Set Cascades and Vector Valleys in Pascal's Triangle

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Abstract

Let N be the positive integers. Let $T(n)$ be the set of all $F \subseteq N$ of size $|F| = n$. The shadow ΔF of F is the subset of $T(n-1)$ obtained by deleting an element of F in all n possible ways. If $R \subseteq T(n)$ then $\Delta R = \cup\{F \in R\}\Delta F$. If we write $|R|$ as an n-Cascade we can immediately write down the best possible lower bound for $|\Delta R|$.

Let $V(n)$ be the set of all vectors of dimension n with coordinates 0 or 1. The shadow ΔA of $A \in V(n)$ is the subset of $V(n-1)$ obtained by deleting a coordinate of A in all ways. If $W \subseteq V(n)$ we write $|W|$ as an n-Valley and get the best possible lower bound for $|\Delta W|$.

1 Shadows of sets.

Let N be the set of all positive integers. For $n \in N$ let $T(n)$ be the set of all $F \subseteq N$ of size $|F| = n$.

Ex.1. We have $T(1) = \{1, 2, 3, ...\}$ and $T(2) = \{12, 13, 23, 14, 24, 34, 15, ...\}$ and $T(3) = \{123, 124, 134, 234, 125, ...\}$.

If $F \in T(n)$ the *shadow* ΔF of F is the $\Delta F \subseteq T(n-1)$ obtained by deleting an element from F in all n possible ways, so $|\Delta F| = n$.

Ex.2. If $F = 123$ then ΔF is $\{12, 13, 23\}$.

If $R \subseteq T(n)$ then $\Delta R = \cup\{F \in R\}\Delta F$.

Ex.3. Let $P = \{12, 13, 23, 14, 24, 34\}$ and $Q = \{12, 13, 23, 24, 25\}$. Let $R_1 =$

$\{123, 124, 134\}$, $R_2 = \{123, 124, 234\}$, $R_3 = \{123, 124, 125\}$ and $R_4 = \{123, 246, 256\}$. Then $\Delta R_1 = \Delta R_2 = P$ and $\Delta R_3 = Q \cup \{14, 15\}$ and $\Delta R_4 = Q \cup \{26, 46, 56\}$.

If $R \subseteq T(n)$ then clearly $|\Delta R| \leq n|R|$ but we want to know *how small* $|\Delta R|$ *can be in terms of* $|R|$. To answer we order each $T(n)$. We do this by the rule $F < G$, in words F is before G, if $max\{F \setminus G\} < max\{G \setminus F\}$.

Ex.4. Since $max\{124 \setminus 345\} = max\{12\} = 2 < 5 = max\{35\} = max\{345 \setminus 124\}$ we have $124 < 345$ in $T(3)$.

We see this order in Ex.1,3. By an *initial section* IS we mean the first so many sets in order of $T(n)$. Thus P, R_1 are IS in Ex.3. An important fact is that the shadow of an IS is an IS.

Theorem 1 *(Kruskal 1963, Katona 1966). Let* $R \subseteq T(n)$. *If* S *is the* IS *of* $T(n)$ *with* $|R| = |S|$ *then* $|\Delta R| \geq |\Delta S|$.

This theorem has been proved in various ways. Here we will explain the idea of the proof in [3].

2 Shifts of sets.

Given $R \subseteq T(n)$ we want to shift it step by step towards the beginning of the order, until it becomes an IS. Each shift must not increase $|\Delta R|$ or change $|R|$, and Theorem 1 will be proved. If $i, j \in N$ and $i < j$ then the *shift* $i \leftarrow j$ changes a j into an i in R wherever possible. So for $F \in R$ the shift $(i \leftarrow j)F$ is $(F \setminus j) \cup i$ if $i \notin F$ and $j \in F$ and $(F \setminus j) \cup i \notin R$, otherwise it is F. Of course $(i \leftarrow j)R = \{(i \leftarrow j)F : F \in R\}$. In Ex.3 we have $R_1 = (1 \leftarrow 2)R_2$ and $R_3 = (1 \leftarrow 6)R_4$ with $7 = |\Delta R_3| < |\Delta R_4| = 8$. In fact $\Delta(1 \leftarrow 6)R_4 = \Delta R_3 \subsetneq (1 \leftarrow 6)\Delta R_4$ which is a special case of

Lemma 1 $\qquad$ *Shadow*$(shift R) \subseteq$ *Shift*$(shadow R)$.

To prove this for $i \leftarrow j$ you consider an arbitrary $G \in \Delta(i \leftarrow j)R$. There is an $h \in N$ with $G \cup h \in (i \leftarrow j)R$. So there is an $F \in R$ with $G \cup h = (i \leftarrow j)F$. Suppose $j \notin F$. Then $G \cup h = F$ so $j \notin G \in \Delta R$ and $G \in (i \leftarrow j)\Delta R$ as required. An interested reader can easily sort out the other cases.

Now in Ex.3 the IS is R_1 but we cannot get it from R_3 by $i \leftarrow j$ shifts. We need more general shifts. We take sets $I, J \subseteq N$ with $|I| = |J|$ and $I < J$ and $I \cap J$ empty. The shift $I \leftarrow J$ of $F \in R$ is $(F \setminus J) \cup I$ if $I \cap F$ is empty, $J \subseteq F$ and $(F \setminus J) \cup I \notin R$, otherwise it is F. We have $R_2 = (34 \leftarrow 15)R_3$ in Ex.3. If R is not an IS there exist such I, J with $|I|$ minimal such that $(I \leftarrow J)R \neq R$. For this shift Lemma 1 still holds, but it is a little harder to write out the proof. We trust that the reader trusts Lemma 1 and Theorem 1.

3 Cascade Representations of Integers.

To get numbers out of Theorem 1 we let S be an arbitrary IS of $T(n)$. There is a largest $m_n \in N$ such that S contains

$$all \binom{m_n}{n} \ sets \ F \ with \ F \subseteq \{1, ..., m_n\}, \ |F| = n.$$

If there are more sets in S they must possess the number $m_n + 1$. Also there must be a largest m_{n-1} so that S contains

$$all \binom{m_{n-1}}{n-1} \ sets \ G \cup \{m_n + 1\} \ with \ G \subseteq \{1, ..., m_{n-1}\}, \ |G| = n - 1.$$

If there are more sets in S there is a largest m_{n-2} so that S contains

$$all \binom{m_{n-2}}{n-2} \ sets \ H \cup \{m_{n-1} + 1, m_n + 1\} \ with \ H \subseteq \{1, ..., m_{n-2}\}, \ |H| = n - 2,$$

and so on. Assume we end with m_e and put

$$k = \binom{m_n}{n} + \binom{m_{n-1}}{n-1} + ... + \binom{m_e}{e} \ where \ m_n > m_{n-1} > ... > m_e \geq e \geq 1, \tag{1}$$

so $k = |S|$. We call (1) an n-Cascade for k. If we plot (1) on Figure 1, the successive binomial coefficients are in successive columns, each not below the one before. So the plot resembles a cascade of water getting deeper as it runs.

$$\begin{array}{ccccc} \cdots & \binom{3}{3} & \binom{2}{2} & \binom{1}{1} & \binom{0}{0} \\ \cdots & \binom{4}{3} & \binom{3}{2} & \binom{2}{1} & \binom{1}{0} \\ & \vdots & \vdots & \vdots & \vdots \end{array} \qquad \begin{array}{ccccc} \cdots & \binom{3}{0} & \binom{2}{0} & \binom{1}{0} & \binom{0}{0} \\ \cdots & \binom{4}{1} & \binom{3}{1} & \binom{2}{1} & \binom{1}{1} \\ & \vdots & \vdots & \vdots & \vdots \end{array}$$

$$For \ sets \ \binom{r}{s} \rightarrow \binom{r}{s-1} \qquad For \ vectors \ \binom{r}{s} \rightarrow \binom{r-1}{s}$$

Figure 1. *Two orientations of Pascal's Triangle.*

From the above construction we see that ΔS contains

$$all \binom{m_n}{n-1} \ sets \ G \ with \ G \subseteq \{1, ..., m_n\}, \ |G| = n - 1,$$

$$all \binom{m_{n-1}}{n-2} \ sets \ H \cup \{m_n + 1\} \ with \ H \subseteq \{1, .., m_{n-1}\}, \ |H| = n - 2,$$

and so on. Thus we just move the n-Cascade (1), using the rule in Figure 1, to get the (n-1)-Cascade for $|\Delta S|$. There may be one other $m_1, ..., m_e$ which satisfies (1), it has $e = 0$. We can use either. Also because ΔS is an IS we can repeat to get $|\Delta\Delta S|$,...

Ex.5. (Lehmer 1964). The 9167-th in $S(12)$ is $1,2,4,5,8,10,11,12,13,15,16,18$.

Ex.6. (Clements 1974). If $P, Q, S \subseteq T(n)$ and S is the IS with $|P| + |Q| = |S|$ then $|\Delta P| + |\Delta Q| \geq |\Delta S|$. Just add a large h to every integer of Q.

Ex.7. (Hilton 1979). let P, Q, R be IS with $P, Q \subseteq T(n)$, $R \subseteq T(n-1)$ and $|P| = |Q| + |R|$ then $|\Delta P| \leq |\Delta R| + max\{|R|, |\Delta Q|\}$.

4 The V-order for vectors.

For $n \in N$ let $V(n)$ be all n-dimensional 0,1 vectors.

Ex.8. $V(2) = \{00, 10, 01, 11\}$ and $V(3) = \{000, 100, 010, 001, 110, 101, 011, 111\}$.

If $A \in V(n)$ the *shadow* $\Delta A \subseteq V(n-1)$ is obtained by deleting a coordinate of A in all ways.

Ex.9. So $\Delta(011100) = \{01100, 11100, 01110\}$.

Also $\Delta W = \bigcup\{A \in W\}\Delta A$ for $W \subseteq V(n)$. We need some notation. For $A = (a_1, .., a_n) \neq B = (b_1, ..., b_n)$ in $V(n)$ we put $\nu A = a_1 + ... + a_n$ and $\alpha(A, B)$ is the first i with $a_i \neq b_i$. Also "$A = (a_1, ..., a_{n-1})$ and A" $= a_n$ and $dA = (d, a_1, ..., a_n)$ for $d = 0, 1$.

Definition 1 *(V-order). If $A, B \in V(n)$ then $A < B$ if either (i) $\nu A < \nu B$, or (ii) $\nu A = \nu B$ and $1 = a_j > b_j = 0$ with $j = \alpha(A, B)$.*

This order is seen in Ex.8,9. Using this we proved in [2] the main result, which is Theorem 2 below. We will not attempt the proof of Theorem 2 here. It seems of necessity to be hard due to the fact there are infinitely many other orders equally good for it.

Theorem 2 *Let $W \subseteq V(n)$. If S is the IS of $V(n)$ with $|W| = |S|$ then $|\Delta W| \geq |\Delta S|$.*

5 The sequence of free sequences.

Using V-order let V$(n,t) = \{A \in V(n) : \nu A = t\} = A_1 < A_2 < ... < A_e$ so $e = \binom{n}{t}$. Also let $\Psi(n,t)$ be the 0,1 sequence A_1", A_2", ..., A_e". Here V, Ψ are empty unless $0 \leq t \leq n$. We see that $V(2,0)$ is 00, and $V(2,1)$ is $10 < 01$, and $V(2,2)$ is 11. In general $V(n+1,t)$ is $1V(n,t-1)$ followed by $0V(n,t)$, so $\Psi(n+1,t)$ is $\Psi(n,t-1)$ followed by $\Psi(n,t)$, and we can contruct Ψ without using V-order. The n-th *free sequence* Ψ_n has length 2^n and is $\Psi(n,0), \Psi(n,1), ..., \Psi(n,n)$.

Let $A \in V(n)$ and S be the IS ending A. Next put $R = S \setminus A$ so R is the IS with $|R| = |S| - 1$. If A" $= 1$ then $\Delta R = \Delta S$ and we say that A is *free* (over R), otherwise A" $= 0$ and $1 + |\Delta R| = |\Delta S|$ because ("A) $\cup$ $(\Delta R) = \Delta S$. From Theorem 2 it is clear that $|\Delta S| = |S| - c$, where c is the sum of the first $|S|$ terms of Ψ_n.

6 Valley Representations of Integers.

On Figure 1 please plot the binominal coefficients of

$$f = \binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} \quad and \quad g = \binom{8}{3} + \binom{6}{2} + \binom{5}{2} + \binom{2}{0} + \binom{1}{0}.$$

We think of $f + g$ as a valley, with the 45 degree slope of f as the left side, and g as the right side. Notice that g is like a Cascade because successive coefficients are to the right, but not below, the last. It is easy to see that each k in $0 \le k \le 2^n$ has one or two n-Valley representations.

Next we find the $(f + g) - th$ vector A in $V(9)$. For $i = 0, 1, 2, 3$ the $i - th$ term in f corresponds to $V(9, i)$, so $\nu A = 4$. The number of vectors in $V(9, 4)$ which start 1 or 011 or 0101 or 0100111 or $B = 01001101$ are respectively $\binom{8}{3}$ or $\binom{6}{2}$ or $\binom{5}{2}$ or $\binom{2}{0}$ or $\binom{1}{0}$, hence $A = B0$.

If S is an IS of $V(n)$ we find the n-Valley of $|S|$, then we move the valley by the rule in Figure 1 to get the $(n-1)$-Valley of $|\Delta S|$. Again ΔS is an IS and we repeat to get $|\Delta\Delta S|$,...

7 The T-order for Vectors.

First let p and q be two sequences $p_1, ..., p_e$ and $q_1, ..., q_e$ with $p_i, q_i \in N$. We order $p < q$ if the least j with $p_j \neq q_j$ has $p_j < q_j$.

Second we find the *type* of a vector by replacing 00 by 0 and 11 by 1 as many times as possible.

Ex.10. If $A = 001111011000$ then type$(A) = 01010$.

We order types as follows $0 < 1 < 10 < 01 < 010 < 101 < 1010 < 0101 < ...$ Two vectors of different types are ordered according to their types. It remains to order vectors of the same type.

Case A" = 0. Let z_1 be the number of zeros at the end of A, then n_1 be the number of ones next to those zeros, and so on. The sequence σA of A is $n_1, n_2, ..., z_2, z_1$. In Ex.10 we see A looking like $z_3, n_2, z_2, n_1, z_1 = 2, 4, 1, 2, 3$ but σA is $2, 4, 2, 1, 3$. We order vectors of the same type according to the order of their sequences.

Case A" = 1. We exchange 0 and 1 to get the complements of vectors, then we order them according to the order of their complements.

Having defined T-order we can now point out that the idea behind it is that the vectors in ΔA have the same type as A or earlier. It turns out that the shadow of an IS is an IS, and $A \in V(n)$ is free iff $a_{n-1} \neq a_n$. Most importantly Theorem 2 holds for V-order and for T-order.

8 Open Problems.

1) *What is smallestl* $R \subseteq V(n)$ *with* $\Delta R = V(n-1)$?

2) *How big can shadows of families of vectors be? More precisely for* $1 \le k \le 2^n$

maximize $|\Delta R|$ *over* $R \subseteq V(n)$ *with* $|R| = k$.

References.

[1] I. Anderson, *Combinatorics of finite sets*, Clarendon Press, Oxford (1987).
[2] Tran Ngoc Danh and David E.Daykin, *Ordering integer vectors for coordinate deletions* (Submitted to J. London Math. Soc.,12 Feb 1994.).
[3] David E.Daykin, *A simple proof of Katona's theorem*, J. Combinatorial Theory, series A,17(1974), 252-253.