Higher Teichmüller theory and geodesic currents

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Overview

Ongoing program to extend features of Teichmüller space to more general situations.

In this talk:

- Some aspects of the classical Teichmülller theory
- A structure theorem for geodesic currents
- Higher Teichmülller theory and applications

Joint with M.Burger, A.Parreau and B.Pozzetti (in progress)

Why Teichmüller theory: relations with complex analysis, hyperbolic geometry, the theory of discrete groups, algebraic geometry, low-dimensional topology, differerential geometry, Lie group theory, symplectic geometry, dynamical systems, number theory, TQFT, string theory,...

Teichmüller space

Σ_g closed orientable surface of genus $g \geq 0$ (for simplicity for the moment with $p = 0$ punctures)

$$\mathcal{T}_g \cong \{\text{complete hyperbolic metrics}\}/\mathrm{Diff}_0^+(\Sigma)$$

Characterizations:

1. $\mathcal{T}_g \cong \left\{ \begin{array}{r} \rho\colon \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R}) \\ \text{discrete and injective} \end{array} \right\} / \,\mathrm{PSL}(2,\mathbb{R})$
2. One connected component in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}(2,\mathbb{R}))$;
3. Maximal level set of $\mathrm{eu}(\Sigma_g, \cdot\,)\colon \mathrm{Hom}\left(\pi_1(\Sigma_g), \mathrm{PSL}(2,\mathbb{R})\right)/\,\mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}$, such that $|\,\mathrm{eu}(\Sigma_g, \rho)| \leq |\chi(\Sigma_g)|$.

Thurston compactification: what to look for and why

$\mathcal{MCG}(\Sigma_g) := \mathrm{Aut}\left(\pi_1(\Sigma_g)\right)/\,\mathrm{Inn}\left(\pi_1(\Sigma_g)\right) \curvearrowright \mathcal{T}_g \cong \mathbb{R}^{6g-6}$

Wanted a compactification $\overline{\Theta(\mathcal{T}_g)}$ such that:

1. The boundary $\partial\Theta(\mathcal{T}_g) = \overline{\Theta(\mathcal{T}_g)} \setminus \Theta(\mathcal{T}_g) \cong S^{6g-7} \Rightarrow$

$$\overline{\Theta(\mathcal{T}_g)} \cong \text{closed ball in } \mathbb{R}^{6g-6};$$

2. The action of $\mathcal{MCG}(\Sigma_g)$ extends continuously to $\partial\Theta(\mathcal{T}_g)$.

Then (1)+(2)$\Rightarrow$ $\mathcal{MCG}(\Sigma_g)$ acts continuously on $\overline{\Theta(\mathcal{T}_g)} \rightsquigarrow$ classify mapping classes (Brower fixed point theorem).

[If $g=1$, $\mathcal{MCG}(\Sigma_1) \cong \mathrm{SL}(2,\mathbb{Z})$, $\mathcal{T}_1 \cong \mathbb{H}$, and $\exists$ a classification of isometries and their dynamics by looking at the fixed points in $\overline{\mathbb{H}}$.]

Thurston–Bonahon compactification

$\mathcal{C}$=homotopy classes of closed curves in Σ_g.If $[\rho] \in \mathcal{T}_g$, define

$$\begin{aligned} \ell_{[\rho]} \colon \mathcal{C} &\longrightarrow \mathbb{R}_{\geq 0} \\ [\gamma] &\mapsto \ell(\rho(c)) \end{aligned}$$

where $\ell(\rho(c)) =$ hyperbolic length of the unique ρ-geodesic in $[\gamma]$.
Thus can define

$$\begin{aligned} \Theta \colon \mathcal{T}_g &\to \mathbb{P}\big(\mathbb{R}^{\mathcal{C}}_{\geq 0}\big) \\ [\rho] &\longmapsto \big[\ell_{[\rho]}\big] \end{aligned}$$

with properties:

- Θ is an embedding;
- $\overline{\Theta(\mathcal{T}_g)}$ is compact;
- (1)+(2) from before;
- $\overline{\Theta(\mathcal{T}_g)}$ and $\partial\Theta(\mathcal{T}_g)$ can be described geometrically in terms of *geodesic currents*, *measured laminations* and *intersection numbers*.

Geodesic currents $\mathrm{Curr}(\Sigma)$

Σ oriented surface with a complete finite area hyperbolization, $\Gamma = \pi_1(\Sigma)$ and
$\mathcal{G}(\widetilde{\Sigma})$ = the set of geodesics in $\widetilde{\Sigma} = \mathbb{H}$

Definition

A *geodesic current* on Σ is a positive Radon measure on $\mathcal{G}(\widetilde{\Sigma})$ that is Γ-invariant.

Convenient: Identify $\mathcal{G}(\widetilde{\Sigma}) \cong (\partial\mathbb{H})^{(2)} = \{$pairs of distinct points in $\partial\mathbb{H}\}$.

Example

$c \subset \Sigma$ closed geodesic, $\gamma \simeq (\gamma_-, \gamma_+) \in (\partial\mathbb{H})^{(2)}$ lift of c.If

$$\delta_c := \sum_{\eta \in \Gamma/\langle\gamma\rangle} \delta_{\eta(\gamma_-,\gamma_+)},$$

$\operatorname{supp} \delta_c$ = lifts of c to $\mathbb{H}$.

Geodesic currents $\mathrm{Curr}(\Sigma)$

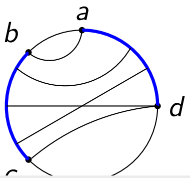
Example

Liouville current $\mathcal{L}$ = the unique $\mathrm{PSL}(2,\mathbb{R})$-invariant measure on $(\partial\mathbb{H})^{(2)}$.
Let $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, so $[x, y]$ is well defined.
If $a, b, c, d \in \partial\mathbb{H}$ are positively oriented,

$$\mathcal{L}\big([d, a] \times [b, c]\big) := \ln[a, b, c, d],$$

where

$$[a, b, c, d] := \frac{(a-c)(b-d)}{(a-b)(c-d)} > 1.$$



Geodesic currents $\mathrm{Curr}(\Sigma)$

Example

Measure geodesic lamination (Λ, m)

- $\Lambda \subset \Sigma =$ closed subset of Σ that is the union of disjoint simple geodesics;
- $m =$ homotopy invariant transverse measure to Λ.

Lift to a Γ-invariant measure geodesic lamination on $\mathbb{H}$.
The associated geodesic current is

$$m([a,b] \times [c,d]) := \tilde{m}(\sigma),$$

where σ is a (geodesic) arc crossing precisely once all leaves connecting $[a,b]$ to $[c,d]$.

Intersection number of two currents

Know: If $\alpha, \beta \in \mathcal{C}$, then $i(\alpha, \beta) = \inf_{\alpha' \in \alpha, \beta' \in \beta} |\alpha' \cap \beta'|$
Want: If $\mu, \nu \in \mathrm{Curr}(\Sigma)$, define $i(\mu, \nu)$ so that $i(\delta_c, \delta_{c'}) = i(c, c')$.

Definition

Let $\mathcal{G}^2_{\pitchfork} := \{(g_1, g_2) \in (\partial\mathbb{H})^{(2)} \times (\partial\mathbb{H})^{(2)} : |g_1 \cap g_2| = 1\}$on which $\mathrm{PSL}(2, \mathbb{R})$ acts properly. Then

$$i(\mu, \nu) := (\mu \times \nu)(\Gamma \backslash \mathcal{G}^2_{\pitchfork})$$

Properties

1. If $\delta_c, \delta_{c'} \in \mathrm{Curr}(\Sigma) \Rightarrow i(\delta_c, \delta_{c'}) = i(c, c')$ and $i(\delta_c, \delta_c) = 0$ if and only if c is simple.
2. $i(\mathcal{L}, \delta_c) = \ell(c) =$ hyperbolic length of c.

Thurston–Bonahon compactification

Theorem (Bonahon, '88)

There is a continuous embedding

$$\begin{aligned} I : \mathbb{P}(\mathrm{Curr}(\Sigma_g)) &\to \quad \mathbb{P}(\mathbb{R}^{\mathcal{C}}_{\geq 0}) \\ [\mu] \qquad &\mapsto \{c \mapsto i(\mu, c)\} \end{aligned}$$

whose image contains the Thurston compactification

$$\overline{\Theta(\mathcal{T}_g)} \subset I\big(\mathbb{P}(\mathrm{Curr}(\Sigma_g))\big).$$

Moreover $\partial\Theta(\mathcal{T}_g)$ corresponds to the geodesic currents coming from measured laminations.

A structure theorem for geodesic currents

Want to generalize to higher rank.
Few observations:

- Intersection can be thought of as length, although more general;
- Geodesic currents can be thought of as some kind of degenerate hyperbolic structure with geodesics of zero length;
- Given $\mu \in \mathrm{Curr}(\Sigma)$, geodesics of zero μ-intersection arrange themselves "nicely" in Σ ([Burger–Pozzetti, '15] for μ-lengths)

A structure theorem for geodesic currents

Definition

Let $\mu \in \mathrm{Curr}(\Sigma)$.A closed geodesic is μ-*special* if

1. $i(\mu, \delta_c) = 0$;
2. $i(\mu, \delta_{c'}) > 0$ for all closed geodesic c' with $c \pitchfork c'$.

In particular:

1. A closed geodesic is simple
2. Special geodesics are pairwise non-intersecting.

Thus if $\mathcal{E}_\mu = \{\text{special geodesics on } \Sigma\}$, $|\mathcal{E}_\mu| \leq \infty$and one can decompose

$$\Sigma = \bigcup_{v \in V_\mu} \Sigma_v,$$

where $\partial \Sigma_v \subset \mathcal{E}_\mu$.

A structure theorem for geodesic currents

Theorem (Burger–I.–Parreau–Pozzetti, '17)

Let $\mu \in \mathrm{Curr}(\Sigma)$*. Then*

$$\mu = \sum_{v \in V_\mu} \mu_v + \sum_{c \in \mathcal{E}_\mu} \lambda_c \delta_c,$$

where μ_v *is supported on geodesics contained in* $\mathring{\Sigma}_v$*.*
Moreover either

1. $i(\mu, \delta_c) = 0$ *<u>for all</u>* $c \in \mathring{\Sigma}_v$*, hence* $\mu_v = 0$*, or*
2. $i(\mu, \delta_c) > 0$ *<u>for all</u>* $c \in \mathring{\Sigma}_v$*.In this case either:*
    1. $\inf_c i(\mu, \delta_c) = 0$*and* $\mathrm{supp}\, \mu$ *is a* $\pi_1(\Sigma_v)$*-invariant lamination that is surface filling and compactly supported, or*
    2. $\inf_c i(\mu, \delta_c) > 0$*.*

$$\mathrm{Syst}_{\Sigma_v}(\mu) := \inf \; i(\mu, \delta_c).$$

Higher Teichmüller theory

Consider representations into a "larger" Lie group.

- real adjoint Lie groups $\rightsquigarrow$ Hitchin component
  e.g. $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$.
  [Techniques: Higgs bundles, hyperbolic dynamics, harmonic maps, cluster algebras (Hitchin, Labourie, Fock–Goncharov)]
- Hermitian Lie groups $\rightsquigarrow$ maximal representations
  Examples: $\mathrm{SU}(p,q)$ (orthogonal group of a Hermitian form of signature (p,q)), $\mathrm{Sp}(2n,\mathbb{R})$
  [Techniques: Bounded cohomology, Higgs bundles, harmonic maps (Toledo, Hernández, Burger–I.–Wienhard, Bradlow–García Prada–Gothen, Koziarz–Maubon)]
- semisimple real algebraic of non-compact type $\rightsquigarrow$ positively ratioed representations (Martone–Zhang)
  Examples: maximal representations and Hitchin components

G real adjoint & Hermitian $\Rightarrow G = \mathrm{Sp}(2n,\mathbb{R})$ and

$$\mathrm{Hom}_{\mathrm{Hitchin}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbb{R})) \subsetneq \mathrm{Hom}_{\mathrm{max}}(\pi_1(\Sigma),\mathrm{Sp}(2n,\mathbb{R}))$$

Maximal representations

Remark

Margulis' superrigidity does not hold.

Can define the *Toledo invariant*

$$\mathsf{T}(\Sigma,\,\cdot\,) : \mathrm{Hom}(\pi_1(\Sigma), \mathsf{PSp}(2n,\mathbb{R}))/\,\mathsf{PSp}(2n,\mathbb{R}) \to \mathbb{R}$$

that is uniformly bounded

$$|\,\mathsf{T}(\Sigma,\,\cdot\,)| \leq |\chi(\Sigma)|\,\mathsf{rank}\,G$$

Definition

ρ is maximal if $\mathsf{T}(\Sigma,\,\cdot\,)$ achieves the maximum value

The Thurston–Parreau compactification

If $g \in \mathrm{Sp}(2n, \mathbb{R})$, it has complex eigenvalues $\lambda_i, \lambda_i^{-1}$, $i = 1, \dots, n$, that we can arrange so that $|\lambda_1| \geq \dots \geq |\lambda_n| \geq 1 \geq |\lambda_n|^{-1} \geq \dots \geq |\lambda_1|^{-1}$.Then we set the *length of g* to be

$$L(g) := \sum_{i=1}^{n} \log |\lambda_i|.$$

Theorem (Martone–Zhang '16, Burger–I.–Parreau–Pozzetti '17)

If $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbb{R})$ is maximal, there exists a geodesic current μ_ρ on Σ such that for every $\gamma \in \pi + 1(\Sigma)$ hyperbolic

$$L(\rho(\gamma)) = i(\mu_\rho, \delta_c),$$

where c is the unique geodesic in the homotopy class of γ.

The Thurston–Parreau compactification

Theorem (Parreau, '14)

The map

$$\Theta\colon \overbrace{\mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{PSp}(2n,\mathbb{R}))/\,\mathrm{PSp}(2n,\mathbb{R})}^{=:\mathrm{Max}(\Sigma,n)} \to \mathbb{P}(\mathbb{R}^{\mathcal{C}}_{\geq 0})$$
$$[\rho] \longmapsto [L_{[\rho]}]$$

is continuous, proper and has relatively compact image (inj. if $n=1$) $\overline{\Theta(\mathrm{Max}(\Sigma,n))}$.

Recall from before that there is a continuous embedding

$$I : \mathbb{P}(\mathrm{Curr}(\Sigma_g)) \to \mathbb{P}(\mathbb{R}^{\mathcal{C}}_{\geq 0})$$
$$[\mu] \mapsto \{c \mapsto i(\mu, c)\}$$

whose image contains the Thurston compactification $\overline{\Theta(\mathcal{T}_g)}$ [Bonahon, '88].We have also:

Length compactification of $\mathrm{Max}(\Sigma, n)$

Theorem (Burger–I.–Parreau–Pozzetti, '17)

$\overline{\Theta(\mathrm{Max}(\Sigma, n))} \subset I\big(\mathbb{P}(\mathrm{Curr}(\Sigma))\big)$.

Corollary (Burger–I.–Parreau–Pozzetti, '17)

If $[L] \in \partial\,\mathrm{Max}(\Sigma, n)$, there is a decomposition of Σ into subsurfaces, where $[L]$ is either the length function associated to a minimal surface filling lamination or it has positive systole.

Example

If $n \geq 2$ positive systole does occur. Can construct $\rho : \pi_1(\Sigma_{0,3}) \to \mathrm{PSp}(4, \mathbb{R})$ maximal. Since $\nexists$ compactly supported laminations on $\Sigma_{0,3} \Rightarrow \mathrm{Syst}_{\Sigma_{0,3}}(\mu_\rho) > 0$.

If $\mathrm{Syst}_\Sigma(\mu) > 0$

Let us assume $\mathrm{Syst}_\Sigma(\mu) > 0$ throughout Σ.

Theorem (Burger–I.–Parreau–Pozzetti, '17)

The set

$$\Omega := \left\{[\mu] \in \mathbb{P}(\mathrm{Curr}\Sigma) : \ \mathrm{Syst}_\Sigma(\mu) > 0\right\}$$

is open and $\mathcal{MCG}(\Sigma)$ acts properly discontinuously on it.

Corollary (Burger–I.–Parreau–Pozzetti, '17)

$$\Omega(\Sigma, n) := \left\{[L \in \overline{\Theta(\mathrm{Max}(\Sigma, n))} : \mathrm{Syst}_\Sigma > 0\right\}$$

is an open set of discontinuity for $\mathcal{MCG}(\Sigma)$.

Remark

- $\Omega(\Sigma_g, 1) = \mathcal{T}_g$;
- $\omega(\Sigma_{0,3}, 2)$ contains boundary points.

If $\mathrm{Syst}_\Sigma > 0$

A geodesic current with $\mathrm{Syst}_\Sigma > 0$ behaves like a Liouville current, that is a current whose intersection computes the length in a hyperbolic structure.

Theorem (Burger–I.–Parreau–Pozzetti, '17)

Assume $\mathrm{Syst}_\Sigma(\mu) > 0$ *and let* $K \subset \Sigma$ *be compact. Then there are constants* $0 < c_1 \leq c_2 < \infty$ *such that*

$$c_1 \ell(c) \leq i(\mu, \delta_c) \leq c_2 \ell(c) \qquad (*)$$

for all $c \subset K \subset \Sigma$*. In particular* $(*)$ *holds for all simple closed geodesics.*

Thank you!