Margulis Superrigidity I & II

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These notes follow the proof of Margulis Superrigidity given in Zimmer's book "Ergodic theory and semisimple groups". In this sketch we omit some technical details, for which the reader is referred to Zimmer's book.

Important concepts

In these talks, most groups are algebraic groups with the Zariski topology. If G is such a group and G is an $\mathbb{R}$-group (i.e. its defining polynomials have coefficients in $\mathbb{R}$), then $G_{\mathbb{R}}$ is the group of $\mathbb{R}$-points $G \cap GL_n(\mathbb{R})$, a Lie group with the Hausdorff topology. Given a subgroup $H \leq G$ we let $H \backslash G$ denote the set (or space) of right cosets of H.

A variety is locally a Zariski-closed subset of some finite-dimensional complex vector space. A map between varieties is called a *morphism* if it is given locally by polynomials in the coordinates (also called a 'regular map' in Zimmer). Similarly, a map is *rational* if it is locally given by rational functions. A rational function is defined only on some Zariski-open subset, where the denominators are all non-zero. A morphism or rational map is called $\mathbb{R}$-defined (or an $\mathbb{R}$-morphism, resp. $\mathbb{R}$-rational map) if these polynomials (resp. rational functions) can be taken to have real coefficients.

An algebraic group is *simple* if it has no (Zariski-)closed, connected normal subgroups. Equivalently, its only normal subgroups are finite and central (thus $\mathrm{SL}_n(\mathbb{C})$ is simple). An algebraic group is *semisimple* if it is connected and has no unipotent normal subgroups; equivalently, it is an almost-direct product of simple algebraic groups. If G is semisimple and $\mathbb{R}$-defined, the Lie group $G_{\mathbb{R}}$ need not be connected in the Hausdorff topology; the notation $G_{\mathbb{R}}^{\circ}$ always denotes the connected component in the Hausdorff topology.

If G is an $\mathbb{R}$-defined algebraic group, then the $\mathbb{R}$-rank of G is the maximal dimension of an $\mathbb{R}$-split torus of G, i.e. the maximal dimension of a connected subgroup of G which can be diagonalised over $\mathbb{R}$.

Finally, if V and W are varieties such that V is defined over $\mathbb{R}$, if $A \subseteq V_{\mathbb{R}}$ is a subset of positive Lebesgue measure, and if f is a measurable function $A \to W$, then f is called *essentially rational* if there is a rational map $V \to W$ which coincides with f almost everywhere on A. If W is defined over $\mathbb{R}$, then we define an *essentially* $\mathbb{R}$*-rational map* similarly.

Prerequisite results

The following is Borel's density theorem in the form we need.

Theorem (3.2.5 in Zimmer)**.** If G is a semisimple $\mathbb{R}$-group such that $G^{\circ}_{\mathbb{R}}$ has no compact factors and if $\Gamma \leq G^{\circ}_{\mathbb{R}}$ is a lattice, then Γ is Zariski-dense in G.

Lemma (Facts regarding varieties and algebraic groups)**.**

(a) If G and H are algebraic groups, and $f : G \to H$ is a group homomorphism and a morphism of varieties, then $f(G)$ is a Zariski-closed subgroup of H [Corollary 3.1.2 in Zimmer].

(b) If $f : G \to H$ is a bijective group homomorphism which is a morphism of varieties (defined over $\mathbb{R}$), then f is an ($\mathbb{R}$-)isomorphism of algebraic groups, i.e. f^{-1} is also a morphism of varieties (defined over $\mathbb{R}$). [Corollary 3.1.11 in Zimmer, and preceding discussion]. *Note: This fails for algebraic groups over fields of positive characteristic.*

(c) Suppose that G acts on varieties X and Y, and thus on the space of rational maps from $X \to Y$. Then the stabiliser of a point $\mathrm{Stab}_G(\phi)$ is a Zariski-closed subgroup [sketched in Proposition 3.3.2 in Zimmer].

(d) If $f : V \to W$ is a morphism of varieties and if $f(A) \subseteq W_{\mathbb{R}}$ for some Zariski-dense subset $A \subseteq V_{\mathbb{R}}$, then f is $\mathbb{R}$-defined [3.1.10 in Zimmer].

In the next four lemmas, G is a locally compact second-countable topological group.

Lemma (Moore's Ergodicity Theorem)**.** If $G = \prod G_i$, where each G_i is a non-compact connected Lie group with finite centre, and if H is a non-compact closed subgroup, then each irreducible lattice in G is ergodic on $H \backslash G$.

Lemma ("Ergodic action + measurable G-map $\implies$ essentially constant")**.** If X is an ergodic G-space and $f : X \to Y$ is G-invariant and measurable (where Y is second-countable), then f is constant on a co-null set.

Lemma (Fürstenberg)**.** If G acts amenably on a measure space S, and if X is a compact metric G-space, then there is a measurable, G-equivariant map from a co-null subset of S into the space $M(X)$ of probability measures on X.

Lemma (Lemma 4.3.7 in Zimmer)**.** If B is a closed subgroup of G, and if $\Gamma \leq G$ is a lattice, then the action of Γ on $B \backslash G$ is amenable if and only if B is amenable.

Some reminders:

Lemma (Page 47 of Zimmer)**.** Let G be a split semisimple $\mathbb{R}$-group. The real points of its parabolic subgroups are non-compact and cocompact. Given a cocompact $\mathbb{R}$-subgroup H of G, the group $G_{\mathbb{R}}$ acts smoothly on the space $M(H_{\mathbb{R}}\backslash G_{\mathbb{R}})$ of probability measures on $H_{\mathbb{R}}\backslash G_{\mathbb{R}}$, and the stabiliser of a measure is the set of real points of an algebraic $\mathbb{R}$-subgroup. Such a stabiliser is either compact or is contained in an algebraic $\mathbb{R}$-subgroup of dimension less than that of G.

Lemma (Margulis; Theorem 3.4.4 in Zimmer)**.** If $f : \mathbb{R}^k \times \mathbb{R}^n \to V$ is measurable function such that

1. for almost every $x \in \mathbb{R}^k$, the map $f_x : \mathbb{R}^n \to V$ given by $f_x(y) = f(x, y)$ is essentially rational, and
2. for almost every $y \in \mathbb{R}^n$, the map $f^y : \mathbb{R}^k \to V$ given by $f^y(x) = f(x, y)$ is essentially rational,

then f is essentially rational. If moreover each f_x and f^y is essentially $\mathbb{R}$-defined, then f is essentially $\mathbb{R}$-defined.

1 Lecture I: Outline and first steps in proving Margulis Superrigidity

Our goal for the two lectures is the following result.

Theorem (Margulis)**.** Let G and H be connected algebraic $\mathbb{R}$-groups, such that:

- G is semisimple of $\mathbb{R}$- rank at least 2 and $G^\circ_{\mathbb{R}}$ has no compact factors, and
- H is simple and centre-free, and $H_{\mathbb{R}}$ is not compact.

If $\Gamma \leq G^\circ_{\mathbb{R}}$ is an irreducible lattice, and if π is a homomorphism $\Gamma \to H_{\mathbb{R}}$ with Zariski-dense image, then π extends to an $\mathbb{R}$-rational homomorphism $G \to H$ (hence defines a continuous homomorphism $G_{\mathbb{R}} \to H_{\mathbb{R}}$).

Remark. Analogous results to the above also hold when we replace $H_{\mathbb{R}}$ with the complex Lie group $H_{\mathbb{C}}$, or with H_k where k is a totally disconnected local field of characteristic zero. This more general result is the full statement of Margulis Superrigidity.

Zimmer's statement of this result assumes H to be 'almost simple over $\mathbb{R}$' and '$\mathbb{R}$-simple', although these terms are never defined in his book. Our assumptions that H is simple and centre-free, are equivalent to the fact that H is simple as an abstract group, which is the statement given elsewhere in the literature.

1.1 Outline of proof

The proof proceeds in a series of steps, as follows.

Step A: Γ acts on G by right multiplication, and also on H (via π), hence on the coset varieties $P\backslash G$ and $L\backslash H$ whenever P and L are Zariski-closed subgroups. In this step, we prove that if we have a Γ-equivariant, $\mathbb{R}$-rational map $P\backslash G \to L\backslash H$ for some choice of P and L, then π extends as required. This step makes use of Borel's density theorem.

Step B: By careful choice of P and L, we can construct a *measurable* Γ-equivariant map from $(P_{\mathbb{R}} \cap G^{\circ}_{\mathbb{R}})\backslash G^{\circ}_{\mathbb{R}}$ to $L_{\mathbb{R}}\backslash H_{\mathbb{R}}$, defined almost everywhere. This step makes use of Fürstenberg's Lemma and Moore's Ergodicity Theorem.

Step C: In this step, the hardest, we show that such a measurable map is essentially $\mathbb{R}$-rational. Thus there exists an $\mathbb{R}$-rational function $P\backslash G \to L\backslash H$ which agrees with the measurable map almost everywhere on the real points, and hence is Γ-equivariant. This also uses Moore's Ergodicity Theorem.

1.2 Step A

Lemma A. Let $P \leq G$ and $L \leq H$ be proper, Zariski-closed $\mathbb{R}$-subgroups, and suppose there exists a Γ-equivariant, $\mathbb{R}$-rational map $\phi : P\backslash G \to L\backslash H$. Then π extends to an $\mathbb{R}$-rational homomorphism $G \to H$.

Proof. Consider the graph

$$\Delta \stackrel{\text{def}}{=} \{(\gamma, \pi(\gamma)) \mid \gamma \in \Gamma\} \leq G \times H$$

of the homomorphism π, and let $\overline{\Delta}$ be its Zariski closure in $G \times H$ (so $\overline{\Delta}$ is an algebraic group with the inherited Zariski topology). We claim that $\overline{\Delta}$ is the graph of an $\mathbb{R}$-rational homomorphism $G \to H$, that is, each $g \in G$ occurs in exactly one pair $(g, h) \in \overline{\Delta}$, and the assignment $g \mapsto h$ gives an $\mathbb{R}$-rational homomorphism.

Firstly, note that the image of the projection $\overline{\Delta} \to G$ contains Γ, and is therefore Zariski-dense in G by the Borel density theorem (recall that $G^{\circ}_{\mathbb{R}}$ has no compact factors). But also, the image is Zariski-closed (Fact (a) above), hence the image of $\overline{\Delta}$ is all of G. Thus every $g \in G$ occurs in some pair $(g, h) \in \overline{\Delta}$.

Next, suppose (g, h_1) and (g, h_2) are elements of $\overline{\Delta}$. We wish to show that $h_1 = h_2$. Let R be the space of all rational maps from $P\backslash G$ to $L\backslash H$. Then $G \times H$ acts on R *on the left*[1] by $[(g, h) \cdot \phi](x) \stackrel{\text{def}}{=} \phi(xg)h^{-1}$. By hypothesis, there exists a Γ-equivariant map $\phi \in R$, which means that $\phi(x\gamma) = \phi(x)\pi(\gamma)$ for all $x \in P\backslash G$ and all $\gamma \in \Gamma$, or in other words $[(\gamma, \pi(\gamma)) \cdot \phi](x) = \phi(x)$ for all x and γ, so ϕ is a fixed point of R under Δ.

[1]**Check:** $[(g_1g_2, h_1h_2) \cdot \phi](x) = \phi(xg_1g_2) \cdot h_2^{-1}h_1^{-1} = \phi'(xg_1)h_1^{-1} = (g_1, h_1) \cdot [(g_2, h_2) \cdot \phi](x)$, where $\phi'(x) \stackrel{\text{def}}{=} \phi(xg_2)g_2^{-1}$, as required. This differs from the (incorrect) action given in Zimmer's book.

By Fact (c) above, the stabiliser $\mathrm{Stab}_{G\times H}(\phi)$ is a Zariski-closed subgroup, and it contains Δ, hence also contains $\overline{\Delta}$. Thus if (g,h_1) and $(g,h_2)\in\overline{\Delta}$ as above, we have

$$\phi(xg)h_1^{-1}=\phi(xg)h_2^{-1}\quad\text{for all } x\in P\backslash G,$$

in particular $\phi(xg)h_1^{-1}h_2=\phi(xg)$ for all $x\in P\backslash G$, so $h_1^{-1}h_2$ fixes $\phi(P\backslash G)$ pointwise. However, since $\phi(x\gamma)=\phi(x)\pi(\gamma)$ for all $\gamma\in\Gamma$, $\pi(\Gamma)$ stabilises $\phi(P\backslash G)$, hence it stabilises the Zariski closure $\overline{\phi(P\backslash G)}$. Since $\pi(\Gamma)$ is Zariski-dense in H, H also stabilises $\overline{\phi(P\backslash G)}$, which implies that $\overline{\phi(P\backslash G)}=L\backslash H$ as the latter has no proper, non-empty H-stable subsets. Therefore $h_1^{-1}h_2$ pointwise fixes all of $L\backslash H$, and therefore lies in the intersection $\bigcap_{h\in H}hLh^{-1}$, a normal subgroup of H, which is proper as L is proper, hence trivial as H is abstractly simple. Thus $h_1^{-1}h_2=e$, as required.

Thus $\overline{\Delta}$ is the graph of a map $G\to H$. Now, the projection $\overline{\Delta}\to G$ is bijective, and is the restriction of the projection homomorphism $G\times H\to G$, hence is a morphism of varieties. By Fact (b) above, this is an isomorphism of algebraic groups, so its inverse is a morphism $G\to\overline{\Delta}$. Composing this with the projection map onto H, we get a morphism $G\to H$, which extends π.

Finally, since π sends the Zariski-dense real set $\Gamma\leq G_{\mathbb{R}}$ into the real set $H_{\mathbb{R}}$, by Fact (d) above we deduce that π is defined over $\mathbb{R}$. $\square$

2 Lecture II: Finding a measuarable map and proving it is essentially $\mathbb{R}$-rational

Since we are now concerned with finding a *rational* map, we shall often replace the given domain by some Zariski-dense subset without further comments.

2.1 Step B

Lemma B. Let $P\stackrel{\text{def}}{=}B\leq G$ be a Borel subgroup and write $B_\circ=G_{\mathbb{R}}^\circ\cap B$. There exist a proper $\mathbb{R}$-subgroup $L\leq H$ and a measurable Γ-equivariant map $\phi: B_\circ\backslash G_{\mathbb{R}}^\circ\to L_{\mathbb{R}}\backslash H_{\mathbb{R}}$.

Proof. Recall that soluble groups are amenable, whence so is $B_\circ$. Thus, Γ acts amenably on $B_\circ\backslash G_{\mathbb{R}}^\circ$. Let now $Q\leq H$ be an arbitrary, but fixed, proper parabolic subgroup of H. Since $Q_{\mathbb{R}}$ is cocompact, it follows from Fürstenberg's Lemma that there exists a measurable Γ-equivariant map $\phi_0: B_\circ\backslash G_{\mathbb{R}}^\circ\to M(Q_{\mathbb{R}}\backslash H_{\mathbb{R}})$. Technical fact: $H_{\mathbb{R}}$ acts (smoothly, on the right) on $M(Q_{\mathbb{R}}\backslash H_{\mathbb{R}})$ and the orbit space $(M(Q_{\mathbb{R}}\backslash H_{\mathbb{R}}))/H_{\mathbb{R}}$ is second countable.

Let ϕ_1 be the composition $B_\circ\backslash G_{\mathbb{R}}^\circ\xrightarrow{\phi_0}M(Q_{\mathbb{R}}\backslash H_{\mathbb{R}})\twoheadrightarrow(M(Q_{\mathbb{R}}\backslash H_{\mathbb{R}}))/H_{\mathbb{R}}$. Notice that ϕ_1 is a Γ-invariant map since $\phi_0(x\gamma)=\phi_0(x)\pi(\gamma)$ for all $x\in B_\circ\backslash G_{\mathbb{R}}^\circ,\gamma\in\Gamma$. This implies that ϕ_1 is (essentially) constant, because $B_\circ\backslash G_{\mathbb{R}}^\circ$ is an ergodic Γ-space by Moore's theorem. In other words, there exists a measure $\mu\in M(Q_{\mathbb{R}}\backslash H_{\mathbb{R}})$ such that ϕ_1 (essentially) takes values on the $H_{\mathbb{R}}$-orbit of μ.

Now, let $L_{\mathbb{R}} = \mathrm{Stab}_{H_{\mathbb{R}}}(\mu)$ be the stabiliser of this chosen measure μ—this is the set of real points of a (proper) $\mathbb{R}$-subgroup $L \leq H$ by our reminder (Lemmata of p. 47). So $\mu \cdot H_{\mathbb{R}} = (\mu \cdot L_{\mathbb{R}}) \cdot H_{\mathbb{R}}$ and, for almost every $x \in B_\circ \backslash G^\circ_{\mathbb{R}}$, we can find $h_x \in H_{\mathbb{R}}$ such that $\phi_1(x) = (\mu \cdot L_{\mathbb{R}}) \cdot h_x = \mu \cdot (L_{\mathbb{R}} h_x)$. The map $\phi_2 : B_\circ \backslash G^\circ_{\mathbb{R}} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$ defined almost everywhere by $x \mapsto L_{\mathbb{R}} h_x$, is Γ-equivariant, and we thus obtain a measurable Γ-equivariant map $\phi : B_\circ \backslash G^\circ_{\mathbb{R}} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$ which coincides with ϕ_2 almost everywhere. □

2.2 Step C

Lemma C. If $L \leq H$ is a proper $\mathbb{R}$-subgroup and $\phi : B_\circ \backslash G^\circ_{\mathbb{R}} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$ is a measurable Γ-equivariant map, then ϕ is essentially rational.

We proceed by reducing to a "small" case:

1. In the statement, we shall replace "map from $B_\circ \backslash G^\circ_{\mathbb{R}}$" by "map from $\mathcal{U}_{\mathbb{R}}$" for some unipotent ($\mathbb{R}$-subgroup) $\mathcal{U} \leq G$;
2. Instead of considering the whole unipotent subgroup $\mathcal{U}_{\mathbb{R}}$, it suffices to look at one-parameter subgroups $\mathcal{U}^\alpha_{\mathbb{R}}$;
3. One obtains (essential) rationality for the induced map $\mathcal{U}^\alpha_{\mathbb{R}} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$ by studying the action of Γ on the space of measurable functions from $\mathcal{U}^\alpha_{\mathbb{R}}$ to $L_{\mathbb{R}} \backslash H_{\mathbb{R}}$.

2.2.1 Part 1

Let us look at G as a split, semisimple, linear algebraic $\mathbb{R}$-group. The chosen $B \leq G$ is a Borel subgroup, i.e. a maximal, connected, soluble $\mathbb{R}$-subgroup. It contains a maximal $\mathbb{R}$-split torus $T \leq G$. We take the "opposite" Borel subgroup, i.e. the conjugate of B whose intersection with B is exactly T, and let $\mathcal{U}$ be its unipotent radical. The point now is that, by multiplying $\mathcal{U}$ and B we recover G in some sense. As an example, one can verify this for SL_5 by matrix computations for the groups

$$B = \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \text{ and } \mathcal{U} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & * & 1 \end{pmatrix}$$

of upper triangular and lower unitriangular matrices, respectively. Summarising, we have the following.

Lemma 1. There exists a unipotent $\mathbb{R}$-subgroup $\mathcal{U} \leq G$ such that the product map

$$\begin{aligned} p : B \times \mathcal{U} &\to G, \\ (b, u) &\mapsto bu \end{aligned}$$

gives an isomorphism of varieties $B \times \mathcal{U} \cong \mathrm{Im}(p)$, with the image $\mathrm{Im}(p)$ being (Zariski) open and dense in G. Moreover, the induced map $\mathcal{U}_{\mathbb{R}} \to B_\circ \backslash G^\circ_{\mathbb{R}}$ is (essentially) an isomorphism of measure spaces.

Thus, lifting ϕ back to $G^\circ_{\mathbb{R}}$ and then restricting it to $\mathcal{U}_{\mathbb{R}}$, the above lemma allows us to replace $B_\circ \backslash G^\circ_{\mathbb{R}}$ by $\mathcal{U}_{\mathbb{R}}$ when trying to check rationality of ϕ.

2.2.2 Part 2

Having (roughly) replaced the cosets $B_\circ \backslash G^\circ_{\mathbb{R}}$ by elements of the unipotent subgroup $\mathcal{U}_{\mathbb{R}}$, we now turn to the structure of $\mathcal{U}_{\mathbb{R}}$ in order to reduce the problem even further.

In our previous example, with $\mathcal{U}$ the subgroup of lower unitriangular matrices of SL_5, we can identify many one-parameter subgroups. Namely, the subgroups generated by elementary matrices in a single position, for instance the $(2,1)$ or the $(4,2)$ entries. Pictorially:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & * & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, these (finitely many) such one-parameter subgroups can be ordered in such a way that each subgroup normalises the subgroup generated by all the previous ones. For example, consider the index set $\Phi = \{\alpha_1, \dots, \alpha_{10}\}$ and define

$$\mathcal{U}^{\alpha_1} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{U}^{\alpha_2} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \dots,$$

$$\mathcal{U}^{\alpha_5} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \dots, \text{ and so on.}$$

We see that the last subgroup, $\mathcal{U}^{\alpha_{10}}$, is central, so $\langle \mathcal{U}^{\alpha_{10}}, \mathcal{U}^{\alpha_9} \rangle$ is abelian. By matrix computations, $\mathcal{U}^{\alpha_8}$ commutes with $\mathcal{U}^{\alpha_9}$, so $\langle \mathcal{U}^{\alpha_{10}}, \mathcal{U}^{\alpha_9}, \mathcal{U}^{\alpha_8} \rangle$ is still abelian. We proceed like this, step-by-step, until we find that $[\mathcal{U}^{\alpha_5}, \mathcal{U}^{\alpha_7}] \subseteq \mathcal{U}^{\alpha_{10}}$, thus $\langle \mathcal{U}^{\alpha_{10}}, \dots, \mathcal{U}^{\alpha_6} \rangle \triangleleft \langle \mathcal{U}^{\alpha_{10}}, \dots, \mathcal{U}^{\alpha_5} \rangle$. It is also not hard to obtain the lower central series of $\mathcal{U}$ using the $\mathcal{U}^{\alpha_i}$'s.

The groups in the example above have one further property. Since there is "enough space" on the diagonal (i.e. the torus of $\mathrm{SL}_5(\mathbb{R})$) then, given any $\mathcal{U}^{\alpha_i}_{\mathbb{R}}$, one can easily find a non-trivial diagonal matrix which is centralised by $\mathcal{U}^{\alpha_i}_{\mathbb{R}}$.

For instance,

$$\text{for } \mathcal{U}_{\mathbb{R}}^{\alpha_6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & * & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ pick } t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & r^{-1} \end{pmatrix} \text{ with } r \notin \{0,1\}.$$

All of the above hold, of course, in the more general set-up of the structure theory of split, semisimple, linear algebraic groups. In that context, we view $\mathcal{U}$ as the unipotent part of some Borel subgroup of the split, semisimple group G and the index set Φ will be the set of (negative) roots with respect to the fixed maximal split torus contained in the chosen Borel subgroup $B \leq G$. For our purposes, we get the following.

Lemma 2. There exist a finite set $\Phi = \{\alpha_1, \dots, \alpha_n\}$, whose cardinality $n = |\Phi|$ depends on $\mathbb{R}\text{-}\operatorname{rank}(G)$, and abelian $\mathbb{R}$-subgroups $\mathcal{U}^{\alpha_i} \leq \mathcal{U}$ such that

1. The product map $\prod_{i=1}^{n} \mathcal{U}^{\alpha_i} \to \mathcal{U}$, $(u_1, u_2, \dots, u_n) \mapsto u_1 u_2 \cdots u_n$ is an isomorphism of varieties;
2. The $\mathcal{U}^{\alpha_i}$'s 'generate' the lower central series of $\mathcal{U}$ and each $\mathcal{U}_{k+1} \overset{\text{def}}{=} \langle \mathcal{U}^{\alpha_{k+1}}, \dots, \mathcal{U}^{\alpha_n} \rangle$ is normal in $\mathcal{U}_k$;
3. If $\mathbb{R}\text{-}\operatorname{rank}(G) \geq 2$, then every $\mathcal{U}_{\mathbb{R}}^{\alpha_i}$ centralises some (non-trivial) torus element.

Therefore, one can verify by induction (and using Margulis' Lemma) that if a map $\phi : \mathcal{U}_{\mathbb{R}} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$ restricts to an essentially $\mathbb{R}$-rational map $\mathcal{U}_{\mathbb{R}}^{\alpha} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$ for each $\alpha \in \Phi$, then in fact ϕ itself is essentially $\mathbb{R}$-rational.

2.2.3 Part 3

Let $F \overset{\text{def}}{=} F(\mathcal{U}_{\mathbb{R}}^{\alpha}, L_{\mathbb{R}} \backslash H_{\mathbb{R}})/\!\sim$ be the space of measurable functions from $\mathcal{U}_{\mathbb{R}}^{\alpha}$ to $L_{\mathbb{R}} \backslash H_{\mathbb{R}}$, where $f \sim g \iff f = g$ almost everywhere. We remark that $H_{\mathbb{R}}$ acts on F via $f \cdot h : x \mapsto f(x) \cdot h$, thus so does Γ via our original homomorphism π.

Now, for each $g \in G_{\mathbb{R}}^{\circ}$, set $\phi_g : x \mapsto \phi(xg)$.

Claim 3. Almost every map $\phi_g \in F$ as above lies in the same $H_{\mathbb{R}}$-orbit.

Proof. Consider the measurable map $\Psi : G_{\mathbb{R}}^{\circ} \to F$, $g \mapsto \phi_g$. By Lemma 2, there exists $t \neq 1$ a torus element such that $\mathcal{U}_{\mathbb{R}}^{\alpha}$ centralises the (non-compact!) cyclic subgroup $\langle t \rangle \leq G_{\mathbb{R}}^{\circ}$. Furthermore, looking back at our original map $\phi : B_{\circ} \backslash G_{\mathbb{R}}^{\circ} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$ and because $t \in B_{\circ}$, we obtain that $\phi(tg) = \phi(g)$ for all $g \in G_{\mathbb{R}}^{\circ}$. Thus, for every $c \in \mathcal{U}^{\alpha}$, we get

$$\phi_{tg}(c) = \phi(ctg) = \phi(tcg) = \phi(cg) = \phi_g(c).$$

In other words, we can look at Ψ as a (still measurable) map from $\langle t \rangle \backslash G_{\mathbb{R}}^{\circ}$ to F. Since ϕ is originally a Γ-equivariant map, we have for all $c \in \mathcal{U}_{\mathbb{R}}^{\alpha}, \gamma \in \Gamma$ and almost every $g \in G^\circ_{\mathbb{R}}$ that

$$\phi_{g\gamma}(c) = \phi(cg\gamma) = \phi(cg) \cdot \pi(\gamma) = \phi_g(c) \cdot \pi(\gamma),$$

i.e. $\Psi(g\gamma) = \Psi(g) \cdot \pi(\gamma)$. Therefore the induced map $\overline{\Psi} : \langle t\rangle\backslash G^\circ_{\mathbb{R}} \to F\backslash H_{\mathbb{R}}$ is essentially Γ-invariant. Since Γ acts ergodically on $\langle t\rangle\backslash G^\circ_{\mathbb{R}}$ by Moore's theorem, the map $\overline{\Psi}$ is (essentially) constant, that is, almost every ϕ_g lies in the same $H_{\mathbb{R}}$-orbit. □

Finally, using the claim, we can for almost every $c \in \mathcal{U}^\alpha_{\mathbb{R}}$ obtain a formula for $\phi(c)$ which makes explicit its rationality.

Lemma 4. There exist a subvariety $W \subseteq L\backslash H$, a regular action $W \curvearrowleft \mathcal{U}^\alpha$ such that the induced action $W_{\mathbb{R}} \curvearrowleft \mathcal{U}^\alpha_{\mathbb{R}}$ is measurable, and a point $x \in W$ such that $\phi(c) = x \cdot c$ for almost every $c \in \mathcal{U}^\alpha_{\mathbb{R}}$.

(Here, a 'regular action' means that the induced map $W \times \mathcal{U}^\alpha \to W$ is a morphism of varieties).

Proof. Consider the $\mathbb{R}$-subgroup $K \leq H$ fixing the (essential) range of $\phi|_{\mathcal{U}^\alpha_{\mathbb{R}}}$ and let $N(K) \leq H$ be its normaliser, which is also an $\mathbb{R}$-subgroup. Define W to be the $\mathbb{R}$-subvariety of $L\backslash H$ of fixed points of K. So $N(K)$ is the group leaving W invariant and then the quotient $Q \stackrel{\text{def}}{=} K\backslash N(K)$ acts $\mathbb{R}$-regularly on W.

Since almost every ϕ_g lies in a single $H_{\mathbb{R}}$-orbit by Claim 3, we have that ϕ_{ag} lies in the same orbit of ϕ_g under the $H_{\mathbb{R}}$-action for all $a \in \mathcal{U}^\alpha_{\mathbb{R}}$ and almost every $g \in G^\circ_{\mathbb{R}}$. Thus, given $a \in \mathcal{U}^\alpha_{\mathbb{R}}$ we can, for almost every $u \in \mathcal{U}^\alpha_{\mathbb{R}}$, choose an element $h(u,a) \in H_{\mathbb{R}}$ for which

$$\phi_{au} = \phi_u \cdot h(u,a).$$

But with ϕ_{au} we are just translating the elements of $\mathcal{U}^\alpha_{\mathbb{R}}$ by an element of $\mathcal{U}^\alpha_{\mathbb{R}}$ before applying ϕ_u, whence the (essential) ranges of ϕ_{au} and ϕ_u coincide. It follows that the elements $h(u,a)$ chosen above leave W invariant, i.e. $h(u,a) \in N(K)$. Now, for all $a, b \in \mathcal{U}^\alpha_{\mathbb{R}}$ and almost all $c, u \in \mathcal{U}^\alpha_{\mathbb{R}}$,

$$\begin{aligned}\phi_u(c) \cdot h(u,ab) = \phi_{abu}(c) = \phi(cabu) = \phi_{bu}(ca) &= \phi_u(ca) \cdot h(u,b) \\ = \phi(cau) \cdot h(u,b) = \phi_{au}(c) \cdot h(u,b) &= (\phi_u(c) \cdot h(u,a)) \cdot h(u,b) \\ &= \phi_u(c) \cdot (h(u,a)h(u,b)).\end{aligned}$$

This means that for almost every u the element $h(u,ab)h(u,b)^{-1}h(u,a)^{-1} \in H_{\mathbb{R}}$ (essentially) fixes $\mathrm{Im}(\phi_u)$, i.e. it belongs to K by definition. Thus we get a map

$$\begin{aligned} h_u : \mathcal{U}^\alpha_{\mathbb{R}} &\to K\backslash N(K) \\ a &\mapsto h(u,a) \end{aligned}$$

which is essentially a homomorphism of $\mathbb{R}$-groups. Now, by definition, one has $\phi(cau) = \phi(cu) \cdot h(u,a)$. Pick $c, u \in \mathcal{U}^\alpha_{\mathbb{R}}$ for which this equality holds for almost every a and such that $\phi(cu)$ lies in the (essential) range of ϕ_u. By our definition of $h(u,-)$ we obtain

$$\begin{aligned}\phi(a) = \phi(cc^{-1}au^{-1}u) = \phi(cu) \cdot h(u, c^{-1}au^{-1}) &= \phi(cu) \cdot h(u, c^{-1}u^{-1}uau^{-1}) \\ = (\phi(cu) \cdot h(u, c^{-1}u^{-1})) \cdot h(u, uau^{-1}).\end{aligned}$$

So the point $x \stackrel{\text{def}}{=} (\phi(cu) \cdot h(u, c^{-1}u^{-1})) \in W$ is the same for almost every a, whence the lemma. $\square$

3 Finishing off the proof of Superrigidity

We verified in the previous section that Claim 3 together with Lemma 4 imply (essential) rationality for each restriction $\phi : \mathcal{U}_{\mathbb{R}}^{\alpha} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$, so from Lemma 2 combined with Margulis' Lemma we recover (essential) rationality for the whole $\phi : \mathcal{U}_{\mathbb{R}} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$, by induction. Applying Lemma 1 gives (essential) rationality for the *original* measurable Γ-equivariant map $\phi : B_{\circ} \backslash G_{\mathbb{R}}^{\circ} \to L_{\mathbb{R}} \backslash H_{\mathbb{R}}$, which exists in the first place by Lemma B. Since $B_{\circ} \backslash G_{\mathbb{R}}^{\circ}$ is Zariski-dense in $B \backslash G$ we get a Γ-equivariant rational map $\varphi : B \backslash G \to L \backslash H$. The theorem now follows from Lemma A. $\square$