Acylindrical hyperbolicity of non-elementary convergence groups

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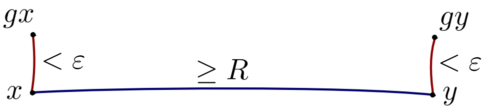
VANDERBILT UNIVERSITY

Acylindrical actions

Definition

An isometric action of a group G on a metric space S is acylindrical if $\forall \epsilon > 0, \exists R, N > 0$ such that $\forall x, y \in S$,

$$d(x,y) \geq R \Rightarrow |\{g \in G \mid d(x,gx) \& d(y,gy) < \epsilon\}| \leqslant N.$$



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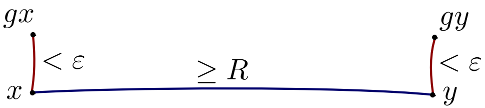

Examples:

- Proper + cobounded $\Rightarrow$ acylindrical
- Any finitely generated group acts on its Cayley graph with respect to any finite generating set
- F_∞ acts on its Cayley graph with respect to any basis

Non-elementariness

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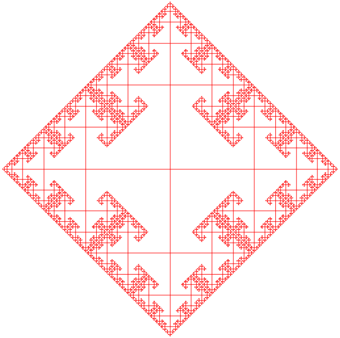
Examples:

- $F(a,b) \curvearrowright \Gamma(F(a,b), \{a,b\})$



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Examples:

- Acylindrical + unbounded orbit + non-virtually cyclic $\Rightarrow$ non-elementary (Osin)

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Examples:

- Non-elementary hyperbolic groups
- Non-elementary relatively hyperbolic groups (Damani-Guirardel-Osin)
- Most mapping class groups of punctured closed orientable surfaces (Bowditch, Mazur-Minsky)
- Outer automorphism groups of non-abelian finite rank free groups (Bestvina-Feighn)
- Many 3-manifold groups (Minasyan-Osin)
- Groups of deficiency at least 2 (Osin)

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Non-examples:

- $G = A \times B$ with $|A| = |B| = \infty$
- $G = A_1 \cdot ... \cdot A_n$ with $A_1, ..., A_n$ amenable (Osin)
- Groups with infinite amenable radicals (Osin)

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Properties:

- $H_b^2(G, \ell^2(G)) \neq 0$ (Hamenstädt,Hull-Osin), Monod-Shalom rigidity theory

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Dehn filling	H/N
M' hyperbolic	$G/\langle\langle N \rangle\rangle$ acylindrically hyperbolic

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- Small cancellation theory (Hull)

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- Space of distinct triples $\Theta_3(M) =$ 3-element subsets of $M = \{(x, y, z) \in M^3 \mid x \neq y, y \neq z, z \neq x\}/S_3$, with quotient topology (non-compact!)

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- Diagonal action: $g\{x, y, z\} = \{gx, gy, gz\}, \forall g \in G, x, y, z \in M$
- Properly discontinuous: $\forall$ compact $K \subset \Theta_3(M)$,

$$|\{g \in G \mid gK \cap K \neq \emptyset\}| < \infty$$

Convergence group

Theorem (Bowditch)

A group G is a convergence group acting on a metrisable compact topological space M if and only if it has the following convergence property: $\forall$ infinite sequence $\{g_n\}$ of distinct elements of G, $\exists$ a subsequence $\{g_{n_k}\}$ and two points $x, y \in M$ such that $g_{n_k}|_{M\setminus\{x\}}$ converges to y locally uniformly.

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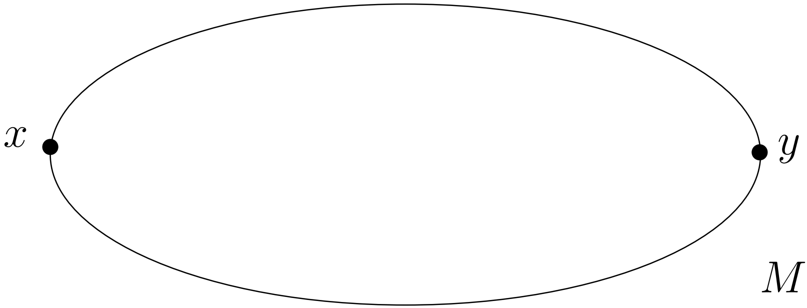

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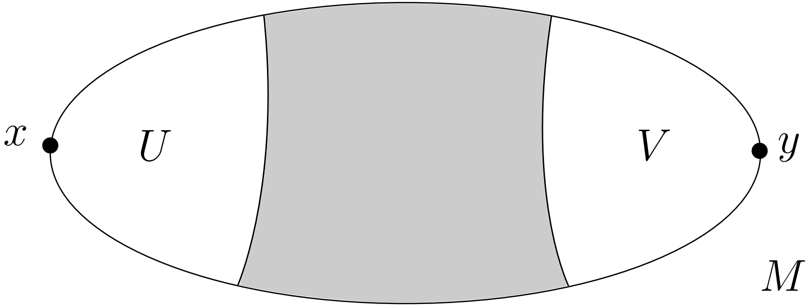

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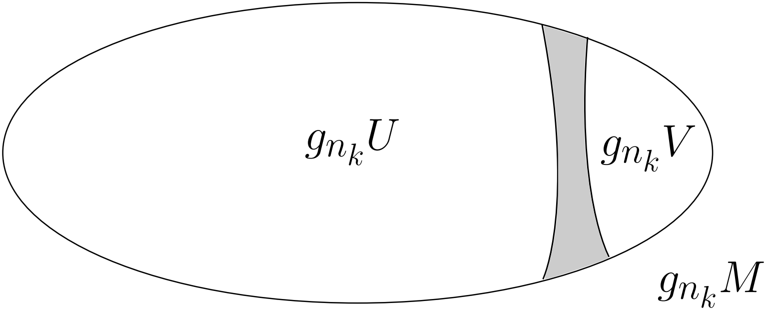

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Examples:

0. $|M| \leqslant 2$
1. Hyperbolic groups acting on their Gromov boundaries (Tukia)
2. Relatively hyperbolic groups acting on their Bowditch boundaries (Bowditch)

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Examples:

- Non-elementary hyperbolic and relatively hyperbolic groups
- Any finitely generated groups whose Floyd boundary has at least 3 points (Karlson)

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Floyd boundary:

Step 1 Pick a Cayley graph $\Gamma(G, X)$ with $|X| < \infty$ and a function $f : \mathbb{N} \to \mathbb{R}_+$ such that $\sum f(n) < \infty, 1 \leqslant f(n)/f(n+1) \leqslant K$
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Step 3 Look at the points added in forming the metric completion

Main result

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Non-elementary convergence groups are acylindrically hyperbolic.

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Common properties proved independently for acylindrically hyperbolic groups and non-elementary convergence groups:

- None of them can be invariably generated. (Hull, Gelander)
- Admits a faithful primitive action if has no non-trivial finite normal subgroup. (Hull-Osin, Gelander-Glasner)
- Has simple reduced C^*-algebra if has no non-trivial finite normal subgroup. (Damani-Guirardel-Osin, Matsuda-Oguni-Yamagata)

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Corollary (Yang)

Let G be a finite generated group whose Floyd boundary has at least 3 points. Then G is acylindrically hyperbolic.

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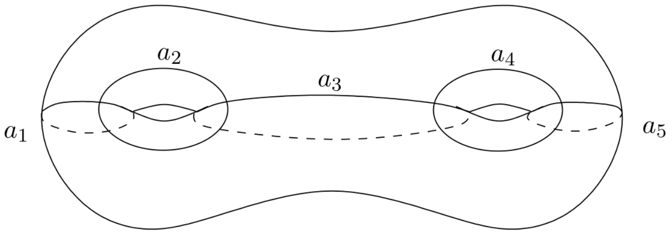
Question

Does every acylindrically hyperbolic group admits a non-elementary convergence action?

A counterexample



Mapping class group of the double torus, generated by $a_1, ..., a_5$, subject to $[a_i, a_j] = 1$ for $|i - j| > 1$ and some other relations

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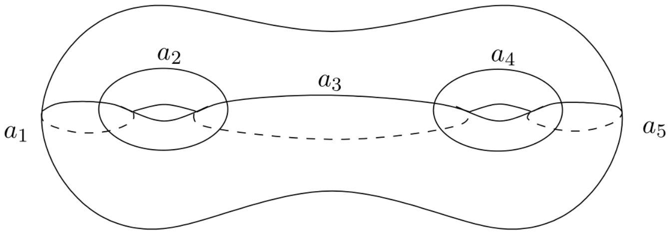

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Facts about convergence groups:

- Any infinite order element fixes one or two points
- Commuting infinite order elements share fixed points

Characterizing acylindrical hyperbolicity

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This action satisfies a generalization of the convergence property, which can be used to characterize acylindrical hyperbolicity.

Thank you for your attention!