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Towards Characterizing Equality in Correlation Inequalities

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1. INTRODUCTION

For most of the basic inequalities in mathematics we know conditions which completely specify the cases of equality. Many combinatorial correlation inequalities are special cases of the AD-inequality, as explained in [3, 8, 10].

However, for this inequality it seems to be difficult to classify the cases of equality. Certainly this is even more difficult for the much more general inequalities of [3] and its relatives, which can be produced by the very same ideas of exploiting notions of expansiveness. In fact, the *equality characterization problem* for these general inequalities constitutes by itself a rich area in combinatorial extremal theory. Closer to home there are the equality characterization problems for inequalities, which are *consequences* of the AD-inequality. Aharoni and Holzman [1] completely settled this for the Marica–Schönheim inequality. Another, though fairly special, still interesting case of AD could be handled by Beck [17].

It seems that the first study of this kind was made by Daykin, Kleitman and West [12], who investigated the inequality

$$|A|\,|B| \leqslant |L|\,|A \wedge B|, \tag{1.1}$$

where the lattice L is a product of finite chains and

$$A \wedge B = \{a \wedge b \colon a \in A,\ b \in B\}.$$

If L is a lattice of subsets of a finite set, then this inequality follows immediately from an inequality known to combinatorialists as Kleitman's inequality [17] and known to probabilists and physicists as Harris's inequality [15]. The more general inequality (1.1) was proved by Anderson [8] and by Greene and Kleitman [14].

Actually, the product of chains is a distributive lattice and (1.1) extends to any distributive lattice, because as such it is a special case of FKG [13]. This was noticed by Seymour and Welsh [19].

FKG in turn is a simple consequence of AD (see [3]). Our renewed interest in correlation inequalities came with our introduction and study of cloud-antichains [5, 6] and the connection to inequality (1.1), which we established in [4].

The main contributions of the present paper are two equality characterization results. They both continue and complete the basic investigations of Daykin, Kleitman and West [12]:

I. On pages 142–143 of [12] there is a detailed discussion about the difficulties in extending the results (Theorems 4 and 5) basic for equality characterization in (1.1) for lattices, which are products of chains of equal length k, to lattices, which are products of chains of varying lengths, say $k_1, k_2, \ldots, k_n$. We overcome these difficulties and also obtain the desired equality characterizations in Theorems 1 and 2 (Section 3). Actually, the corresponding statement (Theorem 6 of [12]) for equal lengths chains contains a flaw (see Example 1 in Section 2). The statement holds, however, if k is a prime.

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II. Hilton [16] proved that if A and B are subsets of a boolean algebra each not containing an element and its complement, and if no element of A is related to any element of B, then $|A \cup B| \leq \frac{1}{2}|L|$. In [12] this was generalized to lattices with a polarity (Theorem 8). Amongst others, the authors called for solution of the equality problem. Our answer is Theorems 3 and 4 of Section 5.

2. Previous Results

We repeat results of Daykin, Kleitman and West [12], which are described in the abstract of [12]. Except for a reference to these theorems in square brackets, we will literally repeat the main part of the abstract:

> 'Let L be a lattice of divisors of an integer (isomorphically, a direct product of chains). We prove $|A|\,|B| \leq |L|\,|A \cap B|$ for any $A,\ B \subset L$ where $|\cdot|$ denotes cardinality and $A \cap B = \{a \cap b : a \in A, b \in B\}$. $|A \cap B|$ attains its minimum for fixed $|A|$, $|B|$ when A and B are ideals [Theorem 2]. $|\cdot|$ can be replaced by certain other weight functions [Theorem 3]. When the n chains are of equal size k, the elements may be viewed as n-digit k-ary numbers. Then for fixed $|A|$, $|B|$, $|A \cap B|$ is minimized when A and B are $|A|$ and $|B|$ smallest n-digit k-ary numbers written backwards and forwards, respectively [Theorem 4]. $|A \cap B|$ for these sets is determined and bounded [Theorem 5]'.

We do not need Theorem 3. Whereas Theorems 2 and 4 are self-explanatory, we give the details of Theorem 5 for the orientation of the reader, even though we do not rely upon it.

THEOREM 5 [12]. *Suppose that L is a product of n chains of size k, $0 \leq \alpha \leq k^n$, $o \leq \beta \leq k^n$. Let $\mu_k(n, \alpha, \beta) = \min\{|A \cap B| : |A| = \alpha, \quad |B| = \beta\}$ and $\varepsilon_k(n, \alpha, \beta) = \mu_k(n, \alpha, \beta) - \alpha\beta/k^n$. If $pk^{n-1} < \alpha \leq (p+1)k^{n-1}$ and $\beta \equiv r \bmod k$, then:*

(i)
$$\mu_k(n, \alpha, \beta) = \mu_k\left(n-1, \alpha - pk^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) + \begin{cases} 0, & p = 0, \\ \sum_{j=0}^{p-1} \left\lceil \frac{\beta - j}{k} \right\rceil, & p > 0; \end{cases}$$

(ii)
$$\varepsilon_k(n, \alpha, \beta) = \varepsilon_k\left(n-1, \alpha - pk^{n-1}, \left\lceil \frac{\beta - p}{k} \right\rceil\right) + \begin{cases} r\left\lceil 1 - \frac{\alpha}{k^n} \right\rceil, & 0 \leq r \leq p, \\ (k-r)\frac{\alpha}{k^n}, & p < r < k. \end{cases}$$

Furthermore,

(iii)
$$\varepsilon_k(n, k^n - \alpha, k^n - \beta) = \varepsilon_k(n, \alpha, \beta);$$

(iv)
$$\mu_k(n, k^n - \alpha, k^n - \beta) = \mu_k(n, \alpha, \beta) + k^n - \alpha - \beta;$$

and, finally,

(v)
$$0 \leq \varepsilon_k(n, \alpha, \beta) \leq kn/4.$$

REMARK. 1. In the notation of this theorem, equality characterization for (1.1) means to find necessary and sufficient conditions for

$$\varepsilon_k(n, \alpha, \beta) = 0. \tag{2.1}$$

Theorem 6 of [12] asserts that (2.1) holds iff
(i) $k^n \mid \alpha\beta$, $k \mid \alpha$ and $k \mid \beta$, or
(ii) trivially, α or β is k^n or 0.
This is true if k is a prime. For composite k the conditions (i) and (ii) are necessary, but not sufficient.

EXAMPLE 1. Choose $n = 3$, $k = 4$ and $\alpha = \beta = 8$. These numbers satisfy (i). However, for all ideals $A, B \subset L$ with $|A| = |B| = 8$, inspection shows that $|A \wedge B| > 1 = |A|\,|B| \cdot 4^{-3}$. We shall see that (i) has to be replaced by
(i*) there are positive integers i, α_1 and β_1 such that

$$\alpha = k^i \cdot \alpha_1 \qquad \text{and} \qquad \beta = k^{n-i}\beta_1.$$

3. EQUALITY CHARACTERIZATION IN $|A \wedge B| \geq |A|\,|B|\,L^{-1}$

Let $L = [k_1] \times \cdots \times [k_n]$ be the lattice defined as direct product of chains $[k_i]$ of length $k_i \geq 2$ ($i = 1, \ldots, n$). For any $I \subset [n] = \{1, 2, \ldots, n\}$, we define the sublattice

$$L_I \triangleq \prod_{i \in I} [k_i]. \tag{3.1}$$

THEOREM 1 (equality characterization within ideals). *For ideals $A, B \subset L$, equality in* (1.1) *holds iff*:
(a) *A or B equals $\varnothing$ or L; or*
(b) *there exists an $I \subset [n]$, $0 < |I| < n$, such that*

$$A = L_I \times A_1 \qquad \text{and} \qquad B = B_1 \times L_{[n]\setminus I}.$$

So, $|A| = \prod_{i \in I} k_i \cdot |A_1|$ and $|B| = \prod_{i \in [n]\setminus I} k_i \cdot |B_1|$, for some ideals $A_1 \subset L_{[n]\setminus I}$ and $B_1 \subset L_I$.

THEOREM 2 (equality characterization for general sets in terms of cardinalities). *Equality in* (1.1) *is assumed for sets of cardinality α and β iff*:
(a) *α or β is 0 or $\prod_{i=1}^n k_i$; or*
(b) *there exists an $I \subset [n]$, $0 < |I| < n$, and there exist positive integers α_1 and β_1 with*

$$\alpha = \prod_{i \in I} k_i \cdot \alpha_1, \qquad \beta = \prod_{i \in [n]\setminus I} k_i \cdot \beta_1.$$

Note that Theorem 2 is an immediate consequence of Theorem 2 of [12], mentioned in Section 2 and Theorem 1. We need here another well-known result, which is now also a child of AD (see [3]).

CHEBYSHEV'S INEQUALITY. *Suppose that we have the two decreasing sequences of non-negative numbers*

$$u_1 \geq u_2 \geq \cdots \geq u_m \geq 0 \qquad \text{and} \qquad x_1 \geq x_2 \geq \cdots \geq x_m \geq 0.$$

Then,

$$\sum_{i=1}^m u_i x_i \geq m^{-1} \sum_{i=1}^m u_i \cdot \sum_{i=1}^m x_i. \tag{3.2}$$

Moreover, equality holds iff at least one of the conditions $u_1 = u_2 = \cdots = u_m$ *or* $x_1 = x_2 = \cdots x_m$ *holds.*

PROOF OF THEOREM 1. Clearly, condition (a), and also condition (b), imply equality in (1.1). The issue is to prove that equality implies (a) or (b).

Suppose then that $A \neq \phi$, $B \neq \phi$ and that (the case $n = 1$ being trivial) $n \geqslant 2$. For any $r \in [n]$ and $i \in [k_r]$, define

$$A_i = \{a^n \in A : a_r = i\}, \qquad B_i = \{b^n \in B : b_r = i\}, \tag{3.3}$$

Clearly,

$$A = \bigcup_{i=1}^{k_r} A_i, \qquad B = \bigcup_{i=1}^{k_r} B_i \tag{3.4}$$

and

$$A_i \cap A_j = \phi, \qquad B_i \cap B_j = \phi \qquad \text{for } i \neq j. \tag{3.5}$$

Therefore

$$|A \cap B| = \sum_{i=1}^{k_r} |A_i \cap B_i|. \tag{3.6}$$

Now set $A_i = \{i\} \times A_i^*$, $B_i = i \times B_i^*$, where $A_i^*, B_i^* \subset L^{(r)} \triangleq \prod_{j \neq r} [k_j]$, $|A_i^*| = |A_i|$, $|B_i^*| = |B_i|$ and $|A_i \cap B_i| = |A_i^* \cap B_i^*|$. Since A and B are ideals, also A_i^*, B_i^* $(i = 1, \ldots, k_r)$ are ideals and

$$A_1^* \supset A_2^* \supset \cdots \supset A_{k_r}^*; \qquad B_1^* \supset B_2^* \supset \cdots \supset B_{k_r}^*. \tag{3.7}$$

Therefore we have

$$|A_1| \geqslant |A_2| \geqslant \cdots \geqslant A_{k_r}|, \qquad |B_1| \geqslant |B_2| \geqslant \cdots \geqslant |B_{k_r}|. \tag{3.8}$$

Since for ideals C and D always

$$C \cap D = C \wedge D, \tag{3.9}$$

we conclude from (1.1) that, for $i = 1, \ldots, k_r$,

$$|A_i^* \cap B_i^*| \geqslant \frac{|A_i^*|\,|B_i^*|}{\prod_{i \neq r} k_j} = \frac{|A_i|\,|B_i|}{\prod_{i \neq r} k_j}. \tag{3.10}$$

Hence, by (3.6) and the following definitions,

$$|A \cap B| = \sum_{i=1}^{k_r} |A_i^* \cap B_i^*| \geqslant \frac{1}{\prod_{j \neq r} k_k} \sum_{i=1}^{k_r} |A_i|\,|B_i|.$$

Under the conditions (3.8) we can now apply Chebyshev's inequality, which yields

$$|A \cap B| \geqslant \frac{1}{\prod_{j \neq r} k_j} \frac{\sum_{i=1}^{k_r} |A_i| \sum_{i=1}^{k_r} |B_i|}{k_r} = \frac{|A|\,|B|}{|L|}.$$

In the case $|A \cap B| = |A|\,|B|/|L|$, therefore, necessarily

$$|A_i^* \cap B_i^*| = \frac{|A_i|\,|B_i|}{\prod_{j \neq r} k_j} \qquad \text{for } i = 1, 2, \ldots, k_r$$

and by the equality characterization in Chebysev's inequality

$$|A_1| = |A_2| = \cdots = |A_{k_r}| = |A|/k_r \qquad \text{or} \qquad |B_1| = |B_2| = \cdots = |B_{k_r}| = |B|/k_r$$

holds. Then define $I \subset [n]$ as the set of all positions for which $|A_1| = \cdots = |A_{k_i}|$ $(i \in I)$. Clearly, then, $|B_1| = \cdots = |B_{k_j}|$ $(j \in [n] \setminus I)$.

If now $I = [n]$, then $A = L$, and if $I = \phi$, then $B = L$, and we are not under our supposition.

Finally, if $0 < |I| < n$, we conclude with (3.7) that $A_1^* = A_2^* = \cdots = A_{k_r}^*$ for $r \in I$ and that $B_1^* = B_2^* = \cdots = B_{k_r}^*$ for $r \in [n] \setminus I$.

Therefore we must have

$$A = L_I \times A_1 \qquad \text{and} \qquad B = B_1 \times L_{[n]\setminus I},$$

where $A_1 \subset L_{[n]\setminus I}$ and $B_1 \subset L_I$ are ideals. □

4. Auxiliary Results for Equality Characterization for Cloud-antichains of Length 2 Satisfying a Polarity Constraint

As indicated under II of the Introduction, we have obtained a second equality characterization in Theorem 2. We introduce first some notions from [4] and [12].

Let L be a distributive lattice. For a subset C of L let $u(C)$ and $l(C)$ denote the filter and the ideal generated by C; that is,

$$u(C) = \{c \in L: \exists a \in C, a \leqslant c\}, \tag{4.1}$$

$$l(C) = \{x \in L: \exists a \in C, a \geqslant c\}. \tag{4.2}$$

By a polarity σ of the lattice L (in the sense of [11]) is meant an order-reversing bijection, the square of which is the identity: that is, $a \leqslant b$ implies $\sigma b \leqslant \sigma a$ and $\sigma(\sigma(a)) = a$. For example, complementation is a polarity. For $A \subset L$ we set $\sigma(A) = \{\sigma a: a \in A\}$. If $a \nleqslant b$ and $b \nleqslant a$ we write $a \supset\!\!\!\mid\!\!\!\subset b$. If for $A, B \subset L$ and for all $a \in A$, $b \in B$, we have $a \supset\!\!\!\mid\!\!\!\subset b$, then we write $A \supset\!\!\!\mid\!\!\!\subset B$.

Let us consider a problem studied in [12], which generalizes the problem considered by Hilton [16] and which is mentioned under II in the Introduction.

For $A, B \subset L$ we write $A \Rightarrow\!\!\!\mid\!\!\!\Leftarrow B$, if

$$A \supset\!\!\!\mid\!\!\!\subset B \tag{4.3}$$

and if

$$a \in A \text{ implies } \sigma(a) \notin A \text{ and } b \in B \text{ implies } \sigma(b) \notin B. \tag{4.4}$$

We also speak of a polar image free cloud-antichain.

Theorem 8 of [12] says that $A \Rightarrow\!\!\!\mid\!\!\!\Leftarrow B$ implies

$$|A| + |B| \leqslant \pi \leqslant \tfrac{1}{2}|L|, \tag{4.5}$$

when π is the number of non-trivial orbits of σ (i.e. unordered pairs $\{e, \sigma e\}$ with $e \neq \sigma(e)$).

It was asked in [12]: 'Which A, B achieve the maximum π?'.

Here we completely answer this question, when L is a direct product of chains of arbitrary lengths and polarity is complementation.

At first we present auxiliary results, which are true for any distributive lattice and any polarity σ.

Suppose that for $A, B \subset L$, $A \Rightarrow\!\!\!\mid\!\!\!\Leftarrow B$ and

$$|A| + |B| = \pi. \tag{4.6}$$

Let (A^*, B^*) be any pair of bisaturated extensions of (A, B) with respect to (4.3); that is, $A \subseteq A^*$, $B \subseteq B^*$, $A^* \supset\!\!\!\mid\!\!\!\subset B^*$ and A^*, B^* are maximal. obviously, A^* and B^* are both convex. Note that the pair (A^*, B^*) is not uniquely defined.

However, we can write

$$A^* = A \cup \sigma(A_1) \cup D_1, \qquad B^* = B \cup \sigma(B_1) \cup D_2,$$

where $D_1 \cup D_2 \subset D = \{a \in L: \sigma(a) = a\}$, $(A_1 \cup B_1) \cap D = \varnothing$ and $A_1 \subset A$, $B_1 \subset B$, since if, say, $a \in \sigma(A_1)$ and $\sigma(a) \notin A$, we could take sets $A' = A \cup \{a\}$, B for which (4.3), (4.4) hold and $|A'| + |B| = \pi + 1$, in contradiction to (4.5).

So A^* and B^* can be represented as

$$A^* = A_1 \cup \sigma(A_1) \cup A_2 \cup C \cup D_1, \qquad B^* = B_1 \cup \sigma(B_1) \cup B_2 \cup \sigma(C) \cup D_2,$$

where $\sigma(A_2 \cup B_2) \cap (A^* \cup B^*) = \varnothing$.

Since (A^*, B^*) satisfies (4.3) and is bisaturated, necessarily

$$E = l(A^*) \setminus A^* = l(B^*) \setminus B^* = l(A^*) \cap l(B^*)$$

and

$$F = u(A^*) \setminus A^* = u(B^*) \setminus B^* = u(A^*) \cap u(B^*)$$

(see also [4]).

Clearly, no element of E is greater than an element from $L \setminus E$, because E is an ideal, and no element of F is smaller than an element from $L \setminus F$, because F is a filter. Formally,

$$E \cap (u(A^*) \cup u(B^*)) = \varnothing \qquad \text{and} \qquad F \cap (l(A^*) \cup l(B^*)) = \varnothing.$$

E and F are unions of the following sets:

$$E = R \cup D_3 \cup \sigma(A_2^\circ) \cup \sigma(B_2^\circ) \qquad \text{and} \qquad F = \sigma(R) \cup D_4 \cup \sigma(A_2^1) \cup \sigma(B_2^1),$$

where

$$R \subset L \setminus D, \qquad D_3 \subset D, \qquad D_4 \subset D, \qquad A_2^\circ \cup A_2^1 = A_2,$$
$$A_2^\circ \cap A_2^1 = \varnothing, \qquad B_2^\circ \cup B_2^1 = B_2, \qquad B_2^\circ \cap B_2^1 = \varnothing$$

Lemma 1.

$$A_2^\circ \supset\!\!\!|\!\!\!\subset \sigma(A_2^1), \qquad A_2^\circ \supset\!\!\!|\!\!\!\subset \sigma(B_2^1), \qquad A_2^1 \supset\!\!\!|\!\!\!\subset \sigma(B_2^\circ), \qquad B_2^\circ \supset\!\!\!|\!\!\!\subset \sigma(B_2^1),$$
$$(A^* \cup B^*) \setminus (A_2^\circ \cup B_2^\circ) \supset\!\!\!|\!\!\!\subset D_3 \qquad \textit{and} \qquad (A^* \cup B^*) \setminus (A_2^1 \cup B_2^1) \supset\!\!\!|\!\!\!\subset D_4.$$

Proof. Suppose that there exists an $a \in A_2^\circ$ and an $a_1 \in \sigma(A_2^1)$ for which $a > a_1$ or $a < a_1$. $a > a_1$ is impossible, because $a \in A_2^\circ \subset A^*$ and $a_1 \in \sigma(A_2^1) \subset F$. Also, $a < a_1$ or, equivalently, $\sigma(a) > \sigma(a_1)$, is impossible, because $\sigma(a) \in \sigma(A_2^\circ) \subset E$ and $\sigma(a_1) \in A_2^1 \subset A^*$. Hence $A_2^\circ \supset\!\!\!|\!\!\!\subset \sigma(A_2^1)$. One proves the other relations similarly. □

We have

$$\pi = |C| + |A_1| + |A_2| + |B_1| + |B_2| + |R|, \qquad D = D_1 \cup D_2 \cup D_3 \cup D_4$$

and

$$|L| = 2\pi + |D|.$$

From assumption (4.6) we have $\pi = |A| + |B| = |A_1| + |A_2| + 2\,|C| + |B_1| + |B_2|$ and hence

$$|R| = |C|. \tag{4.7}$$

We now consider $l(C) \cap l(\sigma C)$. In Theorem 8 of [12] it is shown that

$$l(C) \cap l(\sigma C) \subset R, \tag{4.8}$$

and so $|l(C) \cap l(\sigma C)| \leq |R| = C$, by (4.7).

Also (see [12, Lemma 2]) it has been proved that

$$|C| \leq \frac{|l(C)| \cdot |l(\sigma C)|}{|L|} \leq |l(C) \cap l(\sigma C)|,$$

which, together with (4.7) and (4.8), gives us

$$|R| = |C| = \frac{|l(C)| \cdot |l(\sigma C)|}{|L|} = |l(C) \cap l(\sigma C)| \tag{4.9}$$

and

$$l(C) \cap l(\sigma C) = R. \tag{4.10}$$

LEMMA 2. *Suppose that* (4.6) *holds. Then*:

(i) $$l(C) = C \cup A_2^1 \cup \sigma(B_2^\circ) \cup R, \qquad |l(C)| = 2\,|C| + |A_2^1| + |B_2^\circ|,$$

$$l(\sigma C) = \sigma(C) \cup \sigma(A_2^\circ) \cup B_2^1 \cup R, \qquad |l(\sigma C)| = 2\,|C| + |A_2^\circ| + |B_2^1|.$$

(ii) $$(|A_2^1| + |B_2^\circ|)(|A_2^\circ| + |B_2^1|) = 2 \cdot |C| \cdot |A_1| + 2 \cdot |C| \cdot |B_1| + |C| \times (|D_1| + |D_2| + |D_3| + |D_4|).$$

PROOF. (i) Let us introduce $T = C \cup A_2^1 \cup \sigma(B_2^\circ)$, $S = \sigma(C) \cup \sigma(A_2^\circ) \cup B_2^1$ and show that $T \supset\!\!\!\!\!\times\!\!\!\!\subset S$. Since $A^* \supset\!\!\!\!\!\times\!\!\!\!\subset B^*$ and $\sigma(A^*) \supset\!\!\!\!\!\times\!\!\!\!\subset \sigma(B^*)$, we have $C \supset\!\!\!\!\!\times\!\!\!\!\subset \sigma(C)$, $C \supset\!\!\!\!\!\times\!\!\!\!\subset B_2^1$, $C \supset\!\!\!\!\!\times\!\!\!\!\subset \sigma(A_2^\circ)$, $A_2^1 \supset\!\!\!\!\!\times\!\!\!\!\subset \sigma(C)$, $A_2^1 \supset\!\!\!\!\!\times\!\!\!\!\subset B_2^1$, $\sigma(B_2^\circ) \supset\!\!\!\!\!\times\!\!\!\!\subset \sigma(C)$ and $\sigma(B_2^\circ) \supset\!\!\!\!\!\times\!\!\!\!\subset \sigma(A_2^\circ)$. Also, according to Lemma 1, $A_2^1 \supset\!\!\!\!\!\times\!\!\!\!\subset \sigma(A_2^\circ)$ and $\sigma(B_2^\circ) \supset\!\!\!\!\!\times\!\!\!\!\subset B_2^1$. Hence $T \supset\!\!\!\!\!\times\!\!\!\!\subset S$.

We now consider $l(T)$ and $l(S)$. Clearly, $l(C) \subseteq l(T)$ and $l(\sigma(C)) \subseteq l(S)$.

Let $l(T) = T \cup W_1$ and $l(S) = S \cup W_2$ for some $W_1, W_2 \subset L$. Let us prove that $W_1 \cup W_2 \subset R$. For this it is sufficient to show that

$$(l(S) \cup l(T)) \cap (L \setminus (T \cup S \cup R)) = \varnothing, \qquad \text{since } T \supset\!\!\!\!\!\times\!\!\!\!\subset S.$$

One has

$$L \setminus (T \cup S \cup R) = F \cup A_1 \cup \sigma(A_1) \cup B_1 \cup \sigma(B_1) \cup A_2^\circ \cup B_2^\circ \cup D_1 \cup D_2 \cup D_3.$$

Since $T \cap F = \varnothing$, here $l(T) \cap F = \varnothing$.

Suppose that $a \in A_1 \cup \sigma(A_1)$ and $a \in l(T) = l(C) \cup l(A_2^1) \cup l(\sigma(B_2^\circ))$. Then $a \notin l(C) \cup l(\sigma(B_2^\circ))$, because $(A_1 \cup \sigma(A_1)) \supset\!\!\!\!\!\times\!\!\!\!\subset C \cup \sigma(B_2)$. If $a \in l(A_2^1)$, then there exists an $a_1 \in A_2^1$ and an $a < a_1$ with $\sigma(a) > \sigma(a_1)$. This is impossible, because $\sigma(a) \in A_1 \cup \sigma(A_1) \subset A^*$ and $\sigma(a_1) \in \sigma(A_2^1) \subset F$. Hence, $l(T) \cap (A_1 \cup \sigma(A_1)) = \varnothing$. Similarly, $l(T) \cap (B_1 \cup \sigma(B_1)) = \varnothing$.

Suppose that $a \in A_2^\circ$ and $a \in l(T) = l(C) \cup l(A_2^1) \cup l(\sigma(B_2^\circ))$. This means that there exists an $a_1 \in C \cup A_2^1 \cup \sigma(B_2^\circ)$ for which $a < a_1$ or (equivalently) $\sigma(a) > \sigma(a_1)$, which is impossible, because $\sigma(a) \in \sigma(A_2^\circ) \subset E$ and $\sigma(a_1) \in \sigma(C) \cup \sigma(A_2^1) \cup B_2^\circ \subset L \setminus E$. Therefore we have $l(T) \cap A_2^\circ = \varnothing$ and, similarly, $l(T) \cap B_2^\circ = \varnothing$.

Suppose that $a \in D_1$ and $a \in l(T)$. This means that there exists an $a_1 \in C \cup A_2^1 \cup \sigma(B_2^\circ)$ for which $a < a_1$. Clearly, $a_1 \notin C \cup \sigma(B_2^\circ)$, because $D_1 \supset\!\!\!\!\!\times\!\!\!\!\subset (C \cup \sigma(C) \cup B_2 \cup \sigma(B_2))$. If $a_1 \in A_2^1$ and $a < a_1$, then $\sigma(a) > \sigma(a_1)$, which is impossible, because $\sigma(a) = a \in D_1 \subset A^*$ and $\sigma(a_1) \in \sigma(A_2^1) \subset F$. Therefore $l(T) \cap D_1 = \varnothing$ and, similarly, $l(T) \cap D_2 = \varnothing$, $l(T) \cap D_3 = \varnothing$.

Thus $l(T) \cap (L \setminus (T \cup S \cup R)) = \varnothing$ and hence $W_1 \subset R$. Similarly, it can be proved that $l(S) \cap (L \setminus (T \cup S \cup R)) = \varnothing$ and $W_2 \subset R$. Therefore we have

$$l(T) \cap l(S) \subset R.$$

However, since $l(C) \subseteq l(T)$ and $l(\sigma(C)) \subseteq l(S)$, from (4.10) we conclude that

$$l(T) \cap l(S) = R.$$

Now we apply (4.9) and obtain

$$|C| = |R| = \frac{|l(C)| \cdot |l(\sigma C)|}{|L|} \leqslant \frac{|l(T)| \cdot |l(S)|}{|L|} \leqslant |l(T) \cap l(S)|$$
$$= |l(C) \cap l(\sigma(C))| = |R| = |C|.$$

Therefore $|l(C)| = |l(T)|$, $|l(\sigma(C)| = |l(S)|$ and since $l(C) \subseteq l(T)$ and $l(\sigma(C)) \subseteq l(S)$, necessarily

$$l(C) = l(T) = C \cup A_2^1 \cup \sigma(B_2^\circ) \cup R, \qquad |l(C)| = 2\,|C| + |A_2^1\} + |B_2^\circ|$$

and

$$l(\sigma(C)) = l(S) = \sigma(C) \cup \sigma(A_2^\circ) \cup B_2^1 \cup R, \qquad |l(\sigma C)| = 2\,|C| + |A_2^\circ| + |B_2^1|.$$

This proves (i).

(ii) follows from (4.9) and (i) after simplification. □

LEMMA 3. *Suppose that* (4.6) *holds. Then*:

(i)
$$|A_2^1| \cdot |B_2^1| = |C| \cdot |D_4|, \qquad |A_2^\circ| \cdot |B_2^\circ| = |C| \cdot |D_3|,$$
$$|A_2^\circ| \cdot |A_2^1| = 2 \cdot |C| \cdot |A_1| + |C| \cdot |D_1|,$$
$$|B_2^\circ| \cdot |B_2^1| = 2 \cdot |C| \cdot |B_1| + |C| \cdot |D_2|,$$

(ii)
$$|l(A^*) \cap l(B^*)| = |C| + |D_3| + |A_2^\circ| + |B_2^\circ| = \frac{|l(A^*)| \cdot |l(B^*)|}{|L|}.$$

PROOF. We consider the sets

$$P_1 = C \cup A_2^1, \qquad P_2 = C \cup A_2^\circ, \qquad P_3 = C \cup A_2^\circ, \qquad P_4 = C \cup \sigma(B_2^\circ),$$
$$Q_1 = \sigma(C) \cup B_2^1, \qquad Q_2 = \sigma(C) \cup B_2^\circ, \qquad Q_3 = \sigma(C) \cup \sigma(A_2^1), \qquad Q_4 = \sigma(C) \cup B_2^1.$$

It can be verified (using $A^* \ni\!\!\!\in B^*$ and Lemma 1) that $P_i \ni\!\!\!\in Q_i$ $(i = 1, 2, 3, 4)$.

We are interested in $|l(P_i) \cap l(Q_i)|$ and $|u(P_i) \cap u(Q_i)|$, for $i = 1, 2, 3, 4$. Since $P_1 \subset A^*$ and $Q_1 \subset B^*$, we have

$$l(P_1) \cap l(Q_1) \subset E = \sigma(A_2^\circ) \cup \sigma(B_2^\circ) \cup D_3 \cup R$$

and

$$u(P_1) \cap u(Q_1) \subset F = \sigma(A_2^1) \cup \sigma(B_2^1) \cup D_4 \cup \sigma(R).$$

According to Lemma 1, $P_1 \ni\!\!\!\in \sigma(A_\circ) \cup D_3$ and $Q_1 \ni\!\!\!\in \sigma(B_\circ) \cup D_3$. Therefore

$$|l(P_1) \cap l(Q_1)| = C \quad \text{and} \quad |u(P_1) \cap u(Q_1)| \leqslant |A_2^1| + |B_2^1| + |D_4| + |C|. \tag{4.11}$$

Similarly,

$$|l(P_2) \cap l(Q_2)| \leqslant |A_2^\circ| + |B_2^\circ| + |D_3| + |C| \quad \text{and} \quad |u(P_2) \cap u(Q_2)| = |C|. \tag{4.12}$$

We also verify that

$$l(P_3) \cap l(Q_3) \subset A_1 \cup \sigma(A_1) \cup \sigma(A_2^\circ) \cup A_2^1 \cup D_1 \cup R \quad \text{and} \quad u(P_3) \cap u(Q_3) = \sigma(R)$$

or

$$|l(P_3) \cap l(Q_3)| \leq 2 \cdot |A_1| + |A_2^0| + |A_2^1| + |D_1| + |C|$$
$$= 2\,|A_1| + |A_2| + |D_1| + |C| \quad \text{and} \quad |u(P_3) \cap u(Q_3)| = |C|. \quad (4.13)$$

Furthermore

$$l(P_4) \cap l(Q_4) = R \quad \text{and} \quad u(P_4) \cap u(Q_4) \subset B_1 \cup \sigma(B_1) \cup B_2^0 \cup \sigma(B_2^1) \cup D_2$$

or

$$|l(P_4) \cap l(Q_4)| = |C| \quad \text{and} \quad |u(P_4) \cap u(Q_4)| \leq 2\,|B_1| + |B_2| + |D_2| + |C|. \quad (4.14)$$

Now, since L is a distributive lattice, we can apply the AD inequality and obtain

$$|P_i| \cdot |Q_i| \leq |P_i \vee Q_i| \cdot |P_i \wedge Q_i| \leq |u(P_i) \cap u(Q_i)| \cdot |l(P_i) \cap l(Q_i)| \quad \text{for } i = 1, 2, 3, 4.$$

From (4.11)–)4.14) we have that

$$\begin{aligned} &|A_2^1| \cdot |B_2^1| \leq |C| \cdot |D_4|, \qquad |A_2^0| \cdot |B_2^0| \leq |C| \cdot |D_3|, \\ &|A_2^0| \cdot |A_2^1| \leq 2\,|C| \cdot |A_1| + |C| \cdot |D_1|, \qquad |B_2^0| \cdot |C_2^1| \leq 2\,|C| \cdot |B_1| + |C| \cdot |D_2|. \end{aligned} \quad (4.15)$$

Now (i) follows from (4.15) and (ii) in Lemma 2. (ii) follows from (i) after simplification. □

REMARK. 2. Let us define $s^*(L)$ as the smallest real number s^* such that $|M| \cdot |N| \leq S^*\,|M \cap N|$ for all ideals $M, N \subset L$ with $M \nsubseteq N$, $N \nsubseteq M$. From (ii) in Lemma 3 we draw a simple conclusion.

COROLLARY. *Assume that* $s^* < |L|$. *Then* (4.6) *holds iff* $|A| \cdot |B| = 0$, *i.e. one of* A, B *is* $\varnothing$, *and the other consists of* π *non-trivial orbits.*

EXAMPLE 2. Let L be any lattice for which (1.1) holds. We consider a new lattice $L' = L \cup \{\xi\}$, where element ξ is defined to satisfy $\xi \geq u$ for all $u \in L$. Clearly, L' is a lattice for which $|M| \cdot |N| \leq |L'| \cdot |M \cap N|$ for all ideals $M, N \subset L'$, but $s^* < |L'|$.

We present our last important auxiliary result.

LEMMA 4. *Suppose that* (4.6) *holds,* $0 < |A| \leq |B|$ *and* $|S| \leq 1$. *Then*

$$A^* = A.$$

PROOF. Let $|D| = 0$ or, equivalently, $D_1 = D_2 = D_3 = D_4 = 0$. We apply Lemma 3:

$$|A_2^1| \cdot |B_2^1| = 0, \qquad |A_2^0| \cdot |B_2^0| = 0, \qquad |A_2^0| \cdot |A_2^1| = 2\,|C| \cdot |A_1|,$$
$$|B_2^0| \cdot |B_2^1| = 2\,|C| \cdot |B_1|.$$

Suppose that $|A_1| \neq 0$. Then $|A_2^0| \neq 0$, $|A_2^1| \neq 0$ (since always $C \neq \varnothing$, if $|A| > 0$). Hence $|B_2^1| = |B_2^0| = |B_1| = 0$, which contradicts $|A| \leq |B|$. Therefore, if $|D| = 0$, then $|A_1| = 0$ and hence $A^* = A$.

Now let $|D| = 1$. There are four possibilities:

(i) Suppose first that $D_1 = 1$ and $D_2 = D_3 = D_4 = 0$. Then Lemma 3 gives

$$|A_2^1| \cdot |B_2^1| = 0, \qquad |A_2^0| \cdot |B_2^0| = 0, \qquad |A_2^0| \cdot |A_2^1| = 2\,|C| \cdot |A_1| + |C| > 0,$$
$$|B_2^0| \cdot |B_2^1| = 2\,|C| \cdot |B_1|.$$

We have $|A_2^0| \neq 0$, $|A_2^1| \neq 0$ and hence $|B_2^0| = |B_2^1| = |B_1| = 0$, which contradicts $|A| \leq |B|$. Therefore this case is impossible.
(ii) Next, suppose that $D_1 = 0$. $D_2 = 1$ and $D_3 = D_4 = 0$. then we have

$$|A_2^1| \cdot |B_2^1| = 0, \qquad |A_2^0| \cdot |B_2^0| = 0, \qquad |A_2^0| \cdot |A_2^1| = 2\,|C| \cdot |A_1|,$$
$$|B_2^0| \cdot |B_2^1| = 2\,|C| \cdot |B_1| + |C| > 0.$$

Hence $|B_2^0| \neq 0$ and $|B_2^1| \neq 0$ imply that $|A_2^0| = |A_2^1| = |A_1| = 0$ and $A^* = A$.
(iii) Now suppose that

$$D_1 = D_2 = 0, \qquad D_3 = 1, \qquad D_1 = 0.$$

Then we have

$$|A_2^1| \cdot |B_2^1| = 0, \qquad |A_2^0| \cdot |B_2^1| = |C| > 0, \qquad |A_2^0| \cdot |A_2^1| = 2\,|C| \cdot |A_1|,$$
$$|B_2^0| \cdot |B_2^1| = 2\,|C| \cdot |B_1|.$$

(iv) In the case $|A_1| \neq 0$ necessarily $|A_2^1| \neq 0$ and $|B_2^1| = |B_1| = 0$. From $|A_2^0| \cdot |A_2^1| = 2\,|C| \cdot |A_1| > 0$ and $|A_2^0| \cdot |B_2^0| = |C| > 0$ we conclude that $|B_2^0| = |A_2^1|/2\,|A_1| < |A_2^1|$ and hence $|B| = |C| + |B_2^0| < |C| + |A_2^1| < |A|$, which is a contradiction.

Therefore, $|A_1| = 0$ and hence $A^* = A$. Finally, when $D_1 = D_2 = D_3 = 0$, $D_4 = 1$; similarly, we have $A^* = A$. □

5. The Main Results

Let $L = \prod_{i=1}^n [0, 1, \ldots, k_{i-1}]$ be a direct product of n chains and let the polarity σ be complementation; that is, for $a = (a_1, a_2, \ldots, a_n) \in L$,

$$\sigma(a) = \bar{a} = (k_1 - 1 - a_1, \ldots, k_n - 1 - a_n). \tag{5.1}$$

Obviously, if $2 \mid \prod_1^n k_i$, then $D = \varnothing$ (there are no trivial orbits), and if $2 \nmid \prod_1^n k_i$, then

$$D = \left\{\left(\frac{k_1 - 1}{2}, \ldots, \frac{k_n - 1}{2}\right)\right\}$$

and $|D| = 1$.

Theorem 3 (equality characterization in terms of numbers, $\prod_1^n k_i$ even). *Suppose that $L = \prod_{i=1}^n [0, 1, \ldots, k_{i-1}]$, $2 \mid \prod_1^n k_i$ and that polarity is complementation. Then there exist $A, B \subset L$, for which* (4.3) *and* (4.4) *hold, and*

$$|A| + |B| = \frac{|L|}{2} = \frac{\prod_1^n k_i}{2}, \qquad 0 < |A| \leq |B|$$

iff there exist positive integers a and b and partition $[n] = I_0 \cup J_0$ such that

$$|A| = a \cdot b, \qquad a \leq \frac{\prod_{i \in I_0} k_i}{2} \qquad \textit{and} \qquad b \leq \frac{\prod_{i \in J_0} k_i}{2}.$$

Proof. Let (A, B) be a pair for which (4.3) and (4.4) hold, $|A| + |B| = L/2$ and $0 < |A| \leq |B|$.

Let (A^*, B^*) be a bisaturated extension of (A, B). Thus, by definition, $A^* \supset\!\!\!\subset B^*$ and according to Lemma 4, we have $A^* = A$.

Therefore $A = l(A) \setminus (l(A) \cap l(B^*))$ and $B^* = l(B^*) \setminus (l(A) \cap l(B^*))$.

We set $\alpha = |l(A)|$, $\beta = |l(B^*)|$, apply Lemma 3(ii) and obtain

$$|A| = |l(A)| - |l(A) \cap l(B^*)| = |l(A)| - \frac{|l(A)| \cdot |l(B^*)|}{|L|} = \alpha - \frac{\alpha\beta}{|L|}$$

and $|B^*| = \beta - \alpha\beta/|L|$. Therefore the ideals $l(A)$ and $l(B^*)$ minimize $|l(A) \cap l(B^*)|$ for fixed $|l(A)| = \alpha$ and $|l(B^*)| = \beta$.

Since $|A| + |B| = |L|/2$, $|A| \leqslant |B|$, necessarily $\alpha \leqslant \beta$, $|A| + |B^*| \geqslant |L|/2$ and hence

$$\alpha - \frac{\alpha\beta}{|L|} + \beta - \frac{\alpha\beta}{|L|} \geqslant \frac{|L|}{2},$$

which is equivalent to

$$(|L| - 2\alpha)(|L| - 2\beta) \leqslant 0.$$

Therefore

$$\alpha \leqslant |L|/2, \qquad \beta \geqslant |L|/2. \tag{5.2}$$

Since the ideals $l(A)$ and $l(B^*)$ minimize $|l(A) \cap l(B^*)|$ we apply Theorem 2 to the cardinalities $|l(A)| = \alpha$ and $|l(B^*)| = \beta$:

(a) α or β is 0 or $\prod_{i=1}^{n} k_i = |L|$;

(b) there exists an $I \subset [n]$, $o < |I| < n$, and there exists positive integers α_1 and β_1 with

$$\alpha = \prod_{i \in I} k_i \cdot \alpha_1, \qquad \beta = \prod_{i \in [n] \setminus I} k_i \cdot \beta_1.$$

We omit point (a), because $0 < |A| \leqslant |B|$.

With (5.2) we conclude from (b) that

$$\prod_{i \in I} k_i \cdot \alpha_1 = \alpha \leqslant |L|/2 = \prod_{1}^{n} k_i \Big/ 2,$$

thus

$$\alpha_1 \leqslant \prod_{i \in [n]} k_i \Big/ 2, \qquad \prod_{i \in [n] \setminus I} k_i \cdot \beta_1 = \beta \geqslant |L| \Big/ 2$$

and thus

$$\beta_1 \geqslant \prod_{i \in I} k_i \Big/ 2, \qquad \prod_{i \in I} k_i - \beta_1 \leqslant \prod_{i \in I} k_i \Big/ 2.$$

Hence, $|A| = \alpha - \alpha\beta/|L| = \alpha_1 \cdot \prod_{i \in I} k_i - \alpha_1\beta_1 = \alpha_1(\prod_{i \in I} k_i - \beta_1)$ and as a, b, I_0 and J_0 we can take

$$a = \alpha_1, \qquad b = \prod_{i \in I} k_i - \beta_1, \qquad I_0 = [n] \setminus I, \qquad J_0 = I.$$

This proves necessity.

Now suppose that $|A| = a \cdot b$, $[n] = I_0 \cup J_0$, $I_0 \cap J_0 = \varnothing$, $a \leqslant \prod_{i \in I_0} k_i/2$, $b \leqslant \prod_{i \in J_0} k_i/2$ and let us construct a pair (A, B) with properties (4.3), (4.4) and with $|A| + |B| = |L|/2$.

Let A_1 be the set of the first a lexicographically smallest vectors of length $|I_0|$ in sublattice L_{I_0} and let A_2 be the set of the b lexicographically largest vectors of length $|J_0| = n - |I_0|$ in sublattice L_{J_0}. We consider $A, B^* \subset L$, where

$$A = A_1 \times A_2, \qquad B^* = (L_{I_0} \setminus A_1) \times (L_{J_0} \setminus A_2).$$

It is clear that:

(a) $A \supset\!\!\!\!\subset B^*$;

(b) the sets A, B^* are bisaturated with respect to the relation 'incomparable';

(c) $|A| = a \cdot b$ and $|B^*| = (\prod_{i\in I_0} k_i - a)(\prod_{i\in J_0} k_i - b)$.
Since $2 \mid \prod_1^n k_i$, then at least one of the integers $|L_{I_0}| = \prod_{i\in I_0} k_i$ and $|L_{J_0}| = \prod_{i\in J_0} k_i$ is even.

Furthermore, since $a \leq |L_{I_0}|/2$ and $b \leq |L_{J_0}|/2$, and A_1 and A_2 have lexicographic order, then necessarily at least one of the following holds:
(1) $\overline{a_1} \in L_{L_0} \setminus A_1$ for all $a_1 \in A_1$;
(2) $\overline{a_2} \in L_{J_0} \setminus A_2$ for all $a_2 \in A_2$.
Hence $\bar{A} \subset B^*$. It is easy to verify that in B^* there are exactly $(|L_{I_0}| - 2a)(|L_{J_0}| - 2b)/2$ unordered pairs $\{c, \bar{c}\}$; $c, \bar{c} \in B^*$. Therefore, $B^* = B \cup \overline{B_1}$, where $B_1 \subset B$, $|B_1| = (|L_{I_0}| - 2a)(|L_{J_0}| - 2b)/2$ and B contains no element and its complement. Therefore (A, B) satisfies both (4.3) and (4.4), and we verify that

$$|A| + |B| = a \cdot b + (|L_{I_0}| - a)(|L_{J_0}| - b) - \frac{(|L_{I_0}| - 2a)(|L_{J_0}| - 2b)}{2} = \frac{|L|}{2}. \qquad \square$$

THEOREM 4 (equality characterization in terms of numbers, $\prod_{i=1}^n k_i$ is odd). *Suppose that* $L = \prod_{i=1}^n [0, 1, \ldots, k_i - 1]$, $2 \nmid \prod_1^n k_i$ *and that polarity is complementation. Then there exist* $A, B \subset L$ *for which* (4.3) *and* (4.4) *hold, and*

$$|A| + |B| = \frac{|L| - 1}{2}, \qquad |A| \leq |B|$$

iff:
(i) *there exist positive integers a and b and a partition* $[n] = I_0 \cup J_0$, $I_0, J_0 \neq \varnothing$ *such that*

$$|A| = a \cdot b, \qquad a < |L_{I_0}|/2, \qquad b < |L_{J_0}|/2;$$

or

(ii) $$|A| = (|L_{I_0}| \pm 1)(|L_{J_0}| \mp 1)/4 \quad \textit{and} \quad |B| = |L_{I_0}| \mp 1)(|L_{J_0}| \pm)/4$$

for all I_0 *and* J_0, $I_0 \cup J_0 = [n]$, $I_0, J_0 \neq \varnothing$.

PROOF. Let (A, B) be a pair for which (4.3), (4.4), $|A| + |B| = (|L| - 1)/2$ and $0 < |A| \leq |B|$ hold. Let (A^*, B^*) be a bisaturated extension of (A, B) and again apply Lemma 4 to obtain $A^* = A$.

As in the proof of Theorem 3, $|l(A)| = \alpha$ and $|l(B^*)| = \beta$;

$$|A| = \alpha - \alpha\beta/|L|, \qquad |B^*| = \beta - \alpha\beta/|L|, \qquad \alpha = \prod_{i\in I} k_i \cdot \alpha_1, \qquad \beta = \prod_{i\in[n]\setminus I} k_i \cdot \beta_1.$$

Furthermore, $|A| + |B^*| \geq |A| + |B| = (|L| - 1)/2$, and hence

$$\begin{aligned} |A| + |B^*| &= \alpha - \alpha\beta/|L| + \beta - \alpha\beta/|L| \\ &= \prod_{i\in I} k_i \cdot \alpha_1 - \alpha_1 \cdot \beta_1 + \prod_{i\in[n]\setminus I} k_i \cdot \beta_1 - \alpha_1\beta_1 \geq (|L| - 1)/2 \end{aligned}$$

or, equivalently, $(\prod_{i\in[n]\setminus I} k_i - 2\alpha_1)(\prod_{i\in I} k_i - 2\beta_1) - 1 \leq 0$.

This can be true only when:
(a) $2\alpha_1 < \prod_{i\in[n]\setminus I} k_i$, $2\beta_1 > \prod_{i\in I} k_i$;
(b) $2\alpha_1 = \prod_{i\in[n]\setminus I} k_i - 1$, $2\beta_1 = \prod_{i\in I} k_i - 1$;
(c) $2\alpha_1 = \prod_{i\in[n]\setminus I} k_i + 1$, $2\beta_1 = \prod_{i\in I} k_i + 1$.

For the case (a), as in the proof of Theorem 3, we can take integers $a = \alpha_1$, $b = \prod_{i\in I} k_i - \beta_1$, $I_0 = [n] \setminus I$ and $J_0 = I$, and so $|A|$ can have parameters as in (i).

If (b) holds or, equivalently, $\alpha = (|L| - |L_I|)/2$ and $\beta = (|L| - |L_{[n]\setminus I}|)/2$, then A and B can have parameters

$$|A| = (|L_I| + 1)(|L_{[n]\setminus I}| - 1)/4, \qquad |B| \leq |B^*| = (|L_I| - 1)(|L_{[n]\setminus I}| + 1)/4$$

In case (c) one has

$$|A| = (|L_I| - 1)(|L_{[n]\setminus I}| + 1)/4, \qquad |B| \leq |B^*| = (|L_I| + 1)(|L_{[n]\setminus I}| - 1)/4.$$

Therefore $|A|$ can have only parameters as in (i) or (ii).

This proves necessity.

To show sufficiency, suppose that $|A| = a \cdot b$, $[n] = I_0 \dot\cup J_0$, $I_0, J_0 \neq \varnothing$, $a < |I_0|/2$ and $b < |J_0|/2$. We construct (A, B^*) as in the proof of Theorem 3:

$$A = A_1 \times A_2, \qquad B = (L_{I_0} \setminus A_1) \times (L_{J_0} \setminus A_2).$$

We note that $B^* = B \cup \overline{B_1} \cup \{d\}$, where $B_1 \subset B$, $|B_1| = [(|L_{I_0}| - 2a)(|L_{J_0}| - 2b) - 1]/2$ and $d \in L$ is an element with $d = \bar{d}$; i.e.

$$d = \left(\frac{k_1 - 1}{2}, \ldots, \frac{k_n - 1}{2}\right).$$

We verify that A and B satisfy (4.3) and (4.4) and

$$|A| + |B| = (|L| - 1)/2.$$

Now let $|A_1| = (|L_{I_0}| \pm 1)/2$ and $|A_2| = (|L_{J_0}| \mp 1)/2$ (the sets A_1 and A_2 are defined in the proof of Theorem 3) and consider

$$A = A_1 \times A_2, \qquad B = (L_{I_0} \setminus A_1) \times (L_{J_0} \setminus A_2).$$

It is easy to verify that (A, B) satisfies (4.3) and (4.4):

$$|A| = (|L_{I_0}| \pm 1)(|L_{J_0}| \mp 1)/4, \qquad B = (|L_{I_0}| \mp 1)(|L_{J_0}| \pm 1)/4 \qquad \text{and}$$

$$|A| + |B| = (|L| - 1)/2. \quad \square$$

Corollary. (i) *Suppose that* $k_1 \geq k_2 \geq \cdots \geq k_n$. *Then, for all* $r, r \leq \prod_1^{n-1} k_i)/2$, *there exists a pair* (A, B), $A, B \subset L$, *for which* (4.3) *and* (4.4) *hold*, $|A| + |B| = \lfloor |L|/2 \rfloor$ *and* $|A| = r$.

(ii) *Suppose that* $k_1 = k_2 = \cdots = k_n = 2$ (*Hilton's results in* [16]). *Then, for all* r, $r \leq 2^{n-1}$, *there exists a pair* $(A, B) \subset L$ *for which* (4.3) *and* (4.4) *hold,*

$$|A| + |B| = 2^{n-1} \qquad \text{and} \qquad |A| = r.$$

Proof. (i) We put $a = 1$, $b = r$, $I_0 = \{n\}$, $J_0 = \{1, 2, \ldots, n-1\}$ and apply Theorems 3 and 4.

(ii) follows from (i), because $\min(|A|, |B|) \leq 2^{n-2}$. $\square$

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