SETS OF 0,1 VECTORS WITH MINIMAL SETS OF SUBVECTORS

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Abstract

Let $\mathcal{C}(n)$ be the 0,1 vectors $\underset{\sim}{a} = a_1...a_n$. To get a subvector of $\underset{\sim}{a}$ delete any a_i. If $\mathcal{A} \subseteq \mathcal{C}(n)$ then $\Delta\mathcal{A}$ is the set of all subvectors of members of $\mathcal{A}$, so $\Delta\mathcal{A} \subseteq \mathcal{C}(n-1)$. Put $W\underset{\sim}{a} = a_1 + ... + a_n$. We order $\mathcal{C}(n)$ by $\underset{\sim}{a} < \underset{\sim}{b}$ if (i) $W\underset{\sim}{a} < W\underset{\sim}{b}$ or (ii) $W\underset{\sim}{a} = W\underset{\sim}{b}$ and $1 = a_i > b_i = 0$ for the least i with $a_i \neq b_i$. We present a completely new proof of our Theorem. If $\mathcal{A} \subseteq \mathcal{C}(n)$ and $\mathcal{J}$ is the first $|\mathcal{A}|$ members in $\mathcal{C}(n)$ then $|\Delta\mathcal{J}| \leq |\Delta\mathcal{A}|$.

1 Introduction.

Let $\mathcal{C}(n)$ be the set of all vectors $\underset{\sim}{a} = a_1...a_n$ having $a_h = 0$ or $a_h = 1$ for $1 \leq h \leq n$. Further let $\delta_h\underset{\sim}{a}$ be the subvector in $\mathcal{C}(n-1)$ obtained from $\underset{\sim}{a}$ by deleting coordinate a_h. The *shadow* $\Delta\underset{\sim}{a}$ of $\underset{\sim}{a}$ is the set $\{\delta_1\underset{\sim}{a}, ..., \delta_n\underset{\sim}{a}\}$. The *shadow* $\Delta\mathcal{A}$ of any $\mathcal{A} \subseteq \mathcal{C}(n)$ is $\bigcup\{\underset{\sim}{a} \in \mathcal{A}\}\Delta\underset{\sim}{a}$.

We are concerned with this problem. Given $0 \leq k \leq 2^n$ find $\mathcal{A} \subseteq \mathcal{C}(n)$ with cardinality $|\mathcal{A}| = k$ and shadow size $|\Delta\mathcal{A}|$ minimal. We solved the problem in [1] 1993 by Theorem 1 below, where $\mathcal{C}(n)$ is V-ordered. If $\underset{\sim}{a}, \underset{\sim}{b} \in \mathcal{C}(n)$ and $\underset{\sim}{a} \neq \underset{\sim}{b}$ then $\alpha(\underset{\sim}{a}, \underset{\sim}{b})$ is the first i in $1 \leq i \leq n$ with $a_i \neq b_i$, also $W\underset{\sim}{a} = a_1 + ... + a_n$.

Definition 1 *(V-order). Let $\underset{\sim}{a}, \underset{\sim}{b} \in \mathcal{C}(n)$ with $\underset{\sim}{a} \neq \underset{\sim}{b}$. Then $\underset{\sim}{a} < \underset{\sim}{b}$ if (i) $W\underset{\sim}{a} < W\underset{\sim}{b}$ or (ii) $W\underset{\sim}{a} = W\underset{\sim}{b}$ and $a_j > b_j$ with $j = \alpha(\underset{\sim}{a}, \underset{\sim}{b})$.*

By an *initial section* IS we mean the first so many members of $\mathcal{C}(n)$ in V-order. In particular $IS(a) = \{x \in \mathcal{C}(n) : x \le a\}$.

Theorem 1 *If $\mathcal{J}, \mathcal{A} \subseteq \mathcal{C}(n)$ have $|\mathcal{J}| = |A|$ and $\mathcal{J}$ is an IS then $|\Delta\mathcal{J}| \le |\Delta\mathcal{A}|$.*

In this note we present a completely new proof of Theorem 1. Any statement not proved in full detail is easy to verify. We use condensed notation. If $u = u_1 ... u_r$ and $v = v_1 ... v_s$ are vectors then uv is the vector $u_1 ... u_r v_1 ... v_s$. If $\mathcal{A} \subseteq \mathcal{C}(n)$ and $e \in C(1)$ then $\mathcal{A}e = \{ae : a \in \mathcal{A}\}$. By $\mathcal{A} + \mathcal{B}$ we mean the union $\mathcal{A} \cup \mathcal{B}$ and are saying that $\mathcal{A} \cap \mathcal{B} = \emptyset$, the empty set. Each $\mathcal{A} \subseteq \mathcal{C}(n+1)$ has a partition $\mathcal{A} = A0 + B1$ where $A, B \subseteq \mathcal{C}(n)$. When A, B are IS we say $\mathcal{A}$ is *part compressed* PC.

We tacitly use the fact that for $x, y \in \mathcal{C}(n)$ and any vectors a, b we have $axb < ayb$ iff $x < y$. Given $a \in \mathcal{C}(n)$, if $a = 1...1$ put $Va = 1$, otherwise let Va denote the last h in $1 \le h \le n$ with $a_h = 0$. Then $a^\star = \delta_j a$ where $j = Va$. It is important that $a^\star = max\{\Delta a\}$. Of course $\mathcal{A}^\star = \{a^\star : a \in \mathcal{A}\}$.

2 Initial sections of V-order.

Let $\mathcal{J}$ be an IS. Put $\mathcal{J} = G0 + H1$ and $G = G_0 0 + G_1 1$ and $H = H_0 0 + H_1 1$. Then
(1) G, H are IS,
(2) $H \subseteq G$ and $H_0 \subseteq G_1$ so $H_1 \subseteq H_0 \subseteq G_1 \subseteq G_0$,
(3) $\Delta\mathcal{J} = G = \mathcal{J}^\star$,
(4) $\Delta(\mathcal{J}0) = \mathcal{J}$,
(5) $\Delta(\mathcal{J}1) = G0 + G1$ not usually IS.

G_1	$G0$	$H1$	H_0
	0000		
	1000		
	0100		
00	0010		
		0001	00
	1100		
10	1010		
		1001	10
01	0110		
		0101	01
		0011	
11	1110		
		1101	11
		1011	
		0111	
		1111	

Figure 1. The V-order of $\mathcal{C}(4)$.

In Figure 1 we see the V-orders (i) of $\mathcal{C}(2)$ in G_1, H_0, (ii) of $\mathcal{C}(3)$ in G, H, and (iii) of $\mathcal{C}(4)$ in $G0 + H1$. In the figure $\mathcal{C}(4)$ is cut into slices by the last coordinates of its vectors. Imagine we have this figure for every $\mathcal{C}(n)$.

Consider $IS(a)$ growing with a. We only get an increase in G_1 (resp. H_0) when $a = x10$ (resp. $a = y01$) at the end of a 0-slice (resp. at the beginning of a 1-slice) and the increase is x (resp. y). Moreover if the 0-slice is immediately before the 1-slice then $x = y$. All other vectors in a 0-slice (resp. 1-slice) end 00 (resp. 11). Therefore

(6) if $\mathcal{J} = IS(x10)$ then $G_1 = H_0 + x$ otherwise $G_1 = H_0$.

Let $n \geq 3$ and S be the 0-slice just before a 1-slice T. If $|T| \geq 2$ then $|S| = 1$, but if $|T| = 1$ then $|S| \geq 2$.

3 Part compressed families.

Let D, E be any IS of $\mathcal{C}(n)$ and put $\mathcal{B} = D0 + E1$ so $\mathcal{B}$ is PC. Next Γ is the set of $a \in \mathcal{C}(n+1)$ of the form $a = uz1$, where $z = 0...0$ of $dimz \geq 1$, and either $u = \emptyset$ or u ends 1. Also $\gamma a = u1z$ if $a = uz1 \in \Gamma$. We can now define a map $\psi : \mathcal{B} \longrightarrow \mathcal{C}(n+1)$ by

$$\psi a = \begin{cases} \gamma a & \textit{if } a \in \Gamma \textit{ and } \gamma a \notin \mathcal{B}, \\ a & \textit{otherwise.} \end{cases}$$

Theorem 2 *If $\mathcal{B}$ is PC then $\Delta\psi\mathcal{B} \subseteq \psi\Delta\mathcal{B}$.*

Proof. Observe that the two ψ are different. We can have $\mathcal{B} = D0 + E1$ as above. Let $x \in \Delta\psi\mathcal{B}$. There is a $b \in \psi\mathcal{B}$ with $x \in \Delta b$ and an $a \in \mathcal{B}$ with $b = \psi a$. We distinguish three possibilities. Case 1. $a \notin \Gamma$. Here $b = a$ and a ends 0 or 11. Case 2. $a \in \Gamma$ and $b \neq a$. So $b = u1z$ and $a = uz1$. Case 3. $a \in \Gamma$ and $b = a$. Here $a = uz1$ and $u1z$, a are both in $\mathcal{B}$.

Please look at the last line of Case 1 in Figure 2. There we are given $a = w11$, $x = w1$ and w ends 0. So $w = uz$ where $u = \emptyset$ or u ends 1. As shown $u1z1 \in \mathcal{B}$ yielding $u1z$, $uz1 \in \Delta\mathcal{B}$ and $x = \psi x \in \psi\Delta\mathcal{B}$ as required. In the last line of Case 2 we are given a, b, x, u' as shown for some w. Now u ends 1 so $x = w01z \in \Delta\mathcal{B}$ and $x = \psi x \in \psi\Delta\mathcal{B}$ again. On the final line we are in Case 3 with $x = u'z1$ and $u' = wz_1$ where w ends 1 or $w = \emptyset$. Hence $u = wz_1 1$ and $c = wz_1 11z \in \mathcal{B}$ and $c > d = w1z_1 1z$. Because D is IS this gives a, $d \in \mathcal{B}$ and $w1z_1z, wz_1z1 = x \in \Delta\mathcal{B}$ so $x \in \psi\Delta\mathcal{B}$.$\diamond$

	x	ψa	a	$\in \Delta\mathcal{B}$	Comment
Case 1.	$w'0$	$w0$		$w'0$	
	$w \notin \Gamma$	"		w	
	$w \in \Gamma$	"		$u1z,\ uz1$	$a = uz10 > u1z0 \in \mathcal{B}$ as $D = IS$
	$w'11$	$w11$		$w'11$	
	$w1,\ w_n = 1$	"		$w1$	
	$w1,\ w_n = 0$	"		$u1z,\ uz1$	$a = uz11 > u1z1 \in \mathcal{B}$ as $E = IS$
Case 2.	$u1z'$	$u1z$	$uz1$	$uz'1$	
	uz	"	"	uz	
	$u'1z$	"	"	$u'z1$	u' ends 1 or $u' = \emptyset$
	$u'1z$	"	"	$w01z$	$u' = w0,\ u = w01,\ a = w01z1$
Case 3	uz	$uz1$		uz	
	$uz'1$	"		$u1z',\ uz'1$	
	$u'z1$	"		$u'1z,\ u'z1$	u' ends 1 or $u' = \emptyset$
	$u'z1$	"		" "	u' ends 0. See text.

Figure 2. Proof table for Theorem 2.

Theorem 3 [1] *Let D, E be IS of $\mathcal{C}(n)$ with $E \subseteq D$. Put $D = D_0 0 + D_1 1$ and $E = E_0 0 + E_1 1$ and $\mathcal{B} = D0 + E1$. Then*

$$\Delta\mathcal{B} = \begin{cases} D & \text{if } E_0 \subseteq D_1, \\ D_0 0 + E_0 1 & \text{If } D_1 \subseteq E_0. \end{cases}$$

Proof. The result holds as $\Delta\mathcal{B} = D_0 0 \cup D_1 1 \cup E_0 1$ by (3).◇

Example. To show that $D_1 = E_0$ does not imply $\mathcal{B} = IS$, let $D = IS(1110)$ and $E = IS(0010)$ then $D_1 = E_0 = IS(001)$ but $\mathcal{B} = IS(11100) \setminus 00011$ and $\mathcal{B} \neq IS$.

4 Proof of Theorem 1.

The result holds trivially for $n = 1, 2$. To use induction we assume it holds for $n \leq m$. Let $\mathcal{A} \subseteq C(m+1)$ and put $\mathcal{A} = A0 + B1$ so

$$\Delta\mathcal{A} = A \cup (\Delta A)0 \cup B \cup (\Delta B)1.$$

[1] Remark on Theorem 3. Let f be the final vector in $\mathcal{B}$. Let $\mathcal{J}$ be the largest IS in $\mathcal{B}$ and put $\mathcal{F} = \mathcal{B} \setminus \mathcal{J}$. If $D_1 \subseteq E_0$ then $\mathcal{B} \neq IS$ and f ends 1 and $\Delta\mathcal{B} = \mathcal{J}^\star + \mathcal{F}^\star$. If $E_0 \subseteq D_1$ and $\mathcal{B} \neq IS$ then f ends 0. We do not use these facts in this note.

By exchanging 0 and 1 we may assume $|A| \geq |B|$. Let D, E be IS of $\mathcal{C}(m)$ with $|D| = |A|$ and $|E| = |B|$ so $E \subseteq D$. By induction $|\Delta D| \leq |\Delta A|$ and $|\Delta E| \leq |\Delta B|$. Put $D = D_0 0 + D_1 1$ and $E = E_0 0 + E_1 1$ and $\mathcal{B} = D0 + E1$ as usual so $\mathcal{B}$ is PC.

Case 1. $E_0 \subseteq D_1$. By Theorem 3 we have $|\Delta\mathcal{B}| = |D| = |A| \leq |\Delta\mathcal{A}|$. We forget $\mathcal{A}$ and study the slices of $\mathcal{B}$ in Figure 1. Let R, S, T, U be the consecutive slices with S the last 0-slice containing a vector of $\mathcal{B}$. Thus R, T are 1-slices and $U \cap \mathcal{B} = \emptyset$. Let $\underset{\sim}{x}10$ be the last vector of S, then $\underset{\sim}{x}01$ is the first vector of T.

Case 1.1. $\underset{\sim}{x}01 \in \mathcal{B}$. Here $R, S \subseteq \mathcal{B}$ and $\mathcal{B} = IS$.

Case 1.2. $\underset{\sim}{x}01 \notin \mathcal{B}$. Here $T \cap \mathcal{B} = \emptyset$. If $\mathcal{B} = IS$ we are done, so assume $\mathcal{B} \neq IS$. Let $\underset{\sim}{f}$ be the final vector of $\mathcal{B}$ so $\underset{\sim}{f} \in S$. There is a first $\underset{\sim}{e} \notin \mathcal{B}$ and $\underset{\sim}{e} < \underset{\sim}{f}$ and $\underset{\sim}{e}$ ends 1. We exchange $\mathcal{B}$ for $\mathcal{D} = (\mathcal{B} \setminus \underset{\sim}{f}) + \underset{\sim}{e}$. Then $\mathcal{D}$ is in Case 1. Also $|\Delta\mathcal{D}| = |\Delta\mathcal{B}| - 1$ by Theorem 3, so we start on $\mathcal{D}$. Repetition proves Theorem 1 holds in Case 1.

Case 2. $D_1 \subseteq E_0$. By Theorem 3 and induction

$$|\Delta\mathcal{B}| = |D_0| + |E_0| = |\Delta D| + |\Delta E| \leq |\Delta A| + |\Delta B| \leq |\Delta\mathcal{A}|.$$

So we deal with $\mathcal{B}$. Let T be the last 1-slice containing a vector of $\mathcal{B}$. Again let R, S, T, U be consecutive slices. Further let $\underset{\sim}{x}10, \underset{\sim}{x}01$ be as before. Then $\underset{\sim}{x}01 \in \mathcal{B}$ but by definition of this case $\underset{\sim}{x}10 \notin \mathcal{B}$. Let $\underset{\sim}{g}$ be the first vector of S.

Case 2.1. $\underset{\sim}{g} \in \mathcal{B}$. Here $E_0 = D_1 + \underset{\sim}{x}$ and $|\Delta\mathcal{B}| = |D_0| + |E_0| = |D| + 1$ by Theorem 3. We have $|S| \geq 2$ and this implies $|T| = 1$. If $\underset{\sim}{e}$ is the first vector not in $\mathcal{B}$, then $\underset{\sim}{e} \in S$, and $\mathcal{D} = (\mathcal{B} \setminus \underset{\sim}{x}01) + \underset{\sim}{e}$ is an IS. Again using Theorem 3 we get $|\Delta\mathcal{D}| = |\mathcal{D}_0| = |D| + 1 = |\Delta\mathcal{B}|$. So Theorem 1 holds in this Case 2.1.

Case 2.2. $\underset{\sim}{g} \notin \mathcal{B}$. Put $\mathcal{D} = \psi\mathcal{B}$ and note that $\mathcal{D} \neq \mathcal{B}$ because $\underset{\sim}{g} = \psi(\underset{\sim}{x}01)$, except in the trivial case $\underset{\sim}{g} = \underset{\sim}{z}$. But $|\mathcal{D}| = |\mathcal{B}|$ and Theorem 2 tells that $|\Delta\mathcal{D}| \leq |\Delta\mathcal{B}|$. If $\mathcal{D} = A0 + B1$ then $B \subset E \subseteq D \subset A$ so we can only enter this Case 2.2 a finite number of times. We forget $\mathcal{B}$ and work on $\mathcal{D}$. Since $\mathcal{D}$ may not be PC we must start at the very beginning.$\diamondsuit$

References.

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