# Moore's Ergodicity Theorem

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# Table of Contents

# Ergodic Theory

## Definition
A *standard Borel space* $S$ is a measure space, which is isomorphic to a measurable subset of a complete separable metric space.

## Definition
Two measures $\mu$ and $\mu'$ are said to be *in the same measure class*, $\mu \sim \mu'$, if they have the same null sets.

## Definition
Let $G$ be a locally compact second countable group. Let $G$ act on a standard Borel space $(S, \mu)$. The measure $\mu$ is said to be

- *invariant*, if $\mu(A) = \mu(Ag)$ for all $g \in G$ and $A \subseteq S$ Borel.
- *quasi-invariant*, if $\mu(A) = 0$ if and only if $\mu(Ag) = 0$.

### Definition

An action of $G$ on $(S, \mu)$, with $\mu$ quasi-invariant, is called *ergodic*, if every $G$-invariant measurable subset of $S$ is either null or conull.

### Remark

The action is called *essentially transitive*, if there is a conull orbit.

transitive $\Rightarrow$ essentially transitive $\Rightarrow$ ergodic

An action is called *properly ergodic*, if it is ergodic, but not essentially transitive. In this case, every orbit is a null set.

## Example

Let $S := \{z \in \mathbb{C} \mid |z| = 1\}$, and let $\mathbb{Z}$ act on $S$ via

$$\begin{array}{rcl} T : S & \rightarrow & S \\ z & \mapsto & e^{i\alpha} z \end{array}$$

with $\frac{\alpha}{2\pi} \in \mathbb{R}$ irrational. This action

- preserves the arc-length measure
- has countable (so null) orbits
- and is ergodic, so properly ergodic:

If $A \subseteq S$ is invariant, let $\chi_A = \sum a_n z^n$ denote the $L^2$-Fourier expansion of its characteristic function. By invariance,

$$\sum a_n z^n = \chi_A(z) = \chi_A(Tz) = \sum a_n e^{in\alpha} z^n,$$

and hence $a_n e^{in\alpha} = a_n$, so $a_n = 0$ for $n \neq 0$, as $\frac{\alpha}{2\pi}$ is irrational. This implies $\chi_A$ is constant and verifies ergodicity.

## Proposition

*Let $G$ act continuously on a second countable topological space $S$. Let $\mu$ be a quasi-invariant measure that is positive on open sets. If the action is properly ergodic, then for almost every $s \in S$ the orbit of $s$ is a dense null set.*

## Proof.

If $W \subset S$ open, then

$$\bigcup_{g \in G} Wg$$

is an open invariant set. So by ergodicity it is conull. Thus, if $\{W_i\}$ is a countable basis for the topology

$$D := \bigcap_i (\bigcup_g W_i g)$$

is also conull and every $x \in D$ has a dense orbit, since any such orbit intersects every $W_i$. $\square$

# Smoothness

## Definition

Let $S$ be a separable, metrizable $G$-space. The action of $G$ on $S$ is called *smooth*, if its orbits are locally closed, i.e. if any orbit is open in its closure.

## Proposition

*Let $G$ act smoothly on $S$. Let $\mu$ be a quasi-invariant ergodic measure on $S$. Then $\mu$ is supported on an orbit. In particular, the action is essentially transitive.*

### Proposition

*Suppose $S$ is an ergodic $G$-space and $Y$ is a countably separated space. If $f : S \to Y$ is $G$-invariant, then $f$ is essentially constant.*

### Remark

The proposition is also true, if $G$-invariance of $f$ is replaced by *essential* $G$-invariance, i.e. $f(sg) = f(s)$ for all $g \in G$ and almost all $s \in S$.

## Proposition

*Any continuous action of a compact group on a separable metrizable space is smooth.*

The fact that continuous actions of compact groups are smooth and that smooth ergodic actions are essentially transitive implies the converse statement of Moore's Ergodicity Theorem:

## Proposition

*Let $G = \prod G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let $\Gamma \subset G$ be an irreducible lattice. If $H \subset G$ is a closed subgroup and $H$ is compact, then $H$ is not ergodic on $G/\Gamma$.*

## Proof.

Since $H$ is compact, then by the previous corollary, the action of $H$ on $G/\Gamma$ is smooth. If it were ergodic, then by the previous proposition it would have a conull orbit. But the $H$-orbits are closed submanifolds of strictly lower dimension. $\square$

# Moore's Ergodicity Theorem

In the following section, $G$ is always a finite product $G = \prod G_i$ of connected non-compact simple Lie groups with finite center.

**Theorem (Moore's Ergodicity Theorem)**

*Let $\Gamma \subset G$ be an irreducible lattice. If $H \subset G$ is a closed subgroup and $H$ is not compact, then $H$ is ergodic on $G/\Gamma$.*

### Proposition

*If $H_1, H_2 \subset G$ are two closed subgroups, then $H_1$ is ergodic on $G/H_2$ if and only if $H_2$ is ergodic on $G/H_1$.*

### Theorem (Moore's Ergodicity Theorem)

*Let $\Gamma \subset G$ be an irreducible lattice. If $H \subset G$ is a closed subgroup and $H$ is not compact, then $\Gamma$ is ergodic on $G/H$.*

### Example

- Let $H := \{z \in \mathbb{C} \mid \Re(z) > 0\}$.
  $\mathrm{SL}_2(\mathbb{R})$ acts on $H$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$.
  This action can be extended to the boundary circle $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ by the same formula. Since $\overline{\mathbb{R}} \cong \mathrm{SL}_2(\mathbb{R})/P$, where $P$ is the subgroup of upper triangular matrices, any lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ acts ergodically on $\overline{\mathbb{R}}$.
- Any lattice $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ acts ergodically on $\mathbb{R}\mathrm{P}^{n-1}$, as $\mathbb{R}\mathrm{P}^{n-1}$ can be realised as a homogeneous space of $\mathrm{SL}_n(\mathbb{R})$.

### Definition

Let $S$ be an ergodic $G$-space with finite invariant measure. Then the action is called *irreducible* if for every non-central normal subgroup $N \subset G$, $N$ is ergodic on $S$.

### Proposition

*Let $\Gamma \subset G$ be a lattice. Then*

$$\Gamma \textit{ is irreducible} \Leftrightarrow G/\Gamma \textit{ is an irreducible } G\textit{-space.}$$

### Theorem (Moore's Ergodicity Theorem)

*Let $S$ be an ergodic irreducible $G$-space with finite invariant measure. If $H \subset G$ is a closed subgroup and $H$ is not compact, then $H$ is ergodic on $S$.*

## Definition

A *unitary representation* of $G$ is a homomorphism $\pi : G \to U(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $U(\mathcal{H})$ carries the strong operator topology. (Recall that this is the topology of pointwise convergence: $T_i \to T \Leftrightarrow \|T_i(x) - T(x)\| \to 0$.)

Let $S$ be a $G$-space with finite invariant measure. Define $\pi : G \to U(L^2(S))$ via

$$(\pi(g)f)(s) = f(sg).$$

By general point-set-topological and measure-theoretical arguments, this $\pi$ is continuous.

### Proposition

*If $S$ is a $G$-space with finite invariant measure, then $G$ is ergodic on $S$ if and only if there are no non-trivial $G$-invariant vectors in $L^2(S) \ominus \mathbb{C}$.*

### Proof.

If $A \subset S$ is $G$-invariant, then $\chi_A \in L^2(S)$ will be $G$-invariant and so will be the projection $f_A \in L^2(S) \ominus \mathbb{C}$. If $A$ is neither null nor conull, then $f_A \neq 0$. Thus if $G$ is not ergodic, then there is a non-zero invariant vector in $L^2(S) \ominus \mathbb{C}$.

Conversely suppose $G$ acts ergodically and $f \in L^2(S) \ominus \mathbb{C}$ is $G$-invariant. This means that for all $g \in G$ and almost all $s \in S$ that $f(sg) = f(s)$. But we have already seen that such an $f$ must be essentially constant, so its projection onto $L^2(S) \ominus \mathbb{C}$ is 0. $\square$

**Theorem (Moore)**

*Let $\pi$ be a unitary representation of $G$ so that $\pi\big|_{G_i}$ has no non-trivial invariant vectors for all $i$. If $H \subset G$ is a closed subgroup and $H$ is not compact, then $\pi\big|_H$ has no non-trivial invariant vectors.*

This theorem is implied by the even more general Vanishing of Matrix Coefficients Theorem.

Definition

Let $\pi$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Let $v, w \in \mathcal{H}$ be unit vectors. Then the map $f(g) := \langle \pi(g)v \mid w \rangle$ is called a *matrix coefficient* of $\pi$.

Theorem (Vanishing of Matrix Coefficients)

*Let $\pi$ be a unitary representation of $G$. Then all matrix coefficients vanish at $\infty$, i.e., $f(g) \to 0$ as $g$ leaves compact subsets of $G$.*

This implies Moore's theorem, because if $\pi\big|_H$ has an invariant vector $v$, the matrix coefficient $f(g) := \langle \pi(g)v \mid v \rangle$ is identically 1 along $H$, and hence $H$ is compact.

The Vanishing of Matrix Coefficients for $G = \mathrm{SL}_2(\mathbb{R})$ is implied by the following two facts:

**Lemma**

*Let $\pi$ be a unitary representation of a group $G$. Suppose $G = KAK$ ('Cartan decomposition'), where $K$ is a compact group. Then all matrix coefficients of $\pi$ vanish at $\infty$ if and only if all matrix coefficients of $\pi\big|_A$ vanish at $\infty$.*

**Theorem (Unitary representations of $P$)**

*Let $P \subset \mathrm{SL}_2(\mathbb{R})$ denote the subgroup of upper triangular matrices. Then $P = AN$, where $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$ and $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$. Let $\pi$ be a unitary representation of $P$. Then either*

1. *$\pi\big|_N$ has non-trivial invariant vectors or*
2. *Any matrix coefficient of $\pi\big|_A$ vanishes at $\infty$.*

# Proof of the Vanishing of Matrix Coefficients Theorem

## Proof in the case $G = \mathrm{SL}_2(\mathbb{R})$.

As $\mathrm{SL}_2(\mathbb{R}) = KAK$, with $K = \mathrm{SO}(2)$ and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$, it suffices to show that all matrix coefficients vanish at $\infty$ along $A$.
By the theorem about unitary representations of $P$, it suffices to see that there are no $N$-invariant vectors, $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$.
Suppose there is an $N$-invariant $v$ and let $f(g) := \langle \pi(g)v \mid v \rangle$.
Then $f$ is continuous and bi-invariant under $N$, so lifts from a continuous $N$-invariant $\overline{f} : G/N \to \mathbb{C}$.
Now $G/N \cong \mathbb{R}^2 \setminus \{0\}$ with the action of $G$ being ordinary matrix multiplication (to see this, observe that $N$ is the stabiliser of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^2$.)
This action has two types of orbits

- Lines parallel to the x-axis
- Points on the x-axis (without 0)

Since $\overline{f}$ is continuous and constant along these orbits, it must be constant along the x-axis.
But the x-axis is identified with $P/N \subset G/N$, hence $f$ is constant along $P$, so $v$ must be $P$-invariant.
Therefore $f$ is bi-invariant under $P$ and hence lifts from a continuous $\tilde{f} : G/P \to \mathbb{C}$.
But $G/P \cong P(\mathbb{R}^2)$ $(\cong \overline{\mathbb{R}})$ with ordinary matrix multiplication and $P$ has a dense orbit in $P(\mathbb{R}^2)$.
Thus $f$ is actually constant on $G$ and $v$ is $G$-invariant. $\square$

## Theorem (Unitary representations of $P$)

*Let $\pi$ be a unitary representation of $P$. Then either*

1. $\pi\big|_N$ *has non-trivial invariant vectors or*
2. *All matrix coefficients of* $\pi\big|_A$ *vanish at* $\infty$.

## Sketch.

- Any unitary representation $\pi$ of $N \cong \mathbb{R}$ is unitarily equivalent to a canonical representation on the space $L^2(\widehat{\mathbb{R}}, \mu, \{\mathcal{H}_\lambda\}) = \int^{\oplus} \mathcal{H}_\lambda$, where
    - $\widehat{\mathbb{R}}$ denotes the set of characters of $\mathbb{R}$,
    - $\mu$ is some ($\sigma$-finite) measure on $\widehat{\mathbb{R}}$ and
    - $\lambda \mapsto \mathcal{H}_\lambda$ is a piecewise constant assignment of Hilbert spaces to any $\lambda \in \widehat{\mathbb{R}}$.
- Distinguish the cases $\mu(\{0\}) > 0$ and $= 0$
- $\mu(\{0\}) > 0$ implies that $\mathcal{H}_0$ contains $N$-invariant vectors

Sketch (continued).

- If $\mu(\{0\}) = 0$ we approximate vectors $f, h \in L^2(\widehat{\mathbb{R}}, \mu, \{\mathcal{H}_\lambda\})$ by truncations:

$$\|\chi_E \cdot f - f\| < \varepsilon \text{ and } \|\chi_F \cdot h - h\| < \varepsilon,$$

  with $E, F \subset \widehat{\mathbb{R}} \setminus \{0\}$ compact.
- Now $\langle \pi(g) f \mid h \rangle \leq 2\varepsilon + |\langle \pi(g)(\chi_E \cdot f) \mid \chi_F \cdot h \rangle|$.
- Use that $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in A$ acts on $\widehat{N} \cong \widehat{\mathbb{R}}$ via multiplication with $a^2$, so if $a \gg 0$, $gE \cap F = \emptyset$, so that $\chi_E \cdot f \perp \chi_F \cdot h$.
- It follows that $\langle \pi(g) f \mid h \rangle \leq 2\pi$.

$\square$

# Proof of the Vanishing of Matrix Coefficients

## Proof for $G = \mathrm{SL}_n(\mathbb{R})$ and general semisimple $G$.

$G = \mathrm{SL}_n(\mathbb{R})$ or any semisimple $G$ has a Cartan decomposition, where $A$ is the subgroup of diagonal matrices, resp. a maximal $\mathbb{R}$-split torus.
Instead of looking at $P = AN$, we look at $H = AB$, where

$$B = \left\{ \begin{pmatrix} 1 & b_2 & \dots & b_n \\ 0 & & & \\ \vdots & & \mathrm{id}_{n-1} & \\ 0 & & & \end{pmatrix} \right\},$$

denoting $b \in B$ by $b = (1, b_2, \dots, b_n)$.
Note that $B \cong \mathbb{R}^{n-1}$ is a normal subgroup of $H$.
For semisimple $G$, we use the fact that $A \subset G' \subset G$, where $G'$ is semisimple and $\mathbb{R}$-split. Now we take a set of simple roots, and $B$ is then defined as the subgroup corresponding to the direct sum of the root spaces.

### Proof (continued).

Now a case distinction can be made similar to the case where $G = \mathrm{SL}_2(\mathbb{R})$:

- In the one case the matrix coefficients of $A$ vanish at $\infty$
- In the other case, the subgroup $B_i = \{b \in B \mid b_j = 0 \text{ for } i \neq j\} \subset B$ has non-trivial invariant vectors, this leads to a contradiction:

$B_i$ is contained in a subgroup $H_i$ isomorphic to $\mathrm{SL}_2(\mathbb{R})$.
From the vanishing of matrix coefficients for $\mathrm{SL}_2(\mathbb{R})$, we get that $H_i$ has a non-trivial invariant vector, since $B_i$ isn't compact. In particular $A_i = H_i \cap A$ has an invariant vector.
Let $W := \{v \in \mathcal{H} \mid \pi(a)v = v \text{ for all } a \in A_i\}$. It suffices to show that $W$ is $G$-invariant, since then the representation $\pi_W$ of $G$ on $W$ has $\ker(\pi_W) \supset A_i$, which by simplicity of $G$ implies $\ker(\pi_W) = G$, so that $G$ fixes $W$ pointwise.

### Proof (continued).

It remains to show that $W$ is $G$-invariant.
$G$ is generated by the subgroups $B_{kj}$, $k \neq j$, of matrices that are the identity matrix except for the $(k, j)$-entry, which is arbitrary.
Now there are two possibilities:

- $k, j \notin \{1, i\}$, then $B_{kj}$ commutes with $A_i$ and hence leaves $W$ invariant.
- $\{k, j\} \cap \{1, i\} \neq \emptyset$, then $A_i$ normalizes $B_{kj}$. Hence $A_i B_{kj}$ is a subgroup isomorphic to $P$ but with the theorem about unitary representations of $P$ it is not difficult to prove that any $A$-invariant vector is also $P$-invariant.

□