Uncountably many quasi-isometry classes of groups of type FP

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Joint work with

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Bielefeld U., April 3–6, 2018
TOPOLOGY $\leadsto$ ALGEBRA
Space $X \leadsto \pi_1(X), H_n(X), \pi_n(X)$, etc.
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ALGEBRA $\leadsto$ TOPOLOGY
Group $G \leadsto$ Eilenberg–Mac Lane space $X = K(G, 1)$:
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Group $G \implies$ Eilenberg–Mac Lane space $X = K(G, 1)$:
- $X$ is a CW-complex,
- $\pi_1(X) = G$,
- $\tilde{X}$ is contractible.
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Group $G \leadsto$ Eilenberg–Mac Lane space $X = K(G, 1)$:

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We build $X = K(G, 1)$ as follows:

- $X$ has a single 0–cell,
- 1–cells of $X$ correspond to generators of $G$,
- 2–cells of $X$ correspond to relations of $G$,
- 3–cells of $X$ are added to kill $\pi_2(X)$,
- 4–cells of $X$ are added to kill $\pi_3(X)$,
- etc. . .
If the $n$–skeleton of $K(G, 1)$ has finitely many cells, group $G$ is of type $F_n$: 

$F_1 = \text{finitely generated groups},$ 
$F_2 = \text{finitely presented groups}.$

If $K(G, 1)$ has finitely many cells, group $G$ is of type $F_n$.

If $X = K(G, 1)$, $G$ acts cellularly on $\tilde{X}$ and we have a long exact sequence 

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\cdots \rightarrow C_i(\tilde{X}) \rightarrow \cdots \rightarrow C_1(\tilde{X}) \rightarrow C_0(\tilde{X}) \rightarrow \mathbb{Z} \rightarrow 0
$$

consisting of free $\mathbb{Z}G$–modules. This leads to a definition:

A group $G$ is of type $FP_n$ if the trivial $\mathbb{Z}G$–module $\mathbb{Z}$ has a projective resolution which is finitely generated in dimensions 0 to $n$:

$$
\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0
$$

If, in addition, all $P_i = 0$ for $i > N$, for some $N$, group $G$ is of type $FP$. Clearly, $FP_n \supset FP_{n+1}$ and $F_n \supset F_{n+1}$, and $FP \supset F$, 

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Clearly,

\[ FP_n \supset FP_{n+1} \quad \text{and} \quad F_n \supset F_{n+1}. \]
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**Bestvina–Brady machine**:

**Input**: A flag simplicial complex $L$.

**Output**: A group $BB_L$ with nice properties:

- $L$ is $(n - 1)$–connected $\iff BB_L$ is of type $F_n$,
- $L$ is $(n - 1)$–acyclic $\iff BB_L$ is of type $FP_n$. 

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**Question 2**: How many groups are there of type $FP_2$?

**Answer 1**: Up to isomorphism: $2^\mathbb{N}_0$ (I. Leary’15)

**Answer 2**: Up to quasi-isometry: $2^\mathbb{N}_0$ (R. Kropholler–I. Leary–S.’17)

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$L$ is octahedron: $\pi_1(L) = 1$, $\pi_2(L) \neq 0$, $\implies$ Stallings’s example.

$L$ is $n$–dimensional octahedron (orthoplex) $\implies$ Bieri’s example.

$L$ has $\pi_1(L) \neq 1$, but $H_1(L) = 0$ $\implies$ $BB_L$ of type $FP_2 \setminus F_2$. 
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Answer 1: Up to isomorphism: $2^{\aleph_0}$ (I.Leary’15)  
Answer 2: Up to quasi-isometry: $2^{\aleph_0}$ (R.Kropholler–I.Leary–S.’17)
I.J. Leary’s groups $G_L(S)$

**Input:** A flag simplicial complex $L$, a finite collection $\Gamma$ of directed edge loops in $L$ that normally generates $\pi_1(L)$, a subset $S \subset \mathbb{Z}$.

**Output:** Group $G_L(S)$ defined as:

- **Generators:** directed edges of $L$, the opposite edge to $a$ being $a^{-1}$.
- **Triangle relations** For each directed triangle $(a, b, c)$ in $L$, two relations: $abc = 1$ and $a^{-1}b^{-1}c^{-1} = 1$.
- **Long cycle relations** For each $n \in S \setminus \{0\}$ and each $(a_1, \ldots, a_\ell) \in \Gamma$, a relation: $a_1^n a_2^n \ldots a_\ell^n = 1$.

**Theorem (I.J. Leary)**

If $L$ is a flag complex with $\pi_1(L) \neq 1$, then groups $G_L(S)$ form $2^{\aleph_0}$ isomorphism classes. If, in addition, $L$ is aspherical and acyclic, then groups $G_L(S)$ are all of type FP.

What is a possible example of an aspherical and acyclic flag simplicial complex $L$?
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What is a possible example of an aspherical and acyclic flag simplicial complex \( L \)?
Take the famous Higman’s group:

\[ H = \langle a, b, c, d \mid a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2 \rangle. \]

Let \( K \) be its presentation complex. It is aspherical and acyclic. Take \( L \) to be the 2nd barycentric subdivision of \( K \). Then \( L \) is a flag simplicial complex with 97 vertices, 336 edges and 240 triangles. Thus,

\[ G_L(S) = \langle 336 \text{ gen’s} \mid 240 \times 2 \text{ triangle relators, } 1 \text{ long relator } \forall n \in S \rangle. \]
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**Theorem (R.Kropholler–Leary–S.)**

*Groups $G_L(S)$ form $2^{\aleph_0}$ classes up to quasi-isometry.*
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**Theorem (R.Kropholler–Leary–S.)**

*Groups \( G_L(S) \) form \( 2^{\aleph_0} \) classes up to quasi-isometry.*

Recall that groups \( G_1, G_2 \) are **quasi-isometric** (qi), if their Cayley graphs are qi as metric spaces, i.e. there exists \( f : \text{Cay}(G_1, d_1) \to \text{Cay}(G_2, d_2) \), and \( A \geq 1, B \geq 0, C \geq 0 \) such that for all \( x, y \in \text{Cay}(G_1) \):

\[ \frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + B, \]

and for all \( z \in \text{Cay}(G_2) \) there exists \( x \in \text{Cay}(G_1) \) such that \( d_2(z, f(x)) \leq C \).
How to distinguish groups up to qi?

Grigorchuk'84: growth functions of groups.

Bowditch'98: a concept of taut loops in Cayley graphs. These are the loops which are not consequences of shorter loops. More formally:

If $\Gamma$ is the Cayley graph of $G$, we can form a sequence of 2–complexes $\Gamma \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \ldots$, where $\Gamma_\ell = \Gamma_{\ell-1} \cup \bigcup |\gamma| \leq \ell \text{Cone}(\gamma)$.

We get $\pi_1(\Gamma) \to \pi_1(\Gamma_1) \to \pi_1(\Gamma_2) \to \ldots$. A loop $\gamma \subset \Gamma$ of length $\ell$ is taut if it lies in the kernel $\ker(\pi_1(\Gamma_\ell) \to \pi_1(\Gamma_{\ell+1}))$.

Let $TL(G) \subset \mathbb{N}$ be the spectrum of lengths of taut loops in the Cayley graph of a group $G$.

Bowditch: Groups $G_1$ and $G_2$ quasi-isometric $\implies TL(G_1)$ and $TL(G_2)$ quasi-isometric in $\mathbb{R}$.

I.e. there exist constants $A, B, N > 0$ such that for every $l_1 \in TL(G_1)$, $l_1 > N$, there exist an $l_2 \in TL(G_2)$ such that $l_1 \in [Al_2, Bl_2]$ and vice versa.
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We get $\pi_1(\Gamma) \rightarrow \pi_1(\Gamma_1) \rightarrow \pi_1(\Gamma_2) \rightarrow \ldots$. A loop $\gamma \subset \Gamma$ of length $\ell$ is **taut** if it lies in the kernel $\ker(\pi_1(\Gamma_\ell) \rightarrow \pi_1(\Gamma_{\ell+1}))$.

Let $TL(G) \subset \mathbb{N}$ be the spectrum of lengths of taut loops in the Cayley graph of a group $G$.

**Bowditch**: Groups $G_1$ and $G_2$ quasi-isometric $\implies$ $TL(G_1)$ and $TL(G_2)$ quasi-isometric in $\mathbb{R}$.

I.e. there exist constants $A, B, N > 0$ such that for every $l_1 \in TL(G_1)$, $l_1 > N$, there exist an $l_2 \in TL(G_2)$ such that $l_1 \in [Al_2, Bl_2]$ and vice versa.
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- (Triangle relations) For each directed triangle \((a, b, c)\) in \( L \), two relations: \( abc = 1 \) and \( a^{-1}b^{-1}c^{-1} = 1 \).
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Now there are uncountably many subsets \( S \) in the above set, and these give \( 2^{\aleph_0} \) quasi-isometry classes of groups \( G_L(S) \).
Connection to the Relation Gap problem

If $G$ is an arbitrary group, $G = \langle a_1, \ldots, a_m | r_1, \ldots, r_n \rangle = F/R$, where $F = F(a_1, \ldots, a_m)$ and $R = \langle \langle r_1, \ldots, r_n \rangle \rangle$. $F$ acts on $R$ by conjugation, so it induces an action of $G$ on $R_{ab} = R/R[R,R]$, the relation module. 

$\text{Rank}(R_{ab})$ as a $ZG$–module $\leq$ min number of normal generators of $R$. The difference of the two is the relation gap.

Bestvina–Brady kernels $BB_L$ have infinite relation gap, and so do $G_L(S)$. 

Open Question: Are there groups with nonzero finite relation gap? Take our group $G = G_L(S)$ with infinite $S$. 

Exhaust $S$ by finite sets: $\emptyset \subset S_1 \subset S_2 \subset \cdots \subset S_G = G_L(\emptyset) \to G_L(S_1) \to G_L(S_2) \to \cdots \to G_L(S)$. 

Fact: they all have the same relation module! Their relation gaps are: $0 \cdots \infty$. 

So groups $G_L(S_i)$ for finite $S_i$ are candidates to have finite relation gap!
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