q_2 \xrightarrow{\beta_2 := \begin{pmatrix} \beta & \eta_1^{\#}\psi_1 & \eta_2^{\#}\psi_2 \end{pmatrix}} L^{\#}$$

where the three morphisms z_2, α_2, β_2 satisfy the same hypotheses (1), (2) and (3) for the symmetric space

$$\Big(P \oplus Q_1 \oplus Q_2, \varphi\perp\psi_1\perp\psi_2\Big).$$

This space is not neutral. Moreover, Q_1 and Q_2 are not zero.

Then, of course, do this again and again. You will get an infinite tower of spaces always fitting in triangles over $T^{-1}(L^{\#})$ and L and such that you have the compatibility condition of definition 3.3 ($w_{i+1}\, f_i = w_i$). To see this, observe that the first component of β_2 is β (this was wrong for β_1 and is the reason why we used twice lemma 3.11 !).

The noetherianity condition insures that this tower should stop, that is some $Q_i = 0$, which leads to a contradiction. $\sharp$

3.15. Proof of theorem 3.5. Hereafter we shall label by (1) the first hypothesis of theorem 3.5, namely the fact that we have the commutative diagram $\bar{\Omega}$ with exact lines and columns.

The second hypothesis of theorem 3.5 will be labeled as

$$(2) \qquad \eta_1^{\#}\,\psi\,\eta_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#}.$$

Forgetting for a while the form over Q, we have a diagram Ω for (P, φ). Therefore, lemma 3.7 insures the existence of an exact triangle:

$$(3) \qquad T^{-1}(M^{\#}) \xrightarrow{w} M \xrightarrow{\begin{pmatrix} \varphi^{-1}\nu_2^{\#} \\ \eta_1 \end{pmatrix}} P \oplus Q \xrightarrow{\begin{pmatrix} \nu_2 & y \end{pmatrix}} M^{\#}$$

for some morphism $y : Q \to M^{\#}$ such that

$$(4) \qquad T(\nu_0)\, y = \eta_2.$$

Recall that $w = \eta_0\,\nu_0 = w^*$.

By the above proposition, it suffices to prove that:

$$(i) \qquad \alpha^{\#}(\varphi\perp\psi)\,\alpha = 0 \qquad \text{where} \qquad \alpha := \begin{pmatrix} \varphi^{-1}\nu_2^{\#} \\ \eta_1 \end{pmatrix}$$

and

$$(ii) \qquad \beta\,(\varphi\perp\psi)^{-1}\,\beta^{\#} = 0 \qquad \text{where} \qquad \beta := (\,\nu_2 \quad y\,)\,.$$

That is what we are going to verify.

Condition (i) is immediate:

$$\alpha^{\#}(\varphi\perp\psi)\,\alpha = \big(\,\nu_2\,\varphi^{-1} \quad \eta_1^{\#}\,\big)\cdot\begin{pmatrix}\varphi & 0\\ 0 & \psi\end{pmatrix}\cdot\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix} = \nu_2\,\varphi^{-1}\nu_2^{\#} + \eta_1^{\#}\psi\,\eta_1 \overset{(2)}{=} 0.$$

Dualizing triangle (3) gives the first line of the following diagram, which is an isomorphism of triangles:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "w"] \arrow[d, equal] & M \arrow[r, "\begin{pmatrix}\nu_2^{\#}\\ y^{\#}\end{pmatrix}"] \arrow[d, equal] & P^{\#}\oplus Q^{\#} \arrow[r, "(\,\nu_2\,\varphi^{-1} \quad \eta_1^{\#}\,)"] \arrow[d, "\begin{pmatrix}\varphi^{-1} & 0\\ 0 & \psi^{-1}\end{pmatrix}", "\simeq"'] & M^{\#} \arrow[d, equal] \\
T^{-1}(M^{\#}) \arrow[r, "w"'] & M \arrow[r, "\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \psi^{-1}y^{\#}\end{pmatrix}"'] & P\oplus Q \arrow[r, "(\,\nu_2 \quad \eta_1^{\#}\psi\,)"'] & M^{\#}.
\end{tikzcd}$$

Then the second line is also exact. Call this exact triangle (5).

Now, from (4) and the lower commutative square of $\bar{\Omega}$, we have that:

$$T(\nu_0)\,y = \eta_2 = T(\nu_0)\,\eta_1^{\#}\psi.$$

Therefore $T(\nu_0)\,(y - \eta_1^{\#}\psi) = 0$. This gives, using the exact triangle over $T(\nu_0)$, the existence of a morphism

$$(6) \qquad r\,:Q\to P \qquad \text{such that} \qquad y = \eta_1^{\#}\psi + \nu_2\,r.$$

This is the same as saying that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "w"] \arrow[d, equal] & M \arrow[r, "\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix}"] & P\oplus Q \arrow[r, "(\,\nu_2 \quad y\,)"] \arrow[d, "\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}", "\simeq"'] & M^{\#} \arrow[d, equal] \\
T^{-1}(M^{\#}) \arrow[r, "w"'] & M \arrow[r, "\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \psi^{-1}y^{\#}\end{pmatrix}"'] & P\oplus Q \arrow[r, "(\,\nu_2 \quad \eta_1^{\#}\psi\,)"'] & M^{\#}.
\end{tikzcd}$$

The first line of this diagram is exact triangle (3) and the second one is exact triangle (5). It is then an axiom of triangulated categories that you can complete this in a commutative diagram:

$$(7) \qquad \begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "w"] \arrow[d, equal] & M \arrow[r, "\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix}"] \arrow[d, dashed, "\exists\, g"] & P\oplus Q \arrow[r, "(\,\nu_2 \quad y\,)"] \arrow[d, "\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}", "\simeq"'] & M^{\#} \arrow[d, equal] \\
T^{-1}(M^{\#}) \arrow[r, "w"'] & M \arrow[r, "\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \psi^{-1}y^{\#}\end{pmatrix}"'] & P\oplus Q \arrow[r, "(\,\nu_2 \quad \eta_1^{\#}\psi\,)"'] & M^{\#}.
\end{tikzcd}$$

and the morphism g is necessarily an isomorphism. In particular, we have from the middle commutative square:

$$(8) \qquad \varphi^{-1}\nu_2^{\#} + r\,\eta_1 = \varphi^{-1}\nu_2^{\#}\,g$$

$$(9) \qquad \eta_1 = \psi^{-1}y^{\#}\,g.$$

Let $s := \nu_2\,\varphi^{-1}\nu_2^{\#} \overset{(2)}{=} -\eta_1^{\#}\psi\,\eta_1$. Note that $s^{\#} = s$. Directly from (3), we have that

$$0 = (\,\nu_2 \quad y\,)\cdot\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ \eta_1\end{pmatrix} = \nu_2\,\varphi^{-1}\nu_2^{\#} + y\,\eta_1$$

and therefore

$$y\,\eta_1 = -s. \tag{10}$$

Then composing the equation (9) by $\eta_1^{\#}\psi$ on the left and using (10), we obtain:

$$\underbrace{\eta_1^{\#}\psi\,\eta_1}_{=\,-s} = \eta_1^{\#}\psi\,\psi^{-1}y^{\#}g = (\,\underbrace{y\,\eta_1}_{=\,-s}\,)^{\#}g = -s\,g$$

which means that

$$s\,g = s. \tag{11}$$

Now, apply ν_2 on the left to (8) to obtain:

$$\underbrace{\nu_2\,\varphi^{-1}\nu_2^{\#}}_{=\,s} + \nu_2\,r\,\eta_1 = \underbrace{\nu_2\,\varphi^{-1}\nu_2^{\#}}_{=\,s}\,g.$$

This, compared with (11), forces

$$\nu_2\,r\,\eta_1 = 0. \tag{12}$$

We are now able to compute

$$\begin{aligned} y\,\psi^{-1}y^{\#} &\overset{(6)}{=} (\eta_1^{\#}\psi + \nu_2\,r)\,\psi^{-1}y^{\#} \overset{(9)}{=} (\eta_1^{\#}\psi + \nu_2\,r)\,(\eta_1\,g^{-1}) = \\ &= \eta_1^{\#}\psi\,\eta_1\,g^{-1} + \nu_2\,r\,\eta_1\,g^{-1} \overset{(12)}{=} \underbrace{\eta_1^{\#}\psi\,\eta_1}_{=\,-s}\,g^{-1} \overset{(11)}{=} -s = \\ &= -\nu_2\,\varphi^{-1}\nu_2^{\#}. \end{aligned}$$

Using this last equation, we can now check condition (ii):

$$\beta\circ(\varphi\perp\psi)^{-1}\circ\beta^{\#} \overset{(\mathrm{def})}{=} (\,\nu_2 \quad y\,)\cdot\begin{pmatrix}\varphi & 0\\ 0 & \psi\end{pmatrix}^{-1}\cdot\begin{pmatrix}\nu_2^{\#}\\ y^{\#}\end{pmatrix} = \nu_2\,\varphi^{-1}\nu_2^{\#} + y\,\psi^{-1}y^{\#} = 0.$$

This gives the result by proposition 3.13. ♯

3.16. Comment. So far, we have established that given a diagram Ω which has been completed in a diagram $\bar{\Omega}$ (meaning that we have constructed a form on Q), then the new form is Witt-equivalent to (the opposite of) the form we started with.

Unfortunately, this is not enough for the application we want to do of this in the localization exact sequence. We need to be able to construct such a form on the cone of <u>any</u> morphism η_0 such that Ω commutes. We do not know if the latter is true or not! Nevertheless, the following result will be sufficient.

3.17. Theorem. *Suppose that K is a noetherian triangulated category. Let (P,φ) be a symmetric space and (L,ν_1) be a sub-lagrangian of it. Choose an exact triangle over ν_1:*

$$T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}.$$

Let $\eta_0 : L \to M$ be a morphism such that diagram Ω commutes (confer 3.1). Then there exists another morphism $\mu_0 : L \to M$ such that:

1. *there exists a form on the cone of μ_0 Witt-equivalent to (P,φ);*
2. $\mathrm{Cone}(\mu_0)$ *and* $\mathrm{Cone}(\eta_0)$ *are stably isomorphic, namely*

$$P\oplus\mathrm{Cone}(\mu_0) \simeq P\oplus\mathrm{Cone}(\eta_0).$$

3.18. Proof. First we shall consider two intermediate results. Then, see 3.23. Notation: once again $* = T^{-1} \circ \#$.

3.19. Lemma. *Under the hypothesis of theorem 3.17, there exists a morphism $\lambda : L \to T^{-1}(L^{\#})$ such that*

(3) $T^{-1}(\nu_0^{\#})\,\lambda\,\nu_0 = 0$
(4) *setting $\mu_0 := \eta_0 + T^{-1}(\nu_0^{\#})\lambda$, the following triangle is exact:*

$$M \xrightarrow{s} M^{\#} \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}} T(L) \oplus L^{\#} \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}} T(M)$$

where $s = \nu_2\,\varphi^{-1}\nu_2^{\#}$.

3.20. Lemma. *Under the hypothesis of theorem 3.17, let $s = \nu_2\,\varphi^{-1}\nu_2^{\#}$. Given any morphism $\mu_0 : L \to M$ such that the triangle*

$$M \xrightarrow{s} M^{\#} \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}} T(L) \oplus L^{\#} \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}} T(M)$$

is exact and given any exact triangle over μ_0:

$$L \xrightarrow{\mu_0} M \xrightarrow{\mu_1} R \xrightarrow{\mu_2} T(L),$$

there exists a symmetric form ψ on the cone R of μ_0 such that the following diagram commutes:

$$(\bar{\bar{\Omega}}) \qquad \begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\mu_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\mu_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\mu_0^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] \arrow[d, "\mu_2^{\#}"'] & M \arrow[r, "\nu_2^{\#}"'] \arrow[d, "\mu_1"] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#} \\
R^{\#} \arrow[d, "\mu_1^{\#}"'] & R \arrow[l, "\psi = \psi^{\#}"', "\simeq"] \arrow[d, "\mu_2"] & & \\
M^{\#} \arrow[r, "T(\nu_0)"'] & T(L) & &
\end{tikzcd}$$

and such that

$$\mu_1^{\#}\psi\,\mu_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#}.$$

In other words, such a μ_0 is a good way to send L in its orthogonal, M (see 3.2).

3.21. Proof of lemma 3.19. Let us give a number to the upper part of Ω for η_0:

$$(5) \qquad \begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\eta_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\eta_0^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}
\end{tikzcd}$$

and another number, say the next available one, to the exact triangle:

$$L \xrightarrow{\eta_0} M \xrightarrow{\eta_1} Q \xrightarrow{\eta_2} T(L). \tag{6}$$

We are going to study exact triangles over $s = \nu_2\, \varphi^{-1} \nu_2^{\#}$.

Consider the composition axiom for the relation:

$$\nu_1^{\#} \circ (\varphi\, \nu_1) = 0$$

to obtain the following octahedron (the exact triangles over $\nu_1^{\#}$ and over $\varphi\, \nu_1$ are obtained from the one over ν_1):

$$\begin{tikzcd}
 & & L^{\#} \arrow[dl, "\nu_0^{\#}"'] \arrow[ddrr, "\begin{pmatrix}0\\1\end{pmatrix}"] & & \\
P^{\#} \arrow[urr, "\nu_1^{\#}"] \arrow[ddr, "-\nu_2\,\varphi^{-1}"'] & T(M) \arrow[l, "T(\nu_2^{\#})"'] \arrow[dd, "-T(s)"] & 0 & & \\
 & & L \arrow[uu] \arrow[ull, "\varphi\,\nu_1"] & & T(L)\oplus L^{\#} \arrow[ll, "(1\quad 0)"'] \arrow[ulll, "\exists\,(c\quad d)"'] \\
 & M^{\#} \arrow[ur, "T(\nu_0)"] \arrow[urrr, "\exists\begin{pmatrix}a\\b\end{pmatrix}"'] & & &
\end{tikzcd}$$

for some morphisms a, b, c, d. The relations involving the injection $\begin{pmatrix}0\\1\end{pmatrix}$ and the projection $(1\quad 0)$ do force $a = T(\nu_0)$ and $d = \nu_0^{\#}$. As well, the relation

$$b \circ (-\nu_2) = \nu_1^{\#} \varphi \overset{(1)}{=} \eta_0^{\#} \nu_2$$

insures the existence of a morphism $e : T(L) \to L^{\#}$ such that:

$$b = -\eta_0^{\#} + e\, T(\nu_0).$$

As in the proof 3.8, one can use the automorphism

$$\begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix}$$

of $T(L) \oplus L^{\#}$ to replace b by $-\eta_0^{\#}$. Replacing c by $c + \nu_0^{\#} e$, and calling this $T(\beta)$, everything reduces to the existence of an exact triangle:

$$M \xrightarrow{s} M^{\#} \xrightarrow{\begin{pmatrix}T(\nu_0)\\ -\eta_0^{\#}\end{pmatrix}} T(L) \oplus L^{\#} \xrightarrow{\begin{pmatrix}T(\beta) & \nu_0^{\#}\end{pmatrix}} T(M) \tag{7}$$

for some morphism $\beta : L \to M$ such that:

$$T(\nu_2^{\#})\, T(\beta) = T(\varphi)\, T(\nu_1)$$

because this was true for c and because $T(\nu_2^{\#})\, \nu_0^{\#} = 0$. This is equivalent to:

$$\nu_2^{\#} \beta = \varphi\, \nu_1. \tag{8}$$

Of course, the aesthete would prefer to have $\beta = \eta_0$. This is not true in general but it appears that one can replace η_0 by some μ_0 such that the following triangle is exact:

$$M \xrightarrow{s} M^\# \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^\# \end{pmatrix}} T(L) \oplus L^\# \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^\# \end{pmatrix}} T(M). \tag{9}$$

That is what we are going to do now.

¿From property (8) and from (5), we deduce that

$$\nu_2^\# \beta = \varphi \, \nu_1 = \nu_2^\# \eta_0.$$

In other words, $\nu_2^\#(\beta - \eta_0) = 0$. Then there exists a morphism $l : L \to T^{-1}(L^\#)$ such that

$$\beta = \eta_0 + T^{-1}(\nu_0^\#) \, l. \tag{10}$$

Dualizing exact triangle (7), recalling that $s^\# = s$, gives the following exact triangle:

$$M \xrightarrow{s} M^\# \xrightarrow{\begin{pmatrix} \beta^\# \\ T(\nu_0) \end{pmatrix}} L^\# \oplus T(L) \xrightarrow{\begin{pmatrix} -\nu_0^\# & +T(\eta_0) \end{pmatrix}} T(M).$$

Then using the isomorphism $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ from $L^\# \oplus T(L)$ to $T(L) \oplus L^\#$, we get the following exact triangle:

$$M \xrightarrow{s} M^\# \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\beta^\# \end{pmatrix}} T(L) \oplus L^\# \xrightarrow{\begin{pmatrix} T(\eta_0) & \nu_0^\# \end{pmatrix}} T(M).$$

Put this last triangle as the second row of the following diagram, in which the first is simply (7):

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, equal] & M^\# \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^\# \end{pmatrix}"] & T(L) \oplus L^\# \arrow[r, "\begin{pmatrix} T(\beta) & \nu_0^\# \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ T(l) & 1 \end{pmatrix}"] & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^\# \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\beta^\# \end{pmatrix}"'] & T(L) \oplus L^\# \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^\# \end{pmatrix}"'] & T(M)
\end{tikzcd}$$

and which commutes because of (10). Since it has exact rows, it is an axiom that there exists a morphism (and nobody can keep me from calling it $1 + h$) such that:

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, equal] & M^\# \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^\# \end{pmatrix}"] \arrow[d, dashed, "\exists\, 1+h"] & T(L) \oplus L^\# \arrow[r, "\begin{pmatrix} T(\beta) & \nu_0^\# \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ T(l) & 1 \end{pmatrix}"] & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^\# \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\beta^\# \end{pmatrix}"'] & T(L) \oplus L^\# \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^\# \end{pmatrix}"'] & T(M)
\end{tikzcd} \tag{11}$$

commutes, for some morphism $h : M^\# \to M^\#$. We have the following relations:

$$h \, s = 0 \tag{12}$$

$$T(\nu_0) \, h = 0 \tag{13}$$

$$-\beta^\#(1 + h) = T(l) \, T(\nu_0) - \eta_0^\#. \tag{14}$$

Compose (14) on the right with ν_2 gives:

$$-\underbrace{\beta^{\#}\nu_2}_{\overset{(8)}{=}\ \nu_1^{\#}\varphi}\quad -\beta^{\#}h\,\nu_2 = T(l)\,\underbrace{T(\nu_0)\,\nu_2}_{=0} - \underbrace{\eta_0^{\#}\nu_2}_{\overset{(5)}{=}\ \nu_1^{\#}\varphi}$$

which forces obviously

$$\beta^{\#}h\,\nu_2 = 0. \tag{15}$$

¿From (12) and the exact triangle presented in the second line of (11), we deduce that

$$h = \bar h\begin{pmatrix} T(\nu_0)\\ -\beta^{\#}\end{pmatrix} \tag{16}$$

for some morphism $\bar h : T(L)\oplus L^{\#}\to M^{\#}$. Similarly, from (13) and the exact triangle in the first line of (5), we deduce that:

$$h = \nu_2\,\tilde h \tag{17}$$

for some morphism $\tilde h : M^{\#}\to P$.

Now, compute h^3 using (16) and (17):

$$h^3 = \bar h\begin{pmatrix} T(\nu_0)\\ -\beta^{\#}\end{pmatrix}\circ h\circ(\nu_2\,\tilde h) = \bar h\begin{pmatrix} T(\nu_0)\,h\,\nu_2\\ -\beta^{\#}\,h\,\nu_2\end{pmatrix}\tilde h.$$

It follows then immediately from (13) and (15) that $h^3 = 0$. Therefore $1+\frac12 h$ is an isomorphism:

$$1+\frac{1}{2}h\ : M^{\#}\overset{\sim}{\to} M^{\#}.$$

But, dualizing (11) and using as before the isomorphism $\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$, we get the following morphism of exact triangles:

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s"] \arrow[d, "1+h^{\#}"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0)\\ -\eta_0^{\#}\end{pmatrix}"] \arrow[d, equal] & T(L)\oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\beta) & \nu_0^{\#}\end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0\\ -l^{\#} & 1\end{pmatrix}"] & T(M) \arrow[d, "1+T(h^{\#})"] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0)\\ -\beta^{\#}\end{pmatrix}"'] & T(L)\oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^{\#}\end{pmatrix}"'] & T(M).
\end{tikzcd}$$

If the reader prefers, this last diagram can be obtained directly from the relations (10), (12), (13) and (14) by applying the duality. Taking the mean of this last morphism with (11), we get the following:

(18)

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s"] \arrow[d, "1+\frac{1}{2}h^{\#}"', "\simeq"] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0)\\ -\eta_0^{\#}\end{pmatrix}"] \arrow[d, "\simeq"', "1+\frac{1}{2}h"] & T(L)\oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\beta) & \nu_0^{\#}\end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0\\ \frac12(T(l)-l^{\#}) & 1\end{pmatrix}"] & T(M) \arrow[d, "\simeq"', "1+\frac{1}{2}T(h^{\#})"] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0)\\ -\beta^{\#}\end{pmatrix}"'] & T(L)\oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^{\#}\end{pmatrix}"'] & T(M).
\end{tikzcd}$$

Let us give names:

$$f := 1+\frac{1}{2}h^{\#}\ : M\overset{\sim}{\to} M$$

$$\bar{l} := \frac{l - l^*}{2} = -\bar{l}^* : L \to T^{-1}(L^{\#}).$$

Pushing f up and $f^{\#}$ down in (18), we obtain the following morphism of exact triangles:

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s\,f^{-1}"] \arrow[d, equal] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix}"] \arrow[d, equal] & T(L) \oplus L^{\#} \arrow[r, "T(f) \circ \begin{pmatrix} T(\beta) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ T(\bar{l}) & 1 \end{pmatrix}"] & T(M) \arrow[d, equal] \\
M \arrow[r, "(f^{\#})^{-1}s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\beta^{\#} \end{pmatrix} \circ f^{\#}"'] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^{\#} \end{pmatrix}"'] & T(M).
\end{tikzcd}$$

¿From (12) and the definition of f, we have $s\,f = s$ and from (13) we have $T(\nu_0)\,f^{\#} = T(\nu_0)$. Use this in the above diagram to obtain the following one:

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s"] \arrow[d, equal] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix}"] \arrow[d, equal] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\gamma) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ T(\bar{l}) & 1 \end{pmatrix}"] & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\gamma^{\#} \end{pmatrix}"'] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^{\#} \end{pmatrix}"'] & T(M)
\end{tikzcd} \tag{19}$$

where $\gamma := f\,\beta$. Note that the exact triangle of the first line gives:

$$0 = \begin{pmatrix} T(\gamma) & \nu_0^{\#} \end{pmatrix} \cdot \begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix} = T(\gamma\,\nu_0) - \nu_0^{\#}\eta_0^{\#}$$

which implies

$$\gamma\,\nu_0 = \nu_0^*\eta_0^* \overset{(5)}{=} \eta_0\,\nu_0. \tag{20}$$

Now, since the third commutative square of (19) gives

$$\gamma = \eta_0 + \nu_0^*\bar{l} \tag{21}$$

we have, composing this last equation with ν_0 on the right and comparing with (20), that:

$$\nu_0^*\bar{l}\,\nu_0 = 0. \tag{22}$$

Consider now the following commutative diagram:

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s"] \arrow[d, equal] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix}"] \arrow[d, equal] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\gamma) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \frac{1}{2}T(\bar{l}) & 1 \end{pmatrix}", "\simeq"'] & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ \frac{1}{2}T(\bar{l}\,\nu_0) - \eta_0^{\#} \end{pmatrix}"'] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\gamma - \frac{1}{2}\nu_0^*\bar{l}) & \nu_0^{\#} \end{pmatrix}"'] & T(M)
\end{tikzcd} \tag{23}$$

in which the first row (and therefore also the second) is an exact triangle (see 19). Define

$$\mu_0 := \eta_0 + \frac{1}{2}\nu_0^*\bar{l}.$$

Observe that (21) implies

$$\gamma - \frac{1}{2}\nu_0^*\bar{l} = \eta_0 + \frac{1}{2}\nu_0^*\bar{l} = \mu_0$$

and that (here we use $\bar{l} = -\bar{l}^*$ which is immediate from the definition of $\bar{l}$) :

$$\mu_0^\# = \eta_0^\# + \frac{1}{2}\bar{l}^\# T(\nu_0) = \eta_0^\# + \frac{1}{2}T(\bar{l}^*\nu_0) = \eta_0^\# - \frac{1}{2}T(\bar{l}\nu_0).$$

Replace these two last facts in the second line of (23) to obtain the announced exact triangle (4) :

$$M \xrightarrow{\ s\ } M^\# \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^\# \end{pmatrix}} T(L)\oplus L^\# \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^\# \end{pmatrix}} T(M).$$

Define $\lambda = \frac{1}{2}\bar{l}$, so that $\mu_0 = \eta_0 + \nu_0^*\lambda$ as required. Condition (3) asks $\nu_0^*\lambda\,\nu_0 = 0$, which is a direct consequence of (22). $\sharp$

3.22. Proof of lemma 3.20. Recall that $* = T^{-1}\circ \#$ and that s denotes $s = \nu_2\,\varphi^{-1}\nu_2^\# : M \to M^\#$. Note that $s = s^\#$. Call (1) the exact triangle of the hypothesis :

(1)
$$M \xrightarrow{\ s\ } M^\# \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^\# \end{pmatrix}} T(L)\oplus L^\# \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^\# \end{pmatrix}} T(M).$$

Apply the composition axiom to the relation :

$$\mu_0 = \begin{pmatrix} \mu_0 & \nu_0^* \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

to construct the following octahedron (where the exact triangle over $\begin{pmatrix} \mu_0 & \nu_0^* \end{pmatrix}$ comes from (24)) :

$$\begin{tikzcd}
 & & M \arrow[dl, "s"'] \arrow[drr, "\mu_1"] & & \\
L\oplus T^{-1}(L^\#) \arrow[urr, "(\mu_0\ \ \nu_0^*)"] \arrow[ddr, "(0\ \ 1)"'] & M^\# \arrow[l, "\begin{pmatrix} -T(\nu_0) \\ +\mu_0^\# \end{pmatrix}"'] \arrow[dd, "\mu_0^\#"] & & & \\
 & & L \arrow[uu, "\mu_0"] \arrow[ull, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & & R \arrow[ll, "\mu_2"'] \arrow[ulll, "\exists z"'] \\
 & T^{-1}(L^\#) \arrow[ur, "0"] \arrow[urrr, "\exists y"'] & & &
\end{tikzcd}$$

The relation $y\circ\begin{pmatrix} 0 & 1 \end{pmatrix} = \mu_1\circ\begin{pmatrix} \mu_0 & \nu_0^* \end{pmatrix}$ forces $y = \mu_1\,\nu_0^*$. Then, the octahedron reduces to the existence of a morphism $z : R \to M^\#$ such that

(2) the following triangle is exact :

$$T^{-1}(L^\#) \xrightarrow{\mu_1\,\nu_0^*} R \xrightarrow{\ z\ } M^\# \xrightarrow{\mu_0^\#} L^\#$$

(3) $z\,\mu_1 = s$

(4) $-T(\nu_0)\,z = \mu_2$.

Dualizing (2) we obtain the exact triangle:

$$(5) \qquad L \xrightarrow{\mu_0} M \xrightarrow{z^\#} R^\# \xrightarrow{-T(\nu_0)\,\mu_1^\#} T(L).$$

But we already have an exact triangle over μ_0, so let us compare them:

$$(6) \qquad \begin{tikzcd} L \arrow[r, "\mu_0"] \arrow[d, equal] & M \arrow[r, "\mu_1"] \arrow[d, equal] & R \arrow[r, "\mu_2"] \arrow[d, dashed, "\exists\,\chi"', "\simeq"] & T(L) \arrow[d, equal] \\ L \arrow[r, "\mu_0"'] & M \arrow[r, "z^\#"'] & R^\# \arrow[r, "-T(\nu_0)\,\mu_1^\#"'] & T(L). \end{tikzcd}$$

There exists an isomorphism, non necessarily symmetric, $\chi : R \overset{\sim}{\to} R^\#$ making diagram (6) commute. We cannot use the classical argument because $\chi^\#$ does not make diagram (6) commute! So, we have to be more careful.

Since $\chi\,\mu_1 = z^\#$, we obtain by composing on the right with $-\nu_0^*$ that

$$\chi \circ (-\mu_1\,\nu_0^*) = -z^\#\,\nu_0^*.$$

Using the dual of (4), the right hand side of the above equation may be replaced by $\mu_2^\#$. Apply duality to this new equation to get the following commutative square:

$$\begin{tikzcd} R \arrow[r, "\mu_2"] \arrow[d, "\chi^\#"'] & T(L) \arrow[d, equal] \\ R^\# \arrow[r, "-T(\nu_0)\,\mu_1^\#"'] & T(L) \end{tikzcd}$$

which should be compared with (6). We can complete it into:

$$(7) \qquad \begin{tikzcd} L \arrow[r, "\mu_0"] \arrow[d, equal] & M \arrow[r, "\mu_1"] \arrow[d, dashed, "\exists\ 1+h"] & R \arrow[r, "\mu_2"] \arrow[d, "\chi^\#"] & T(L) \arrow[d, equal] \\ L \arrow[r, "\mu_0"'] & M \arrow[r, "z^\#"'] & R^\# \arrow[r, "-T(\nu_0)\,\mu_1^\#"'] & T(L) \end{tikzcd}$$

for some morphism $h : M \to M$. We immediately have:

$$(8) \qquad h\,\mu_0 = 0.$$

¿From (6), we took $\chi\,\mu_1 = z^\#$. Dualize it: $\mu_1^\#\chi^\# = z$. Compose this last equality by μ_1 on the right and use (3) to obtain:

$$(9) \qquad \mu_1^\#\chi^\#\,\mu_1 = z\,\mu_1 \overset{(3)}{=} s.$$

But (7) tells us that $\chi^\#\,\mu_1 = z^\#\,(1+h)$. Replacing this in (9) gives:

$$s = \mu_1^\#\,z^\#\,(1+h) = (\underbrace{z\,\mu_1}_{\overset{(3)}{=}\ s})^\#\,(1+h) = s^\#\,(1+h) = s\,(1+h)$$

and therefore:

$$(10) \qquad s\,h = 0.$$

We still haven't used the third commutative square of (6) which says that $\mu_2 = -T(\nu_0)\,\mu_1^{\#}\,\chi$. Dualizing gives:

$$\mu_2^{\#} = -\chi^{\#}\mu_1\,\nu_0^{*}.$$

We can use (7) to replace $\chi^{\#}\mu_1$ by $z^{\#}\,(1+h)$:

$$\begin{aligned}\mu_2^{\#} &= -z^{\#}\,(1+h)\,\nu_0^{*} = -\underbrace{z^{\#}\,\nu_0^{*}}_{=\,\left(T(\nu_0)\,z\right)^{\#}} \qquad -z^{\#}\,h\,\nu_0^{*}\\ &\overset{(4)}{=} \mu_2^{\#} - z^{\#}\,h\,\nu_0^{*}.\end{aligned}$$

Step back and look! You will see:

$$z^{\#}\,h\,\nu_0^{*} = 0. \tag{11}$$

It is now easy to get $h^3 = 0$. Use (8) and exact triangle (5) to write

$$h = \bar{h}\,z^{\#}$$

for some morphism $\bar{h} : R^{\#} \to M$. Similarly, use (10) and exact triangle (24) to write

$$h = (\,\mu_0 \quad \nu_0^{*}\,) \circ \tilde{h}$$

for some morphism $\tilde{h} : M \to L \oplus T^{-1}(L^{\#})$. Now simply compute

$$h^3 = \bar{h}\,z^{\#} \circ h \circ (\,\mu_0 \quad \nu_0^{*}\,)\,\tilde{h} = \bar{h} \circ (\,z^{\#}\,h\,\mu_0 \quad z^{\#}\,h\,\nu_0^{*}\,) \circ \tilde{h}$$

Equations (8) and (11) insure the central matrix to be zero. Therefore h is nilpotent and then $1 + \frac{1}{2}h$ is an isomorphism. Taking the mean between (6) and (7), we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd} L \arrow[r, "\mu_0"] \arrow[d, equal] & M \arrow[r, "\mu_1"] \arrow[d, "1+\frac{1}{2}h", "\simeq"'] & R \arrow[r, "\mu_2"] \arrow[d, "\frac{1}{2}(\chi+\chi^{\#})"] & T(L) \arrow[d, equal] \\ L \arrow[r, "\mu_0"'] & M \arrow[r, "z^{\#}"'] & R^{\#} \arrow[r, "-T(\nu_0)\,\mu_1^{\#}"'] & T(L). \end{tikzcd} \tag{12}$$

In other words, the symmetric morphism

$$\psi := \;-\; \frac{1}{2}(\chi + \chi^{\#}) : R \to R^{\#}$$

is an isomorphism. Diagram (12) gives:

$$T(\nu_0)\,\mu_1^{\#}\psi = \mu_2.$$

This means that the form ψ makes the diagram $\bar{O}\bar{m}ega$ commute. We still have to check

$$\mu_1^{\#}\psi\mu_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#} \qquad (= -s)$$

but this is immediate from the mean of relation (9) and its dual. $\sharp$

3.23. Proof of theorem 3.17. Choose λ as in lemma 3.19 and define of course

$$\mu_0 = \eta_0 + \nu_0^* \lambda.$$

Lemma 3.20 insures the existence of a completion $\bar{\bar{\Omega}}$ and theorem 3.5 implies that $(R, -\psi)$ is Witt-equivalent to (P, φ) which is expected property (1) of theorem 3.17. To see how to obtain property (2), observe that:

$$\mu_0\,\nu_0 \overset{(\mathrm{def})}{=} (\eta_0 + \nu_0^*\lambda)\,\nu_0 = \eta_0\,\nu_0 + \underbrace{\nu_0^*\lambda\,\nu_0}_{\overset{(3)}{=}\,0} = \eta_0\,\nu_0.$$

Then it suffices to apply twice lemma 3.7, once to η_0 and then to μ_0. But in both case the morphism $w = \eta_0\,\nu_0 = \mu_0\,\nu_0$ is the same. Therefore Cone(w) is $P \oplus Q$ or $P \oplus R$ as well. This gives (2) and then theorem 3.17.

By discussion 3.1, there always exists a morphism η_0 such that Ω commutes. Therefore the above proof gives in particular the following corollary. ♯

3.24. Corollary. *For any symmetric space (P, φ) and any sub-lagrangian (L, ν_1) of (P, φ), for any exact triangle over ν_1 :*

$$T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}$$

there exists a morphism $\mu_0 : L \to M$ and a form ψ on $R :=$ Cone(μ_0) such that diagram $\bar{\bar{\Omega}}$ of lemma 3.20 commutes, has exact lines and columns and such that $\mu_1^{\#}\,\psi\,\mu_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#}$. In particular (R, ψ) is Witt-equivalent to $(P, -\varphi)$. ♯

3.25. Remark. This result, although it sounds very nice, is probably useless in general. In the classical case, it was the ideal formulation because there was only one way to send a sub-lagrangian into its orthogonal. Here, there are several $(\eta_0, \mu_0, \dots)$, some better (μ_0) than others (η_0). Theorem 3.17 will be more useful.

The results of this paragraph might be used to establish some link between the Witt group I propose here and the quotient of it defined by Youssin in [9].

4. Localization context and residue homomorphism.

4.1. Localization of triangulated categories with duality.
Let us consider what we can call an *exact sequence of triangulated categories*:

$$0 \longrightarrow J \longrightarrow K \longrightarrow L \longrightarrow 0.$$

By that, we understand that L is a localization of K, that is $L = S^{-1}K$ for a class S of morphisms. We can always suppose that S is saturated (*i.e.* S is formed by all the morphisms of K which become isomorphisms in L) and that L is obtained by calculus of fraction (nice exercise). Then J is supposed to be the *kernel* of this localization, *i.e.* the full subcategory of K over the objects which become isomorphic to zero in L.

Suppose moreover that K is endowed with a duality (or a skew-duality) $\#$ such that $\#(S) = S$. Then $S^{-1}K$ inherits a unique duality that can be called the localization of $\#$ and that we shall lazily also denote by $\#$. It is then clear that $\#$ restricts to a duality on J. In other words, we have an *exact sequence of triangulated categories with duality*:

$$\text{(LOC)} \qquad \begin{tikzcd} 0 \arrow[r] & J \arrow[r, "j"] \arrow[d, "\#"'] & K \arrow[r, "q"] \arrow[d, "\#"'] & S^{-1}K \arrow[r] \arrow[d, "\#"'] & 0 \\ 0 \arrow[r] & J \arrow[r, "j"'] & K \arrow[r, "q"'] & S^{-1}K \arrow[r] & 0. \end{tikzcd}$$

4.2. Example. Let X be a noetherian regular separated scheme and U an open subscheme. Consider $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(X)$ the derived category of bounded complexes of locally free $\mathcal{O}_X$-modules of finite rank. Then $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(U)$ is a localization of $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(X)$. It is moreover a localization of triangulated categories with duality (*confer* [1, *théorème* 4.17, p. 28]).

The kernel category J is here the (derived) category of complexes whose homology is concentrated on Y, the closed complement of U in X. It is a good question to wonder how the Witt groups of J might be related to the Witt groups of Y.

This very application of the present paper is my main motivation and it will hopefully appear in forthcoming works. However, this seems to me independent of the 12-term localization sequence. In other words, first, there is a localization sequence which is easy to formulate in the triangulated framework and second, we shall try to interpret J in terms of Y. This enlights another point, namely the difficulty usually encountered to define a good residue of a form on U with values in some Witt groups of Y. When existing, this residue will actually be decomposable as the general residue presented hereafter followed by some isomorphism between the Witt groups of J and those of Y.

4.3. Remark. If we have an exact sequence (LOC), functoriality of Witt groups naturally induces short complexes:

$$\mathrm{W}(J) \xrightarrow{\mathrm{W}(j)} \mathrm{W}(K) \xrightarrow{\mathrm{W}(q)} \mathrm{W}(S^{-1}K)$$

$$\mathrm{W}_1^-(J) \xrightarrow{\mathrm{W}_1^-(j)} \mathrm{W}_1^-(K) \xrightarrow{\mathrm{W}_1^-(q)} \mathrm{W}_1^-(S^{-1}K)$$

$$\mathrm{W}^-(J) \xrightarrow{\mathrm{W}^-(j)} \mathrm{W}^-(K) \xrightarrow{\mathrm{W}^-(q)} \mathrm{W}^-(S^{-1}K)$$

$$\mathrm{W}_1(J) \xrightarrow{\mathrm{W}_1(j)} \mathrm{W}_1(K) \xrightarrow{\mathrm{W}_1(q)} \mathrm{W}_1(S^{-1}K).$$

To get a long (periodic) exact sequence, we are going to construct connecting homomorphisms between Witt groups of $S^{-1}K$ and Witt groups of J, which will make appear the above short complexes in the order in which they are presented. This will use the cone construction of 1.6 - 1.7. First of all we shall recall from [2] the construction of $\mathrm{W}(S^{-1}K)$ using S-spaces.

4.4. Definition. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$). Let S be a system of morphisms in K. A *S-space* is a pair (A, s) where A is an object in K and s is a morphism $s : A \to A^{\#}$ such that:

(1) $s = s^{\#}\,\varpi_A$
(2) $s \in S$.

We say that two S-spaces (A, s) and (B, t) are *S-isometric* if there exists an object C and morphisms u, v in S such that $u : C \to A$, $v : C \to B$ and

$$u^{\#} s\, u = v^{\#} t\, v.$$

4.5. Proposition. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with δ-duality in the sense of 4.1. The application $(A, s) \mapsto \big(q(A), q(s)\big)$ induces a well defined monoid homomorphism between the set of S-isometry classes of S-spaces and the Witt monoid* $\mathrm{MW}(S^{-1}K)$ *of* $S^{-1}K$ *(see definition 1.10). This homomorphism is an isomorphism.*

Through this application $\mathrm{NW}(K)$ *maps surjectively onto* $\mathrm{NW}(S^{-1}K)$.

4.6. Proof. See [2, *lemme* 2.11, p. 98]. ♯

4.7. Remark. Of course, a symmetric space is an S-space. The above proposition describes the Witt monoid of $S^{-1}K$ in terms of S-spaces in K. Neutral spaces in $S^{-1}K$ are obtained by the neutral spaces of K. In other words, the Witt group of $S^{-1}K$ can be computed as follows. Take S-isometry classes of S-spaces and divide out by neutral spaces of K. As a consequence of that, to define an group homomorphism

$$W(S^{-1}K) \longrightarrow G$$

to a group G, it suffices to define it additively on S-spaces, to check that it is invariant under S-isometries and that it sends neutral spaces of K to zero in G.

We are going to check that such an homomorphism is induced by the cone construction of 1.6 - 1.7.

4.8. Theorem. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with δ-duality in the sense of 4.1. Let (A,s) and (B,t) be S-isometric S-spaces in K. Then*

$$[\mathrm{Cone}(A,s)] = [\mathrm{Cone}(B,t)] \in \mathrm{W}_1^{-\delta}(J).$$

4.9. Proof. The proof goes in several steps. First of all, observe that the cone construction is additive:

$$\mathrm{Cone}\big((A,s)\perp(B,t)\big) \simeq \mathrm{Cone}(A,s)\perp\mathrm{Cone}(B,t)$$

and is well defined up to strong isometry:

$$\mathrm{Cone}(B,h^{\#}\,s\,h) \simeq \mathrm{Cone}(A,s)$$

if $h : B \overset{\sim}{\to} A$ is an isomorphism in K.

We shall only prove the case where $\#$ is a duality, that is $\delta = +1$. The skew-duality goes as well. It is convenient to abreviate $* = T \circ \#$ the shifted skew-duality. If (A,s) is an S-space for $\#$, then $\mathrm{Cone}(A,s)$ is a skew-symmetric space for $*$ (see if necessary Γ_1 in construction 1.6 and definition 1.7).

4.10. Lemma. *Let (P,φ) be a symmetric space in K and let (L,ν_1) be a sublagrangian of this space such that $L \in J$ the kernel category. Choose an exact triangle*

$$T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}.$$

Then $\nu_2 \in S$ and

$$\mathrm{Cone}(M,\nu_2\,\varphi^{-1}\nu_2^{\#}) = \left(T(L)\oplus L^{\#}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

In particular, $[\mathrm{Cone}(M,\nu_2^{\#}\varphi^{-1}\nu_2)] = 0$ *in* $\mathrm{W}_1^{-}(J)$.

4.11. Proof. Define $s := \nu_2\,\varphi^{-1}\nu_2^{\#} : M \to M^{\#}$. By discussion 3.1, there exists a morphism $\eta_0 : L \to M$ such that diagram Ω commutes (*confer* 3.1). By lemma 3.19, there exists an exact triangle

$$M \xrightarrow{s} M^{\#} \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}} T(L)\oplus L^{\#} \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}} T(M)$$

where $\mu_0 : L \to M$ is obtained by some modification of η_0 of no importance here. Since one can use any exact triangle over s to construct $\mathrm{Cone}(M,s)$, let us use this one.

By definition, $\mathrm{Cone}(M,s) = \big(T(L)\oplus L^{\#},\psi\big)$ for some form $\psi : T(L)\oplus L^{\#} \longrightarrow \big(T(L)\oplus L^{\#}\big)^{*} = L^{\#}\oplus T(L)$ such that $\psi^{*} = -\psi$ and such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, equal] & M^{\#} \arrow[r, "s_1 := \begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}"] \arrow[d, equal] & T(L)\oplus L^{\#} \arrow[r, "s_2 := \begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\psi"] & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "-s_2^{*} = \begin{pmatrix} -\mu_0^{\#} \\ -T(\nu_0) \end{pmatrix}"'] & L^{\#}\oplus T(L) \arrow[r, "s_1^{*} = \begin{pmatrix} \nu_0^{\#} & -T(\mu_0) \end{pmatrix}"'] & T(M).
\end{tikzcd}$$

Clearly, it suffices to take $\psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. ♯

4.12. Lemma. *Let (P,φ) be a symmetric space in K and let $s : C \to P$ be a morphism in S. Then*

$$[\mathrm{Cone}(C, s^{\#}\varphi\, s)] = 0 \quad \textit{in} \quad \mathrm{W}_1^-(J).$$

4.13. Proof. Choose an exact triangle over s :

$$(1) \qquad C \xrightarrow{s} P \xrightarrow{s_1} D \xrightarrow{s_2} T(C).$$

Observe that $s \in S$ forces

$$(2) \qquad D \in J.$$

Define $t := s^{\#}\varphi\, s : C \to C^{\#}$. We want to prove that $\mathrm{Cone}(C,t)$ is stably neutral in $(J,*)$. That is, we have to find some object N in J and a morphism $w : N \to N^{\#}$ (observe that $T^{-1} \circ * = \#$) such that $w = w^{\#}$ and $\mathrm{Cone}(C,t)\perp\mathrm{Cone}(N,w)$ is neutral in J. Choose $N := D^{\#}$ and $w := -s_1\varphi^{-1}s : D^{\#} \to D$. Observe immediately that $w^{\#} = w$.

Choose an exact triangle over t of the form :

$$(3) \qquad C \xrightarrow{t} C^{\#} \xrightarrow{t_1} T(L) \xrightarrow{t_2} T(C)$$

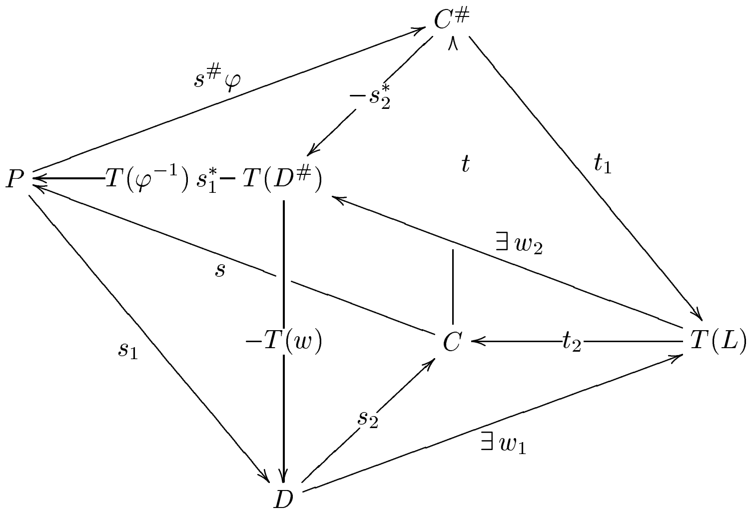
and observe that $t \in S$ implies

$$(4) \qquad L \in J.$$

Write down the composition axiom for the relation

$$t = (s^{\#}\varphi)\, s$$

where the exact triangle over $s^{\#}$ (and then over $s^{\#}\varphi$) is constructed from exact triangle (1). This gives the following octahedron :



That is, we have an exact triangle

$$D^{\#} \xrightarrow{w} D \xrightarrow{w_1} T(L) \xrightarrow{w_2} T(D^{\#})$$

and the relations :

$$(5) \qquad t_2\, w_1 = s_2$$

$$(6) \qquad w_1\, s_1 = t_1\, s^{\#}\varphi$$

$$(7) \qquad w_2\, t_1 = -s_2^*$$

$$(8) \qquad T(\varphi^{-1})\, s_1^* w_2 = T(s)\, t_2.$$

Let us baptize $\nu_1 : L \to P$ to be

$$\nu_1 := s\,T^{-1}(t_2) \overset{(8)}{=} \varphi^{-1} s_1^{\#} T^{-1}(w_2). \tag{9}$$

We are going to use the previous lemma. To do that, observe that (L, ν_1) is a sub-lagrangian of (P, φ) :

$$\nu_1^{\#} \varphi\, \nu_1 \overset{(9)}{=} \left(T(w_2^{\#})\, s_1\, \varphi^{-1}\right) \varphi \left(s\,T^{-1}(t_2)\right) = 0.$$

To apply the previous lemma, we have to construct an exact triangle over ν_1. To do that, observe that

$$t \circ (-T^{-1} s_2) = -s^{\#} \varphi\, s\, T^{-1}(s_2) = 0.$$

The octahedron gives :

$$\begin{tikzcd}
 & & & C^{\#} \arrow[dll, "t_1"'] \arrow[ddr, "\begin{pmatrix}1\\0\end{pmatrix}"] & \\
C \arrow[urrr, "t"] \arrow[ddr, "s"'] & T(L) \arrow[l, "t_2"'] \arrow[dd, "T(\nu_1)"] & & & \\
 & & T^{-1}(D) \arrow[ull, "-T^{-1}(s_2)"'] & & C^{\#} \oplus D \arrow[ll, "(0\ \ 1)"'] \arrow[ulll, "\exists\, T(\nu_0)"'] \\
 & P \arrow[ur, "s_1"] \arrow[urrr, "\exists\, \nu_2"'] & & &
\end{tikzcd}$$

In particular, we have an exact triangle

$$T^{-1}(C^{\#}) \oplus T^{-1}(D) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} C^{\#} \oplus D$$

where $\nu_2 = \begin{pmatrix} x \\ s_1 \end{pmatrix}$ for some morphism $x : P \to C^{\#}$ such that $x\,s = t$. But $t = s^{\#} \varphi\, s$. This means that

$$(s^{\#}\varphi - x)\, s = 0.$$

Using exact triangle (1), we have the existence of some morphism $a : D \to C^{\#}$ such that $s^{\#}\varphi - x = a\, s_1$, which, in turn, can be seen as

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ s_1 \end{pmatrix} = \begin{pmatrix} s^{\#}\varphi \\ s_1 \end{pmatrix}.$$

Using this endomorphism of $C^{\#} \oplus D$, one can then suppose that

$$\nu_2 = \begin{pmatrix} s^{\#}\varphi \\ s_1 \end{pmatrix},$$

changing ν_0 as well, but keeping the ν_1 we liked. Now, having lemma 4.10 in mind, compute

$$\nu_2\, \varphi^{-1} \nu_2^{\#} = \begin{pmatrix} s^{\#}\varphi \\ s_1 \end{pmatrix} \varphi^{-1} \begin{pmatrix} \varphi\, s & s_1^{\#} \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & -w \end{pmatrix}.$$

Lemma 4.10 insures us that the cone of this form is neutral is J. But it is obviously the orthogonal sum of the cone of t and of the cone of $-w$, which is neutral in J because $D^{\#} \in J$ by (2). $\sharp$

4.14. Proof of theorem 4.8. Let (A,s) and (B,t) be S-isometric S-spaces. It suffices to do the case when there exists a morphism $u : B \to A$, $u \in S$, such that

$$t = u^{\#} s\, u. \qquad (1)$$

Recall that the cone construction is additive and invariant under strong isometries in K (see 4.9). Now consider

$$\begin{pmatrix} 1 & 0 \\ u^{\#} & 1 \end{pmatrix} \cdot \begin{pmatrix} s & 0 \\ 0 & -t \end{pmatrix} \cdot \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} s & s\,u \\ u^{\#}\, s & 0 \end{pmatrix}.$$

It suffices to prove that $[\mathrm{Cone}\big(A \oplus B, \begin{pmatrix} s & s\,u \\ u^{\#}\, s & 0 \end{pmatrix}\big)] = 0$ in $\mathrm{W}_1^-(J)$. Now, consider the morphism

$$\begin{pmatrix} 1 & 0 \\ 0 & s\,u \end{pmatrix} : A \oplus B \longrightarrow A \oplus A^{\#}$$

which obviously belongs to S and consider the symmetric form

$$\varphi = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}$$

over $A \oplus A^{\#}$. The result is now a direct consequence of lemma 4.18 since

$$\begin{pmatrix} 1 & 0 \\ 0 & s\,u \end{pmatrix}^{\#} \cdot \varphi \cdot \begin{pmatrix} 1 & 0 \\ 0 & s\,u \end{pmatrix} = \begin{pmatrix} s & s\,u \\ u^{\#}\, s & 0 \end{pmatrix}.$$

$\sharp$

4.15. Mea culpa. The carefull reader of [1] will object that theorem 5.15 there already states that this connecting homomorphism is well defined. The proof of lemma 5.11 *loc. cit.* on which this theorem relays, is incomplete and contains several typographic mistakes, as I noticed too late. The present version should fill this gap.

4.16. Corollary and definition. Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with duality in the sense of 4.1. The cone construction of 1.6 and 1.7 induce well defined homomorphisms

$$\begin{aligned} \gamma_n^{\epsilon} : \mathrm{W}_n^{\epsilon}(S^{-1}K) &\longrightarrow \mathrm{W}_{n+1}^{\epsilon'}(J) \\ x &\mapsto [\mathrm{Cone}(A,s)] \\ &\qquad \text{where } x = [q(A), q(s)] \end{aligned}$$

for all $n \in \mathbb{Z}$ and $\epsilon = \pm 1$, where $\epsilon' = (-1)^{n+1} \cdot \epsilon$.

4.17. Proof. This is immediate from theorem 4.8 and remark 4.7. In fact, the cone of any (neutral) form in K is of course trivial since the cone of an isomorphism is zero.

This corollary is stated for a duality ($\delta = +1$) but involves $T^n \circ \#$ for all $n \in \mathbb{Z}$. If $\#$ was a skew-duality, it would fit in this result as the translated skew-duality of $T^{-1}\#$, which is a duality. $\sharp$

4.18. Definition. Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with duality in the sense of 4.1. Since there are only four Witt groups we want to focus on, we correct a little bit the connecting homomorphism γ in order to let them stay in the set W, W_1^-, W^-, W_1. In other words, since, for instance $\gamma_1 : \mathrm{W}_1(S^{-1}K) \to \mathrm{W}_2(J)$, we compose it with the identification $W_2 \cong \mathrm{W}_0$ to get

$$\begin{aligned} \partial : \mathrm{W}(S^{-1}K) &\xrightarrow{\gamma_0^+} \mathrm{W}_1^-(J) \\ \partial_1^- : \mathrm{W}_1^-(S^{-1}K) &\xrightarrow{\gamma_1^-} \mathrm{W}_2^-(J) \xrightarrow{\sim} \mathrm{W}^-(J) \\ \partial^- : \mathrm{W}^-(S^{-1}K) &\xrightarrow{\gamma_0^-} \mathrm{W}_1(J) \\ \partial_1 : \mathrm{W}_1(S^{-1}K) &\xrightarrow{\gamma_1^+} \mathrm{W}_2(J) \xrightarrow{\sim} \mathrm{W}(J). \end{aligned}$$

They will be called the *residue homomorphisms*.

4.19. Exercise. Check that ∂_1 can also be computed as the composition:

$$\mathrm{W}_1(S^{-1}K) \xrightarrow{\sim} \mathrm{W}_{-1}(S^{-1}K) \xrightarrow{\gamma^+_{-1}} \mathrm{W}(J).$$

More generally, prove that the $\gamma^\pm_*$ commute to the isomorphisms $\mathrm{W}^\pm_* \cong \mathrm{W}^\pm_{*+2}$.

4.20. Exercise. Check that the sequence obtained by using the natural homomorphisms of 4.3 and the residue homomorphisms of 4.18 is a complex (very easy).

4.21. Exercise. Understand the analogy between the residue homomorphisms presented here and the classical (second) residue in the case of a valuation ring, for instance.

5. Main result.

5.1. Theorem. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with duality in the sense of 4.1. Suppose that K and $S^{-1}K$ are noetherian and that $\frac{1}{2} \in K$. Then, using the natural homomorphisms $\mathrm{W}^\pm_*(j)$ and $\mathrm{W}^\pm_*(q)$ and the residue homomorphisms $\partial^\pm_*$ of definition 4.18, we obtain the following:*

$$\begin{tikzcd}
 & \mathrm{W}(J) \arrow[r, "\mathrm{W}(j)"] & \mathrm{W}(K) \arrow[r, "\mathrm{W}(q)"] & \mathrm{W}(S^{-1}K) \arrow[dr, bend left, "\partial"] & \\
\mathrm{W}_1(S^{-1}K) \arrow[ur, bend left, "\partial_1"] & & & & \mathrm{W}_1^-(J) \arrow[d, "\mathrm{W}_1^-(j)"] \\
\mathrm{W}_1(K) \arrow[u, "\mathrm{W}_1(q)"] & & \textit{12-term exact sequence.} & & \mathrm{W}_1^-(K) \arrow[d, "\mathrm{W}_1^-(q)"] \\
\mathrm{W}_1(J) \arrow[u, "\mathrm{W}_1(j)"] & & & & \mathrm{W}_1^-(S^{-1}K) \arrow[dl, bend left, "\partial_1^-"] \\
 & \mathrm{W}^-(S^{-1}K) \arrow[ul, bend left, "\partial^-"] & \mathrm{W}^-(K) \arrow[l, "\mathrm{W}^-(q)"] & \mathrm{W}^-(J) \arrow[l, "\mathrm{W}^-(j)"] &
\end{tikzcd}$$

5.2. Proof. We are now well prepared to prove this theorem. We shall only establish exactness of the sequence at $\mathrm{W}(J)$, $\mathrm{W}(K)$ and $\mathrm{W}(S^{-1}K)$ (the last is already done in [1] and [2]). The same results for the other groups, using the translated duality $T \circ \#$ and/or the opposite identification $-\varpi : \mathrm{Id} \xrightarrow{\sim} \#^2$, go as well. The reader can establish the analogue of the results of paragraphs 2 to 4 for the associated skew-duality. Details are left as an exercise.

Exactness at $\mathrm{W}(J)$.

Let $x \in \mathrm{Ker}\big(\mathrm{W}(J) \to \mathrm{W}(K)\big)$. Choose a symmetric space such that $x = [P, \varphi]$. Then (P, φ) is stably neutral in K. By theorem 2.5, (P, φ) is neutral in K. This means by definition 1.9 that

$$(P, \varphi) = \mathrm{Cone}(A, u)$$

for some object A in K and some morphism $u : A \to T^{-1}(A^\#)$ such that

$$T^{-1}(u^\#) = u.$$

But then $\mathrm{Cone}(u) = P \in J$ and therefore u becomes an isomorphism in $S^{-1}K$ (where P would be zero). Then, since S is saturated, $u \in S$. In other words, (A, u) is an S-space for the skew-duality $T^{-1} \circ \#$ and

$$(P, \varphi) = \mathrm{Cone}(A, u).$$

By definition 4.16,

$$(P, \varphi) = \gamma_1^-\big(q(A), q(u)\big).$$

Since $\mathrm{W}_{-1} \simeq \mathrm{W}_1$, using definition 4.18, this is the announced result (*confer* exercise 4.19 if any confusion occurs). Observe that we did not use that K is noetherian.

Exactness at W(K).

The first step is to give a criterion for symmetric spaces over K to become neutral in $S^{-1}K$. This should be compared with lemma 2.1 condition (3). In $S^{-1}K$ the new isomorphisms are essentially the morphisms of S and therefore we obtain new neutral forms using the criterion of the lemma. We prove hereafter that they are the only one.

5.3. Lemma. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with duality in the sense of 4.1. Suppose that $\frac{1}{2} \in K$. Then a symmetric space (P, φ) in K is neutral in $S^{-1}K$ if and only if there exists a morphism of exact triangles:*

$$\begin{tikzcd}
T^{-1}(M^\#) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(h^\#)"'] & L \arrow[r, "\nu_1"] \arrow[d, "h"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^\# \arrow[d, "h^\#"] \\
T^{-1}(L^\#) \arrow[r, "T^{-1}(\nu_0^\#)"'] & M \arrow[r, "\nu_2^\#"'] & P^\# \arrow[r, "\nu_1^\#"'] & L^\#
\end{tikzcd}$$

with $h \in S$.

5.4. Proof. Suppose that there exists such a diagram. Then localize it (*i.e.* see it in $S^{-1}K$) and apply lemma 2.1 in $S^{-1}K$. You will obtain neutrality of $\big(q(P), q(\varphi)\big)$.

The converse is the interesting part. The reader should pay attention to distinguish diagrams in $S^{-1}K$ from those in K. Hereafter, $*$ shall denote $T^{-1} \circ \#$ from time to time.

Let (P, φ) be a symmetric space in K neutral in $S^{-1}K$. By definition, there exists an exact triangle in $S^{-1}K$:

$$(1) \qquad T^{-1}(A^\#) \xrightarrow{a_0} A \xrightarrow{a_1} P \xrightarrow{a_2} A^\#$$

such that

(2) $a_2 = a_1^\# \varphi$ in $S^{-1}K$;
(3) $a_0 = a_0^*$ in $S^{-1}K$.

Trivial Remark. Consider an exact triangle in $S^{-1}K$ like in (1) satisfying (2) and (3). Choose any morphism $s : B \to A$ in K such that $s \in S$. Set $b_1 = a_1 \circ s$ and $w = s^{-1} a_0 \, (s^{-1})^*$. Then we have an exact triangle in $S^{-1}K$:

$$T^{-1}(B^\#) \xrightarrow{w} B \xrightarrow{b_1} P \xrightarrow{b_1^\# \varphi} B^\#$$

with $w = w^*$ in $S^{-1}K$.

To prove this, it suffices to observe the commutative diagram in $S^{-1}K$:

$$\begin{tikzcd}
T^{-1}(A^\#) \arrow[r, "a_0"] \arrow[d, "s^*"', "\simeq"] & A \arrow[r, "a_1"] & P \arrow[r, "a_2"] \arrow[d, equal] & A^\# \arrow[d, "s^\#"', "\simeq"] \\
T^{-1}(B^\#) \arrow[r, "w"'] & B \arrow[r, "b_1"'] \arrow[u, "s", "\simeq"'] & P \arrow[r, "b_1^\# \varphi"'] & B^\#
\end{tikzcd}$$

which commutes because, in $S^{-1}K$, we have:

$$b_1^{\#}\varphi = (a_1\, s)^{\#}\varphi = s^{\#} a_1^{\#}\varphi \overset{(2)}{=} s^{\#} a_2.$$

Of course (3) implies that $w^* = w$. This finishes the proof of the trivial remark.

Consider the exact triangle (1). Write a_1 as a fraction:

$$\text{(2)} \qquad a_1 = b_1\, s^{-1} \ : \ A \xleftarrow{s\in S} B \xrightarrow{b_1} P$$

for some object B and some morphism $s : B \to A$ in S. We have $b_1 = a_1\, s$. Then the remark tells us that we can replace (1) by an exact triangle in $S^{-1}K$:

$$T^{-1}(B^{\#}) \xrightarrow{w} B \xrightarrow{b_1} P \xrightarrow{b_1^{\#}\varphi} B^{\#}$$

with $w = w^*$ in $S^{-1}K$. Then it is immediate that

$$b_1^{\#}\varphi\, b_1 = 0 \qquad \text{in } S^{-1}K.$$

This implies that there exists a morphism $s' : B' \to B$ such that

$$b_1^{\#}\varphi\, b_1 \circ s' = 0 \qquad \text{in } K.$$

Use again the trivial remark to replace b_1 by $b_1\, s'$. Summarizing up, we have the following situation:

(4) we have an exact triangle in $S^{-1}K$:

$$T^{-1}(B^{\#}) \xrightarrow{w} B \xrightarrow{b_1} P \xrightarrow{b_1^{\#}\varphi} B^{\#},$$

(5) $w = w^*$ in $S^{-1}K$,
(6) b_1 is a morphism in K $\big($meaning $= q(b_1)\big)$
(7) $b_1^{\#}\varphi\, b_1 = 0$ <u>in K</u>.

Since we have (6), we can choose an exact triangle in K containing b_1:

$$\text{(8)} \qquad T^{-1}(C) \xrightarrow{b_0} B \xrightarrow{b_1} P \xrightarrow{b_2} C.$$

In $S^{-1}K$, we can compare (4) and (8), or if you prefer, (4) and the localization of (8):

$$\text{(9)} \qquad \begin{tikzcd} T^{-1}(B^{\#}) \arrow[r, "w"] \arrow[dd, dashed, "\exists\,\simeq"'] & B \arrow[r, "b_1"] \arrow[dd, equal] & P \arrow[r, "b_1^{\#}\varphi"] \arrow[dd, equal] & B^{\#} \arrow[d, dashed, "\exists\, t^{\#}"] \\ & & & D^{\#} \\ T^{-1}(C) \arrow[r, "b_0"'] & B \arrow[r, "b_1"'] & P \arrow[r, "b_2"'] & C \arrow[u, dashed, "\exists\, u"'] \end{tikzcd}$$

There exists an isomorphism $B^{\#} \xrightarrow{\sim} C$, that can be expressed as a fraction like displayed above. Moreover, the relation $t^{\#}b_1^{\#}\varphi = u\, b_2$ is valid in $S^{-1}K$ and therefore in K up to some composition, say on the left, by a morphism in S. Up to a modification of D, t and u, one can suppose that

$$t^{\#}b_1^{\#}\varphi = u\, b_2 \qquad \underline{\text{in } K}.$$

Since S is saturated, we have $t, u \in S$.

Define

$$c_1 := b_1\, t : D \to P$$

so that the above relation becomes:

$$(10) \qquad c_1^{\#}\varphi = u\, b_2 \qquad \underline{\text{in } K}.$$

Choose an exact triangle over c_1 in K and use the definition of c_1, to construct the following morphism of exact triangles $\underline{\text{in } K}$:

$$(11) \qquad \begin{tikzcd} T^{-1}(E) \arrow[r, "c_0"] \arrow[d, dashed, "T^{-1}(v)"'] & D \arrow[r, "c_1"] \arrow[d, "t"] & P \arrow[r, "c_2"] \arrow[d, equal] & E \arrow[d, dashed, "\exists\, v \in S"] \\ T^{-1}(C) \arrow[r, "b_0"'] & B \arrow[r, "b_1"'] & P \arrow[r, "b_2"'] & C. \end{tikzcd}$$

There exists a morphism $v : E \to C$ making (11) commute and it is necessarily in S, which is saturated. Now define

$$l := u\, v : E \to D^{\#}.$$

Note that $l \in S$. Compose on the left the third square of (11) with u to obtain $l\, c_2 = u\, b_2$ and therefore, because of (10), we have:

$$(12) \qquad l\, c_2 = c_1^{\#}\varphi \qquad \underline{\text{in } K}.$$

Define also in $S^{-1}K$:

$$z := t^{-1} \circ w \circ (t^{-1})^* : T^{-1}(D^{\#}) \to D.$$

Relation (5) forces $z^* = z$ in $S^{-1}K$. Now compute:

$$\begin{aligned} z\, T^{-1}(l) = z\, T^{-1}(u\, v) &= t^{-1}\, \underbrace{w\, (t^{-1})^*\, T^{-1}(u)}_{\overset{(9)}{=}\, b_0}\, T^{-1}(v) \\ &= t^{-1}\, b_0\, T^{-1}(v) \overset{(11)}{=} c_0 \qquad\qquad \text{in } S^{-1}K. \end{aligned}$$

Regrouping this last relation with (12), we have the following commutative diagram in $S^{-1}K$:

$$(13) \qquad \begin{tikzcd} T^{-1}(E) \arrow[r, "c_0"] \arrow[d, "T^{-1}(l)"'] & D \arrow[r, "c_1"] \arrow[d, equal] & P \arrow[r, "c_2"] \arrow[d, equal] & E \arrow[d, "l"] \\ T^{-1}(D^{\#}) \arrow[r, "z"'] & D \arrow[r, "c_1"'] & P \arrow[r, "c_1^{\#}\varphi"'] & D^{\#}. \end{tikzcd}$$

But relation (12) was true in K and therefore can be expressed as the commutativity of the following right hand square $\underline{\text{in } K}$:

$$\begin{tikzcd} T^{-1}(E) \arrow[r, "c_0"] \arrow[d, "T^{-1}(l)"'] & D \arrow[r, "c_1"] & P \arrow[r, "c_2"] \arrow[d, "\varphi"] & E \arrow[d, "l"] \\ T^{-1}(D^{\#}) \arrow[r, "c_0^*"'] & E^{\#} \arrow[r, "c_2^{\#}"'] & P^{\#} \arrow[r, "c_1^{\#}"'] & D^{\#} \end{tikzcd}$$

where the first line is the exact triangle over c_1 we chose in (11) and where the second line is simply the dual of the first. Now, choose in K a morphism $m : D \to E^{\#}$ such that the following diagram commutes:

$$(14) \qquad \begin{tikzcd} T^{-1}(E) \arrow[r, "c_0"] \arrow[d, "T^{-1}(l)"'] & D \arrow[r, "c_1"] \arrow[d, dashed, "\exists\, m"] & P \arrow[r, "c_2"] \arrow[d, "\varphi"] & E \arrow[d, "l"] \\ T^{-1}(D^{\#}) \arrow[r, "c_0^*"'] & E^{\#} \arrow[r, "c_2^{\#}"'] & P^{\#} \arrow[r, "c_1^{\#}"'] & D^{\#}. \end{tikzcd}$$

Observe that $m \in S$. Define

$$h := \frac{1}{2}(m + l^{\#}) : D \to E^{\#}.$$

It is obvious from (14) that the following diagram commutes in K (easy exercise):

$$(15) \qquad \begin{array}{ccccccc} T^{-1}(E) & \xrightarrow{c_0} & D & \xrightarrow{c_1} & P & \xrightarrow{c_2} & E \\ \downarrow{\scriptstyle T^{-1}(h)} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle h} \\ T^{-1}(D^{\#}) & \xrightarrow[c_0^*]{} & E^{\#} & \xrightarrow[c_2^{\#}]{} & P^{\#} & \xrightarrow[c_1^{\#}]{} & D^{\#}. \end{array}$$

The only non-trivial point is to check that $h \in S$, which is of course not true simply because $l, m \in S$. To prove this last point, we are going to establish that $h^{\#}$ also makes diagram (13) commute (in place of l). This is enough because S is saturated. In view of the definition of h, it is of course sufficient to establish that $m^{\#}$ makes diagram (13) commute.

In other words, we have to establish:

(16) $z\,T^{-1}(m^{\#}) = c_0$ in $S^{-1}K$
(17) $m^{\#}\,c_2 = c_1^{\#}\varphi$ in $S^{-1}K$.

But (17) is already true in view of the middle square of (14), by applying #. Now simply compute in $S^{-1}K$:

$$z\,m^* = z^*\,m^* = (m \underbrace{z}_{\overset{(13)}{=} c_0\,T^{-1}(l^{-1})})^* = \big(m\,c_0\,T^{-1}(l^{-1})\big)^* \overset{(14)}{=} (c_0^*)^* = c_0.$$

Putting everything together, we have the commutative diagram (15) with exact lines in K, for a morphism $h : D \to E^{\#}$ such that $h \in S$. This is exactly the claim of the lemma. The proof of this lemma is octahedron-free and it is a pity! ♯

It is now easy to establish exactness of the sequence at $\mathrm{W}(K)$. It suffices to prove that a form satisfying the condition of the lemma is Witt-equivalent in K to a form (R, ψ) with $R \in J$. Theorem 3.17 applied to $\eta_0 = h$, precisely asserts that there exists a form on some $R = \mathrm{Cone}(\mu_0)$, Witt-equivalent to (P, φ) and such that

$$P \oplus R \simeq P \oplus \mathrm{Cone}(h).$$

Localize the above isomorphism in $S^{-1}K$:

$$q(P) \oplus q(R) \simeq q(P) \oplus \underbrace{q\big(\mathrm{Cone}(h)\big)}_{=0} \simeq q(P)$$

because $h \in S$. Therefore, since $S^{-1}K$ is noetherian(and we only use condition (1) of definition 3.3),

$$q(R) \simeq 0 \qquad \text{in } S^{-1}K.$$

This means by definition that $R \in J$. ♯

Exactness at $\mathrm{W}(S^{-1}K)$.

This was established in [1, theorem 5.17] and in [2, *théorème* 2.21, p. 103]. ♯

Bibliography

[1] Paul BALMER, *Derived Witt groups of a scheme*, Journal of pure and applied algebra (to appear).

[2] Paul BALMER, *Groupes de Witt dérivés d'un schéma*, Thèse de doctorat, Université de Lausanne (1998).

[3] Max KAROUBI, *Localisation de formes quadratiques I*, Annales scientifiques de l'École Normale Supérieure de Paris, 4^e série, 7, pp. 359-404, (1974).

[4] Max KAROUBI, *Localisation de formes quadratiques II*, Annales scientifiques de l'École Normale Supérieure de Paris, 4^e série, 8, pp. 99-155, (1975).

[5] Manfred KNEBUSCH, *Symmetric bilinear forms over algebraic varieties*, In: Conference on Quadratic Forms, Kingston 1976, Queen's Papers on Pure and Applied Mathematics 46, pp. 103-283, (1977).

[6] William PARDON, *The exact sequence of a localization for Witt groups*, In: Algebraic K-theory, Evanston 1976, Springer Lecture Notes in Mathematics 551, pp. 336-379, (1977).

[7] Andrew RANICKI, *Algebraic L-theory I. Foundations*, Proceedings of the London MAthematical Society 27, pp. 101-125, (1973).

[8] Jean-Louis VERDIER, *Catégories dérivées, État 0*, SGA $4\frac{1}{2}$, Springer Lecture Notes in Mathematics 569, pp. 262-311, (1977).

[9] Boris YOUSSIN, *Witt groups of derived categories*, Journal of K-theory, volume 11, No 4, pp. 373-395, (1997).

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