

SETS OF INTEGERS AND QUASI-INTEGERS
WITH PAIRWISE COMMON DIVISOR

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0. INTRODUCTION

Consider the set $\mathbb{N}_s(n) = \left\{ u \in \mathbb{N} : \left(u, \prod_{i=1}^{s-1} p_i \right) = 1 \right\} \cap \langle 1, n \rangle$ of positive integers between 1 and n , which are not divisible by the first $s-1$ primes p_1, \dots, p_{s-1} .

Erdős introduced in [4] (and also in [5], [6], [7], [9]) the quantity $f(n, k, s)$ as the largest integer ρ for which an $A \subset \mathbb{N}_s(n)$, $|A| = \rho$, exists with no $k+1$ numbers being coprimes. Certainly the set

$$(1) \quad \mathbb{E}(n, k, s) = \left\{ u \in \mathbb{N}_s(n) : u = p_{s+i}v \text{ for some } i = 0, 1, \dots, k-1 \right\}$$

does not have $k+1$ coprimes.

Conjecture 1 (Erdős [4]): $f(n, k, 1) = |\mathbb{E}(n, k, 1)|$ for all $n, k \in \mathbb{N}$ was disproved in [1].

This disproves of course also the General Conjecture (Erdős [7]): for all $n, k, s \in \mathbb{N}$

$$(2) \quad f(n, k, s) = |\mathbb{E}(n, k, s)|.$$

However, in [2] we proved (2) for every k, s and (relative to k, s) large n .

In the present paper we are concerned with the case $k = 1$, which in [1] and [2] we called

Conjecture 2: $f(n, 1, s) = |\mathbb{E}(n, 1, s)|$ for all $n, s \in \mathbb{N}$.

Erdős mentioned in [7] that he did not even succeed in settling this special case of the General Conjecture.

Whereas in [1] we proved this by a completely different approach for $n \geq (p_{s+1} - p_s)^{-1} \prod_{i=1}^{s+1} p_i$, we establish it now for all n (Theorem 2).

We generalize and analyze Conjecture 2 first for quasi-primes in order to understand how the validity of Conjecture 2 depends on the distribution of the quasi-primes and primes. Our main result is a simply structured sufficient condition on this distribution (Theorem 1). Using sharp estimates on the prime number distribution by Rosser and Schoenfeld [14] we show that this condition holds for $\mathbb{Q} = \{p_s, p_{s+1}, \dots\}$, $s \geq 1$, as set of quasi-primes and thus Theorem 2 follows.

1. BASIC DEFINITIONS FOR NATURAL NUMBERS AND QUASI-NUMBERS

Whenever possible we keep the notation of [2]. \mathbb{N} denotes the set of positive integers and $\mathbb{P} = \{p_1, p_2, \dots\} = \{2, 3, 5, \dots\}$ denotes the set of all primes. \mathbb{N}^* is the set of square free numbers.

For two numbers $u, v \in \mathbb{N}$ we write $u \mid v$ (resp. $u \nmid v$) iff u divides v (resp. u doesn't divide v), $[u, v]$ stands for the smallest common multiple of u and v , (u, v) is the largest common divisor of u and v , and we say that u and v have a common divisor, if $(u, v) > 1$. $\langle u, v \rangle$ denotes the interval $\{x \in \mathbb{N} : u \leq x \leq v\}$.

For any set $A \subset \mathbb{N}$ we introduce

$$(1.1) \quad A(n) = A \cap \langle 1, n \rangle$$

and $|A|$ as cardinality of A . The set of multiples of A is

$$(1.2) \quad M(A) = \{m \in \mathbb{N} : a \mid m \text{ for some } a \in A\}.$$

For set $\{a\}$ with one element we also write $M(a)$ instead of $M(\{a\})$. For $u \in \mathbb{N}$, $p^+(u)$ denotes the largest prime in its prime number representation

$$(1.3) \quad u = \prod_{i=1}^{\infty} p_i^{\alpha_i}, \quad \sum_{i=1}^{\infty} \alpha_i < \infty.$$

We also need the function π , where for $y \in \mathbb{N}$

$$(1.4) \quad \pi(y) = |\mathbb{P}(y)|,$$

and the set Φ , where

$$(1.5) \quad \Phi(u, y) = \{x \in \mathbb{N}(u) : (x, p) = 1 \text{ for all } p < y\}.$$

We note that $1 \in \Phi(u, y)$ for all $u \geq y$, $u \geq 1$.

Clearly, by (1.3) $u \in \mathbb{N}$ corresponds to a multiset $(\alpha_1, \alpha_2, \dots)$. Therefore, instead of saying that $A \subset \mathbb{N}(z)$ has pairwise (nontrivial) common divisors, we adapt the following shorter multiset terminology.

Definition 1. $A \subset \mathbb{N}(z)$, $z \geq 1$, is said to be intersecting iff for all $a, b \in A$; $a = \prod_{i=1}^{\infty} p_i^{\alpha_i}$, $b = \prod_{i=1}^{\infty} p_i^{\beta_i}$; $\alpha_j \beta_j \neq 0$ for some j .

In order to better understand, how properties depend on the multiset structure and how on the distribution of primes it is very useful to introduce quasi-(natural) numbers and quasi-primes. Results then also can be applied to a subset of the primes, if it is viewed as the set of quasi-primes.

A set $\mathbb{Q} = \{1 < r_1 < r_2 < \dots\}$ of positive real numbers, $\lim_{i \rightarrow \infty} r_i = \infty$, is called a (complete) set of quasi-prime numbers, if every number in

$$(1.6) \quad \mathbb{X} = \left\{ x \in \mathbb{R}^+ : x = \prod_{i=1}^{\infty} r_i^{\alpha_i}, \alpha_i \in \{0, 1, 2, \dots\}, \sum_{i=1}^{\infty} \alpha_i < \infty \right\}$$

has a unique representation. (See also Remark 1 after Theorem 1.)

The set \mathbb{X} is the set of quasi-numbers corresponding to the set of quasi-primes \mathbb{Q} .

We can now replace \mathbb{P}, \mathbb{N} by \mathbb{Q}, \mathbb{X} in all concepts of this Section up to Definition 1 and thus for any $u, v \in \mathbb{X}$ $u | v$, $u \nmid v$, (u, v) , $[u, v]$, $\langle u, v \rangle$ ($= \{x \in \mathbb{X} : u \leq x \leq v\}$); for any $A \subset \mathbb{X}$ $A(z)$, $M(A)$ ($= \{m \in \mathbb{X} : a | m \text{ for some } a \in A\}$); and “intersecting” are well defined. So are also the function π and the sets $\Phi(u, y)$ for $u \geq y, u \geq 1$.

We study $\mathcal{I}(z)$, the family of all intersecting $A \subset \mathbb{X}(z)$, and

$$(1.7) \quad f(z) = \max_{A \subset \mathcal{I}(z)} |A|, \quad z \in \mathbb{X}.$$

The subfamily $\mathcal{O}(z)$ of $\mathcal{I}(z)$ consists of the optimal sets, that is,

$$(1.8) \quad \mathcal{O}(z) = \{A \in \mathcal{I}(z) : |A| = f(z)\}.$$

A key role is played by the following configuration.

Definition 2. $A \subset \mathbb{X}(z)$ is called star, if

$$A = M(\{r\}) \cap \mathbb{X}(z) \text{ for some } r \in \mathbb{Q}.$$

2. AUXILIARY RESULTS CONCERNING LEFT COMPRESSED SETS, “UPSETS” AND “DOWNSETS”

There is not only one way to define “left pushing” of subsets of \mathbb{X} . Here the following is most convenient.

For any $i, j \in \mathbb{N}$, $j < i$, we define the operation “left pushing” $L_{i,j}$ on subsets of \mathbb{X} . For $A \subset \mathbb{X}$ let

$$A_1 = \{a \in A : a = a_1 \cdot r_i^\alpha, \alpha \geq 1, (a_1, r_i \cdot r_j) = 1, (a_1 \cdot r_j^\alpha) \notin A\} \text{ and}$$

$$L_{i,j}(A) = (A \setminus A_1) \cup A_1^*, \text{ where}$$

$$A_1^* = \{a = a_1 \cdot r_j^\alpha : (a_1, r_i \cdot r_j) = 1 \text{ and } a_1 \cdot r_j^\alpha \in A_1\}.$$

Clearly $|L_{i,j}(A) \cap \mathbb{X}(z)| \geq |A(z)|$ for every $z \in \mathbb{R}^+$.

It is easy to show, that the operation $L_{i,j}$ preserves the property “intersecting”.

By finitely many (resp. countably many) “left pushing” operations $L_{i,j}$ one can transform every $A \subset \mathbb{X}(z)$, $z \in \mathbb{R}^+$, (resp. $A \subset \mathbb{X}$) into a “left compressed” set A' , where the concept of left compressedness is defined as follows:

Definition 3. $A \subset X$ is said to be left compressed if

$$L_{i,j}(A) = A \text{ for all } i, j \text{ with } i > j.$$

We note that there are left compressed sets A' and A'' , which are obtained by left pushing from the same set A .

Lemma 1. For all $z \in \mathbb{X}$

$$f(z) = \max_{A \in \mathcal{C}(z)} |A|.$$

Clearly, any $A \in \mathcal{O}(z)$ is an “upset”:

$$(2.1) \quad A = M(A) \cap \mathbb{X}(z).$$

and it is also a “downset” in the following sense:

$$(2.2) \quad \text{for } a \in A, a = r_{i_1}^{\alpha_1} \dots r_{i_t}^{\alpha_t}, \alpha_i \geq 1 \text{ also } a' = r_{i_1} \dots r_{i_t} \in A.$$

For every $B \subset \mathbb{X}$ we introduce the unique primitive subset $P(B)$, which has the properties

$$(2.3) \quad b_1, b_2 \in P(B) \text{ implies } b_1 \nmid b_2 \text{ and } B \subset M(P(B)).$$

We know from (2.2) that for any $A \in \mathcal{O}(z)$ $P(A)$ consists only of squarefree quasi-numbers and that by (2.1)

$$(2.4) \quad A = M(P(A)) \cap \mathbb{X}(z).$$

From Lemma 1 we know that $\mathcal{O}(z) \cap \mathcal{C}(z) \neq \emptyset$.

Let now $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$ and $P(A) = \{a_1, \dots, a_m\}$, where the a_i 's are written in lexicographic order. The set of multiples of $P(A)$ in $\mathbb{X}(z)$ can be written as a union of disjoint sets $B^i(z)$:

$$(2.5) \quad M(P(A)) \cap \mathbb{X}(z) = \dot{\cup} B^i(z),$$

$$(2.6) \quad B^i(z) = \{x \in M(P(A)) \cap \mathbb{X}(z) : a_i \mid x, a_j \nmid x \text{ for } j = 1, \dots, i-1\}.$$

We can say more about $B^i(z)$, if we use the factorisation of the square free quasi-numbers a_i .

Lemma 2. Let $a_i = r_{j_1}, \dots, r_{j_\ell}; r_{j_1} < r_{j_2} \dots < r_{j_\ell}$, then

$$B^i(z) = \left\{ x \in \mathbb{X}(z) : x = r_{j_1}^{\alpha_1} \dots r_{j_\ell}^{\alpha_\ell} T, \alpha_i \geq 1, \left(T, \prod_{r_i \leq r_{j_\ell}} r_i \right) = 1 \right\}.$$

Proof: This immediately follows from the facts that A is left compressed, “upset” and “downset”.

Finally, a result for stars. Keep in mind that they contain a single prime and that Lemma 1 holds.

Lemma 3. For any $B \subset \mathcal{I}(z)$ and $B' \subset \mathbb{X}(z)$, which is left compressed and obtained from B by left pushing we have: B is a star exactly if B' is a star.

3. THE MAIN RESULT

Theorem 1. *Suppose the quasi-primes \mathbb{Q} satisfy the following condition:
for all $u \in \mathbb{R}^+$ and for all $r_\ell, \ell \geq 2$*

$$(a) \quad 2|\Phi(u, r_\ell)| \leq |\Phi(u \cdot r_\ell, r_\ell)|.$$

Then, for all $z \in \mathbb{R}^+$, every optimal $A \in \mathcal{O}(z)$ is a “star”. In particular

$$f(z) = |M(r_1) \cap X(z)| \text{ for all } z \in \mathbb{X}.$$

Remarks:

1. This result and also Lemma 2 below immediately extend to the case where quasi-primes are defined without the requirement of the uniqueness of the representations in (1.6.), if multiplicities of representations are taken into consideration. \mathbb{X} is thus just a free, discrete commutative semigroup in $\mathbb{R}_{\geq 1}^+$.
2. Without the uniqueness requirement we are led to a new problem by not counting multiplicities.
3. However, without the assumption $\lim_{i \rightarrow \infty} r_i = \infty$ or without the assumption of discreteness the quasi-primes have a cluster point ρ and one can produce infinitely many infinite, intersecting sets in $\mathbb{X}(\rho^3 + \varepsilon)$, which are not stars.
4. In Section 5 we discuss the case of finitely many quasi-primes.

Proof: Let $A \in \mathcal{O}(z)$ and let $P(A) = \{a_1, \dots, a_m\}$ be the primitive subset of A which generates A .

Under condition (a), the Theorem is equivalent to the statement:

for all $z \in \mathbb{X}$, $m = 1, a_1 = r_\ell$ for some quasi-prime r_ℓ .

Suppose, to the opposite, that for some $z \in \mathbb{X}$ there exists $A \in \mathcal{O}(z)$ which is not a star, i.e. if $P(A) = \{a_1, \dots, a_m\}$ is the primitive, generating subset of A , then $m > 1$ and hence every element $a_i \in P(A)$ is a product of at least two different quasi-primes.

According to Lemma 3 we can assume, that $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$, $P(A) = \{a_1, a_2, \dots, a_m\}$; a_i 's are written in lexicographic order, $m > 1$ and

$$p^+(a_m) = r_t, \quad t \geq 2.$$

Write $P(A)$ in the form

$$P(A) = S_1 \cup S_2 \cup \dots \cup S_t, \quad t \geq 2, S_t \neq \emptyset,$$

where

$$S_i = \{a \in P(A) : p^+(a) = r_i\}.$$

Since $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$, we have

$$A = M(P(A)) \cap X(z) = \bigcup_{1 \leq j \leq t} B(S_j), \text{ where}$$

$$B(S_j) = \bigcup_{a_i \in S_j} B^i(z) \text{ and } B^i(z) \text{ are described in Lemma 2.}$$

Now we consider $S_t = \{a_\ell, a_{\ell+1}, \dots, a_m\}$ for some $\ell \leq m$, and let $S_t = S_t^1 \dot{\cup} S_t^2$, where

$$S_t^1 = \{a_i \in S_t : r_{t-1} \mid a_i\}, S_t^2 = S_t \setminus S_t^1.$$

We have

$$(3.1) \quad B(S_t) = B(S_t^1) \dot{\cup} B(S_t^2), \text{ where}$$

$$B(S_t^j) = \bigcup_{a_i \in B_t^j} B^{(i)}(z); \quad j = 1, 2.$$

Let $\tilde{S}_t = \left\{ \frac{a_\ell}{r_t}, \frac{a_{\ell+1}}{r_t}, \dots, \frac{a_m}{r_t} \right\}$ and similarly $\tilde{S}_t^j = \left\{ \frac{a_i}{r_t} : a_i \in S_t^j \right\}; \quad j = 1, 2$.

It is clear, that $\frac{a_i}{r_t} > 1$ for all $a_i \in S_t$.

Obviously $\tilde{S}_t^1 \in \mathcal{I}(z)$, because all elements of \tilde{S}_t^1 have common factor r_{t-1} . Let us show that $\tilde{S}_t^2 \in \mathcal{I}(z)$ as well. Suppose, to the opposite, there exist $b_1, b_2 \in \tilde{S}_t^2$ with $(b_1, b_2) = 1$.

We have $b_1 \cdot r_t, b_2 \cdot r_t \in S_t^2 \subset A$ and $(b_1 \cdot r_1, r_{t-1}) = 1, (b_2 \cdot r_t, r_{t-1}) = 1$.

Since $A \in \mathcal{C}(z)$ and $r_{t-1} \nmid b_1 \cdot b_2$ (see definition of S_t^2), we conclude that $r_{t-1} \cdot b_1 \in A$ as well. Hence the elements $r_{t-1} \cdot b_1, r_t \cdot b_2 \in A$ and at the same time $(r_{t-1} \cdot b_1, r_t \cdot b_2) = 1$, which is a contradiction. So, we have $\tilde{S}_t^j \in \mathcal{I}(z); \quad j = 1, 2$; and hence

$$A_j = M((P(A) \setminus S_t) \cup \tilde{S}_t^i) \cap \mathbb{X}(z) \in \mathcal{I}(z); \quad j = 1, 2.$$

We are going to prove that at least one of the inequalities $|A_1| > |A|$, $|A_2| > |A|$ holds, and this will lead to a contradiction.

From (3.1) we know that

$$\max\{|B(S_t^1)|, |B(S_t^2)|\} \geq \frac{1}{2}|B(S_t)|.$$

Let us assume, say

$$(3.2) \quad |B(S_t^2)| \geq \frac{1}{2}|B(S_t)|,$$

and let us show that

$$(3.3) \quad |A_2| > |A|$$

(if $|B(S_t^1)| \geq \frac{1}{2}|B(S_t)|$ the situation is symmetrically the same).

Let $b \in \widetilde{S}_t^2$ and $b = r_{i_1} \cdot r_{i_2} \dots r_{i_s}$; $r_{i_1} < r_{i_2} < \dots < r_{i_s} < r_t$. We know that

$$a_i = b \cdot r_t = r_{i_1} \dots r_{i_s} \cdot r_t \in S_t^2 \text{ for some } i \leq m,$$

and that (see Lemma 2), the contribution of $M(a_i)$ in $B(S_t)$ (and as well in A) are the elements in the form:

$$B^i(z) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot r_t^{\alpha_t} \cdot T; \text{ where } \alpha_i \geq 1 \text{ and } \left(T, \prod_{i \leq t} r_i \right) = 1 \right\}.$$

We write $B^i(z)$ in the following form:

$$(3.4) \quad B^i(z) = \bigcup_{(\alpha_1, \dots, \alpha_s), \alpha_i \geq 1} D(\alpha_1, \dots, \alpha_s), \text{ where}$$

$$(3.5) \quad D(\alpha_1, \dots, \alpha_s) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot r_t \cdot T_1; \left(T_1, \prod_{i \leq t-1} r_i \right) = 1 \right\}.$$

Now we look at the contribution of $M(b)$ in $A_2 = M((P(A) \setminus S_t) \cup \widetilde{S}_t^2) \cap \mathbb{X}(z)$, namely we look only at the elements in A_2 (denoted by $B(b)$), which are divisible by b , but not divisible by any element from $(P(A) \setminus S_t) \cup (\widetilde{S}_t^2 \setminus b)$.

Since $A \subset C(z)$ and r_t is the largest quasi-prime in $P(A)$, we conclude that

$$B(b) \supseteq B^*(b) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot \widetilde{T}, \alpha_i \geq 1, \text{ where } \left(\widetilde{T}, \prod_{i \leq t-1} r_i \right) = 1 \right\},$$

and we can write

$$(3.6) \quad B^*(b) = \bigcup_{(\alpha_1, \dots, \alpha_s), \alpha_i \geq 1} \widetilde{D}(\alpha_1, \dots, \alpha_s),$$

where

$$(3.7) \quad \widetilde{D}(\alpha_1, \dots, \alpha_s) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot \widetilde{T}, \alpha_i \geq 1, \left(\widetilde{T}, \prod_{i \leq t-1} r_i \right) = 1 \right\}.$$

Hence

$$(3.8) \quad |B(b)| \geq |B^*(b)| = \sum_{(\alpha_1, \dots, \alpha_s), \alpha_i \geq 1} \widetilde{D}(\alpha_1, \dots, \alpha_s).$$

At first we prove that $|A_2| \geq |A|$. In the light of (3.2), (3.4–3.8), for this it is sufficient to show that

$$(3.9) \quad |\widetilde{D}(\alpha_1, \dots, \alpha_s)| \geq 2|D(\alpha_1, \dots, \alpha_s)|,$$

for all $(\alpha_1, \dots, \alpha_s), \alpha_i \geq 1$.

However, this is exactly the condition (a) in the Theorem for $u = \frac{z}{r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot r_t}$ and $\ell = t$. Hence $|A_2| \geq |A|$.

To prove (3.3), that is $|A_2| > |A|$, it is sufficient to show the existence of $(\alpha_1, \dots, \alpha_s), \alpha_i \geq 1$, for which in (3.9), strict inequality holds. For this we take $\beta \in \mathbb{N}$ and $(\alpha_1, \dots, \alpha_s) = (\beta, 1, 1, \dots, 1)$ such that

$$\frac{z}{r_t} < r_{i_1}^\beta \cdot r_{i_2} \cdot r_{i_s} \leq z.$$

This is always possible, because

$$r_{i_1} \cdot r_{i_2} \dots r_{i_s} \cdot r_t \leq z \text{ implies } r_{i_1} \dots r_{i_s} \leq \frac{z}{r_t} \text{ and } r_{i_1} < \dots < r_s < r_t.$$

We have $|\tilde{D}(\beta, 1, \dots, 1)| = 1$ and $|D(\beta, 1, \dots, 1)| = 0$. Hence $|A_2| > |A|$, which is a contradiction, since $A_2 \in \mathcal{I}(z)$. This completes the proof.

Lemma 4. *Sufficient for condition (a) in Theorem 1 to hold is the condition*

$$(b) \quad 2\pi(v) \leq \pi(r_2 \cdot v) \text{ for all } v \in \mathbb{R}^+.$$

Proof: Under condition (b) it is sufficient to prove for every $u \in \mathbb{R}^+, r_\ell, (\ell \geq 2)$ that $|\Phi(u, r_\ell)| \leq |\Phi_1(u \cdot r_\ell, r_\ell)|$ where $\Phi_1(u \cdot r_\ell, r_\ell) = \{x \in \Phi(u \cdot r_\ell, r_\ell) : u < x \leq u \cdot r_\ell\}$.

We avoid the trivial cases $u < 1$, for which $\Phi(u, r_\ell) = \emptyset$, and $1 \leq u < r_\ell$, for which $\Phi(u, r_\ell) = \{1\}$ and $r_\ell \in \Phi_1(u \cdot r_\ell, r_\ell)$. Hence, we assume $u \geq r_\ell$.

Let $F(u, r_\ell) = \{a \in \Phi(u, r_\ell), a \neq 1 : a \cdot p^+(a) \leq u\} \cup \{1\}$. It is clear that for any $b \in \Phi(u, r_\ell), b \neq 1$, we have $\frac{b}{p^+(b)} \in F(u, r_\ell)$ and that

$$(3.10) \quad |\Phi(u, r_\ell)| = 1 + \sum_{a \in F(u, r_\ell)} |\tau(a)|,$$

where $\tau(a) = \{r \in Q : r_\ell \leq p^+(a) \leq r \leq \frac{u}{a}\}$ and integer 1 in (3.10) stands to account for the element $1 \in \Phi(u, r_\ell)$.

On the other hand we have

$$(3.11) \quad |\Phi_1(u \cdot r_\ell, r_\ell)| \geq \sum_{a \in F(u, r_\ell)} |\tau_1(a)|, \text{ where}$$

$$\tau_1(a) = \left\{ r \in Q : \frac{u}{a} < r \leq \frac{u \cdot r_\ell}{a} \right\}.$$

We have

$$|\tau(a)| \leq \pi\left(\frac{u}{a}\right) - \ell + 1 \leq \pi\left(\frac{u}{a}\right) - 1 \quad (\ell \geq 2) \text{ and by condition (b)}$$

$$(3.12) \quad |\tau_1(a)| = \pi\left(\frac{u \cdot r_\ell}{a}\right) - \pi\left(\frac{u}{a}\right) \geq \pi\left(\frac{u}{a}\right).$$

Hence $|\tau_1(a)| > |\tau(a)|$ for all $a \in F(u, r_\ell)$ and, since $F(u, r_\ell) \neq \emptyset$ ($u \geq r_\ell$), from (3.10), (3.11), (3.12) we get

$$|\Phi_1(u \cdot r_\ell, r_\ell)| \geq |\Phi(u, r_\ell)|.$$

4. PROOF OF ERDÖS' "CONJECTURE 2"

For a positive integer s let $\mathbb{N}_s = \left\{ u \in \mathbb{N} : \left(u, \prod_{i=1}^{s-1} p_i \right) = 1 \right\}$ and let $\mathbb{N}_s(n) = \mathbb{N}_s \cap \langle 1, n \rangle$.

Erdős introduced in [4] (and also in [5], [6], [7], [9]) the quantity $f(n, k, s)$ as the largest integer ρ for which an $A \subset \mathbb{N}_s(n)$, $|A| = \rho$, exists with no $k + 1$ numbers being coprimes.

Certainly the set

$$(4.1) \quad \mathbb{E}(n, k, s) = \left\{ u \in \mathbb{N}_s(n) : u = p_{s+i}v \text{ for some } i = 0, 1, \dots, k-1 \right\}$$

does not have $k + 1$ coprimes.

The case $s = 1$, in which $\mathbb{N}_1(n) = \langle 1, n \rangle$, is of particular interest.

Conjecture 1 (Erdős [4]):

$$(4.2) \quad f(n, k, 1) = |\mathbb{E}(n, k, 1)| \text{ for all } n, k \in \mathbb{N}$$

was disproved in [1].

This disproves of course also the **General Conjecture** (Erdős [7]): for all $n, k, s \in \mathbb{N}$

$$(4.3) \quad f(n, k, s) = |\mathbb{E}(n, k, s)|.$$

However, in [2] we proved (4.3) for every k, s and (relative to k, s) large n . For further related work we refer to [8–10].

Erdős mentions in [7] that he did not succeed in settling even the case $k = 1$. This special case of the General Conjecture was called in [1] and [2]

Conjecture 2: $f(n, 1, s) = |\mathbb{E}(n, 1, s)|$ for all $n, s \in \mathbb{N}$.

Notice that

$$(4.4) \quad \mathbb{E}(n, 1, s) = \left\{ u \in \mathbb{N}_s(n) : p_s \mid u \right\}, \text{ i.e. } \mathbb{E}(n, 1, s) \text{ is a star.}$$

In the language of quasi-primes we can define

$$(4.5) \quad \mathbb{Q} = \{r_1, r_2, \dots, r_\ell \dots\} = \{p_s, p_{s+1}, \dots, p_{s+\ell-1}, \dots\}$$

and the corresponding quasi-integers \mathbb{X} .

Now, Conjecture 2 is equivalent to

$$(4.6) \quad f(n, 1, s) = |M(p_s) \cap \mathbb{X}(n)| \text{ for all } n, s \in \mathbb{N}.$$

Notice that $\mathbb{X}(n)$ is the set of those natural numbers not larger than n , which are entirely composed from the primes not smaller than p_s . Clearly, condition (1.6) for quasi-prime is satisfied.

Theorem 2.

- (i) *Conjecture 2 is true.*
- (ii) *For all $s, n \in \mathbb{N}$, every optimal configuration is a “star”.*
- (iii) *The optimal configuration is unique if and only if*

$$|M(p_s) \cap \mathbb{N}_s(n)| > |M(p_{s+1}) \cap \mathbb{N}_s(n)|,$$

which is equivalent to the inequality

$$\left| \Phi \left(\frac{n}{p_s}, p_s \right) \right| > \left| \Phi \left(\frac{n}{p_{s+1}}, p_s \right) \right|.$$

Remark 5: We believe, that also for $k = 2, 3$

$$f(n, k, s) = |\mathbb{E}(n, k, s)| \text{ for all } n, s \in \mathbb{N}.$$

For $k = 4$ our counterexample in [1] applies. Moreover, we believe that every optimal configuration in the case $k = 2$ is a union of two stars. In the case $k = 3$ it is not always true, which shows the following

Example: Let $s \in \mathbb{N}$ be such that $p_s \cdot p_{s+7} > p_{s+5} \cdot p_{s+6}$ (as such primes we can take for instance the primes from the mentioned counterexample) and let $p_{s+5} \cdot p_{s+6} \leq n \leq p_s \cdot p_{s+7}$. We verify that

$$|\mathbb{E}(n, 2, s)| = |M(p_s, p_{s+1}, p_{s+2}) \cap \mathbb{N}(n)| = 21.$$

On the other hand the set

$$A = \{p_i \cdot p_j, s \leq i < j \leq s + 6\}$$

has no 4 coprime elements and is not a union of the stars, but again

$$|A| = 21.$$

Proof of Theorem 2: We prove (ii). Since $M(p_s) \cap \mathbb{N}_s(n)$ is not smaller than any competing star, this implies (i) and (iii). In the light of Theorem 1 and Lemma 4, it is sufficient to show that

$$(4.7) \quad 2\pi(v) \leq \pi(p_{s+1} \cdot v) \text{ for all } v \in \mathbb{R}^+.$$

Since for $v < P_s$, $\pi(v) = 0$, we can assume $v \geq P_s$.

(4.7) is equivalent to

$$(4.8) \quad 2(\Pi(v) - s + 1) \leq \Pi(p_{s+1} \cdot v) - s + 1$$

where $\Pi(\cdot)$ is usual the counting function of primes. To show (4.8) it is sufficient to prove for all $v \in \mathbb{R}^+$

$$(4.9) \quad 2\Pi(v) \leq \Pi(3v).$$

For this it suffices to show (4.9) only for $v \in \mathbb{P}$.

We use the very sharp estimates on the distribution of primes due to Rosser and Schoenfeld [14]:

$$(4.10) \quad \frac{v}{\log v - \frac{1}{2}} < \Pi(v) < \frac{v}{\log v - \frac{3}{2}} \quad \text{for every } v \geq 67.$$

From (4.10) we get

$$2\Pi(v) < \Pi(3v) \quad \text{for all } v > 298.$$

The cases $v < 298$, $v \in \mathbb{P}$ are verified by inspection. We just mention that for $v \in \{3, 5, 7, 13, 19\}$ one has even the equality $2\Pi(v) = \Pi(3v)$.

5. EXAMPLES OF SETS OF QUASI-PRIMES FOR WHICH ALMOST ALL OPTIMAL INTERSECTING SETS OF QUASI-NUMBERS ARE NOT STARS

Suppose we are given only a *finite number* of quasi-primes:

$$1 < r_1 < r_2 < \cdots < r_m, \quad m \geq 3,$$

satisfying (1.6).

The sets $\mathbb{X}, \mathbb{X}^*, \mathbb{X}(z), \mathcal{I}(z), \mathcal{O}(z)$ are defined as in Section 1. Here \mathbb{X}^* has exactly 2^m elements. We are again interested in the quantity

$$f(z) = \max_{A \in \mathcal{I}(z)} |A|, \quad z \in \mathbb{X}.$$

For all $y \in \mathbb{X}^*$, $y = r_1^{\alpha_1} \cdots r_m^{\alpha_m}$; $\alpha_i \in \{0, 1\}$, let $w(y) = \sum_{i=1}^m \alpha_i$ and let

$$T(y) = \{x \in \mathbb{X}, x = r_1^{\beta_1} \cdots r_m^{\beta_m} : \beta_i \geq 1 \text{ iff } \alpha_i = 1\}.$$

We distinguish two cases.

Case I: $m = 2m_1 + 1$

Define $\mathbb{X}_1^* = \{x \in \mathbb{X}^* : w(x) \geq m_1 + 1\}$.

Proposition 1. *Let $m = 2m_1 + 1$ be odd. There exists a constant $z_0 = z(r_1, \dots, r_m)$ such that for all $z > z_0$, $|\mathcal{O}(z)| = 1$ and $A \in \mathcal{O}(z)$ has the form*

$$A = M(\mathbb{X}_1^*) \cap X(z) = \bigcup_{y \in \mathbb{X}_1^*} T(y) \cap \mathbb{X}(z).$$

Proof: Suppose $B \in \mathcal{O}(z)$. Since by optimality B is “downset” and “upset”, we have

$$B = \bigcup_{y \in Y} T(y) \cap X(z) \text{ for some } Y \subset \mathbb{X}^*.$$

It is clear, that $|Y| \leq 2^{m-1}$, because by the intersecting property $y \in Y$ implies $\bar{y} = \frac{r_1 \dots r_m}{y} \notin Y$.

Write $Y = Y_1 \dot{\cup} Y_2$, where

$$Y_1 = \{y \in Y : w(y) \leq m_1\} \text{ and } Y_2 = \{y \in Y : w(y) \geq m_1 + 1\}.$$

Our aim is to prove, that for large enough z one always has $Y_1 = \emptyset$, from where the Proposition follows. Since \mathbb{X}^* is finite, for this it is sufficient to show that for all $y \in \mathbb{X}^*$ with $w(y) \leq m_1$

$$(5.1) \quad |T(y) \cap \mathbb{X}(z)| < |T(\bar{y}) \cap \mathbb{X}(z)|, \text{ if } z > z(y).$$

Let $y = r_1^{\alpha_1} \dots r_m^{\alpha_m}$, $\alpha_i \in \{0, 1\}$, and let $\mathcal{I}(y) \subset \{1, 2, \dots, m\}$, $|\mathcal{I}(y)| = w(y)$, be the positions with $\alpha_i = 1$.

We introduce

$$(5.2) \quad c_i = \log r_i \text{ for } i = 1, \dots, m.$$

Then it is easy to see that $|T(y) \cap \mathbb{X}(z)|$ is the number of solutions of

$$\sum_{i \in \mathcal{I}(y)} c_i \gamma_i \leq \log z \text{ in } \gamma_i \in \mathbb{N}$$

and $|T(\bar{y}) \cap \mathbb{X}(z)|$ is the number of solutions of

$$\sum_{i \in \mathcal{I}(\bar{y})} c_i \delta_i \leq \log z \text{ in } \delta_i \in \mathbb{N}.$$

We verify that

$$|T(y) \cap \mathbb{X}(z)| \sim c_* (\log z)^{w(y)}, \text{ where } c_* = \frac{1}{\prod_{i \in \mathcal{I}(y)} c_i \cdot (w(y))!}$$

and

$$|T(\bar{y}) \cap \mathbb{X}(z)| \sim c_{**} (\log z)^{m-w(y)}, \text{ where } c_{**} = \frac{1}{\prod_{i \in \mathcal{I}(\bar{y})} c_i \cdot (m-w(y))!}$$

Since $w(y) \leq m_1$, $m - w(y) \geq m_1 + 1$, then there exists a $z(y)$ for which (5.1) is satisfied.

Case II: $m = 2m_1$

Let $\mathbb{X}_1^* = \{x \in \mathbb{X}^* : w(x) \geq m_1 + 1\}$ and $\mathbb{X}_0^* = \{x \in \mathbb{X}^* : w(x) = m_1\}$.

For every $y \in \mathbb{X}_0^*$ let

$$g(y) = \prod_{i \in \mathcal{I}(y)} c_i \text{ with } c_i \text{ defined as in (5.2).}$$

Finally, define $\tilde{\mathbb{X}}_0^* = \{y \in \mathbb{X}_1^* : g(y) \leq g(\bar{y})\}$. If $g(y) = g(\bar{y})$ we take as an element of $\tilde{\mathbb{X}}_0^*$ one of them, so $|\mathbb{X}_0^*| = \frac{\binom{2m_1}{m_1}}{2}$.

Using the same approach as in the proof of Proposition 1 we get

Proposition 2. *Let $m = 2m_1$ be even. There exists a constant $z_0 = z(r_1, \dots, r_m)$ such that for all $z > z_0$ an optimal set $A \in \mathcal{O}(z)$ is*

$$A = M(\mathbb{X}_1^* \cup \tilde{\mathbb{X}}_0^*) \cap \mathbb{X}(z) = \bigcup_{x \in \mathbb{X}_1^* \cup \tilde{\mathbb{X}}_0^*} T(y) \cap \mathbb{X}(z)$$

and, if $g(y) \neq g(\bar{y})$ for all $y \in X_0^*$, then the optimal set is unique.

From these Propositions follows that for *finite* sets Q of quasi-primes, for all sufficiently large z , the optimal intersecting sets are *not stars*.

By choosing Q 's of infinitely many quasi-primes, which are sufficiently far from each other, say $r_{i+1} > \exp(r_i)$, (details are omitted), one can make sure, that again for all sufficiently large z , the optimal intersecting sets are *never stars*.

We thank Thomas Zink for valuable suggestions, which helped to improve the presentation.

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