Uncountably many quasi-isometry classes of groups of type FP

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Joint work with

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We build $X = K(G, 1)$ as follows:

- X has a single 0–cell,
- 1–cells of X correspond to generators of G,
- 2–cells of X correspond to relations of G,
- 3–cells of X are added to kill $\pi_2(X)$,
- 4–cells of X are added to kill $\pi_3(X)$,
- etc. . .

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consisting of free $\mathbb{Z}G$–modules. This leads to a definition:

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A group G is **of type $\mathbf{FP_n}$** if the trivial $\mathbb{Z}G$–module $\mathbb{Z}$ has a projective resolution which is **finitely generated** in dimensions 0 to n:

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Input: A flag simplicial complex L.

Output: A group BB_L with nice properties:

- L is $(n-1)$–connected $\iff$ BB_L is of type F_n,
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L is n–dimensional octahedron (orthoplex) $\implies$ Bieri's example.
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Question 2: How many groups are there of type FP_2?
Answer 1: Up to isomorphism: $2^{\aleph_0}$ (I.Leary'15)
Answer 2: Up to quasi-isometry: $2^{\aleph_0}$ (R.Kropholler–I.Leary–S.'17)

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Theorem (I.J.Leary)

If L is a flag complex with $\pi_1(L) \neq 1$, then groups $G_L(S)$ form $2^{\aleph_0}$ isomorphism classes. If, in addition, L is aspherical and acyclic, then groups $G_L(S)$ are all of type FP.

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What is a possible example of an aspherical and acyclic flag simplicial complex L?

Take the famous Higman's group:

$$H = \langle a, b, c, d \mid a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2 \rangle.$$

Let K be its presentation complex. It is aspherical and acyclic. Take L to be the 2nd barycentric subdivision of K. Then L is a flag simplicial complex with 97 vertices, 336 edges and 240 triangles. Thus,

$$G_L(S) = \langle 336 \text{ gen's} \mid 240 \times 2 \text{ triangle relators},\ 1 \text{ long relator } \forall n \in S \rangle.$$

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Recall that groups G_1, G_2 are **quasi-isometric** (qi), if their Cayley graphs are qi as metric spaces, i.e. there exists $f\colon Cay(G_1, d_1) \to Cay(G_2, d_2)$, and $A \geq 1$, $B \geq 0$, $C \geq 0$ such that for all $x, y \in Cay(G_1)$:

$$\frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B,$$

and for all $z \in Cay(G_2)$ there exists $x \in Cay(G_1)$ such that $d_2(z, f(x)) \leq C$.

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We get $\pi_1(\Gamma) \to \pi_1(\Gamma_1) \to \pi_1(\Gamma_2) \to \dots$. A loop $\gamma \subset \Gamma$ of length ℓ is **taut** if it lies in the kernel $\ker\big(\pi_1(\Gamma_\ell) \to \pi_1(\Gamma_{\ell+1})\big)$.

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Bowditch: Groups G_1 and G_2 quasi-isometric $\Longrightarrow$ $TL(G_1)$ and $TL(G_2)$ quasi-isometric in $\mathbb{R}$.
I.e. there exist constants $A, B, N > 0$ such that for every $l_1 \in TL(G_1)$, $l_1 > N$, there exist an $l_2 \in TL(G_2)$ such that $l_1 \in [Al_2, Bl_2]$ and vice versa.

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then $TL(G_L(S))$ lies in some multiplicative $[A, B]$ neighborhood of S. Now there are uncountably many subsets S in the above set, and these give $2^{\aleph_0}$ quasi-isometry classes of groups $G_L(S)$.

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Bestvina–Brady kernels BB_L have infinite relation gap, and so do $G_L(S)$.

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So groups $G_L(S_i)$ for finite S_i are candidates to have finite relation gap!

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Thank you!