Acylindrical hyperbolicity of non-elementary convergence groups

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Acylindrical actions

Definition

An isometric action of a group $G$ on a metric space $S$ is acylindrical if $\forall \epsilon > 0$, $\exists R, N > 0$ such that $\forall x, y \in S$, $d(x, y) \geq R \Rightarrow |\{g \in G | d(x, gx) \& d(y, gy) < \epsilon\}| \leq N$.

Examples:

▶ Proper + cobounded $\Rightarrow$ acylindrical

▶ Any finitely generated group acts on its Cayley graph with respect to any finite generating set

▶ $\mathbb{F}_\infty$ acts on its Cayley graph with respect to any basis
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Non-elementariness

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Examples:

- $F(a, b) \lhd \Gamma(F(a, b), \{a, b\})$
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**Examples:**

- Acylindrical + unbounded orbit + non-virtually cyclic $\Rightarrow$ non-elementary (Osin)
A group is **acylindrically hyperbolic** if it admits a non-elementary acylindrical and isometric action on some hyperbolic space.
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Examples:

- Non-elementary hyperbolic groups
- Non-elementary relatively hyperbolic groups (Damani-Guirardel-Osin)
- Most mapping class groups of punctured closed orientable surfaces (Bowditch, Mazur-Minsky)
- Outer automorphism groups of non-abelian finite rank free groups (Bestvina-Feighn)
- Many 3-manifold groups (Minasyan-Osin)
- Groups of deficiency at least 2 (Osin)
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**Non-examples:**

- $G = A \times B$ with $|A| = |B| = \infty$
- $G = A_1 \cdot \ldots \cdot A_n$ with $A_1, \ldots, A_n$ amenable (Osin)
- Groups with infinite amenable radicals (Osin)
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**Properties:**

- $H^2_b(G, \ell^2(G)) \neq 0$ (Hamenstädt, Hull-Osin), Monod-Shalom rigidity theory
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\[ M \setminus \partial M \text{ hyperbolic} \quad H \text{ hyperbolically embedded into } G \]

Dehn filling

\[ H/N \]

\[ M' \text{ hyperbolic} \quad G/\langle\langle N\rangle\rangle \text{ acylindrically hyperbolic} \]
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- Small cancellation theory (Hull)
Convergence groups

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- Space of distinct triples \( \Theta_3(M) = \) 3-element subsets of \( M = \{(x, y, z) \in M^3 \mid x \neq y, y \neq z, z \neq x\}/S_3 \), with quotient topology (non-compact!)
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- Diagonal action: $g\{x, y, z\} = \{gx, gy, gz\}, \forall g \in G, x, y, z \in M$
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- Diagonal action: $g\{x, y, z\} = \{gx, gy, gz\}, \forall g \in G, x, y, z \in M$
- Properly discontinuous: $\forall$ compact $K \subset \Theta_3(M)$,
  $$|\{g \in G \mid gK \cap K \neq \emptyset\}| < \infty$$
Theorem (Bowditch)

A group $G$ is a convergence group acting on a metrisable compact topological space $M$ if and only if it has the following convergence property: \( \forall \) infinite sequence $\{g_n\}$ of distinct elements of $G$, \( \exists \) a subsequence $\{g_{n_k}\}$ and two points $x, y \in M$ such that $g_{n_k}|_{M\setminus\{x\}}$ converges to $y$ locally uniformly.
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Examples:

0. $|M| \leq 2$
1. Hyperbolic groups acting on their Gromov boundaries (Tukia)
2. Relatively hyperbolic groups acting on their Bowditch boundaries (Bowditch)
A convergence group $G$ acting on a compact metrisable topological space $M$ is **non-elementary** if $G$ does not fix setwise a non-empty subset of $M$ with at most 2 points.
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**Examples:**

- Non-elementary hyperbolic and relatively hyperbolic groups
- Any finitely generated groups whose Floyd boundary has at least 3 points (Karlson)
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Floyd boundary:

**Step 1** Pick a Cayley graph $\Gamma(G, X)$ with $|X| < \infty$ and a function $f : \mathbb{N} \to \mathbb{R}^+$ such that $\sum f(n) < \infty$, $1 \leq f(n)/f(n+1) \leq K$

Example: $f(n) = 1/n^2$
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**Step 2** Rescale $\Gamma(G, X)$ by deeming an edge in $\Gamma(G, X)$ with distance $n$ from 1 to have length $f(n)$
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**Step 2** Rescale $\Gamma(G, X)$ by deeming an edge in $\Gamma(G, X)$ with distance $n$ from 1 to have length $f(n)$.

**Step 3** Look at the points added in forming the metric completion.
Theorem (S)

Non-elementary convergence groups are acylindrically hyperbolic.
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Common properties proved independently for acylindrically hyperbolic groups and non-elementary convergence groups:

- None of them can be invariably generated. (Hull, Gelander)
- Admits a faithful primitive action if has no non-trivial finite normal subgroup. (Hull-Osin, Gelander-Glasner)
- Has simple reduced $C^*$-algebra if has no non-trivial finite normal subgroup. (Damani-Guirardel-Osin, Matsuda-Oguni-Yamagata)
Main result

Theorem (S)
Non-elementary convergence groups are acylindrically hyperbolic.

Corollary (Yang)
Let $G$ be a finite generated group whose Floyd boundary has at least 3 points. Then $G$ is acylindrically hyperbolic.
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Question
Does every acylindrically hyperbolic group admits a non-elementary convergence action?
A counterexample

Mapping class group of the double torus, generated by $a_1, \ldots, a_5$, subject to $[a_i, a_j] = 1$ for $|i - j| > 1$ and some other relations.
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Facts about convergence groups:

- Any infinite order element fixes one or two points
- Commuting infinite order elements share fixed points
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Look at the induced action on $\partial S$. 
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Look at the induced action on $\partial S$.

This action satisfies a generalization of the convergence property, which can be used to characterize acylindrical hyperbolicity.
Thank you for your attention!