A Cascade Proof of a Finite Vectors Theorem

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Abstract

Let $\mathcal{C}(n)$ be the n-dimensional 0,1 vectors. If $\underset{\sim}{a} \in \mathcal{C}(n)$ then $\Delta\underset{\sim}{a} \subseteq \mathcal{C}(n-1)$ is obtained by deleting a coordinate of $\underset{\sim}{a}$ in all n ways. Put $\Delta\mathcal{A} = \bigcup\{\underset{\sim}{a} \in \mathcal{A}\}\Delta\underset{\sim}{a}$ for all $\mathcal{A} \subseteq \mathcal{C}(n)$. Given $k, n \geq 1$ the Danh-Daykin Theorem finds the best possible lower bound for $|\Delta\mathcal{A}|$ over all $\mathcal{A} \subseteq \mathcal{C}(n)$ with $|\mathcal{A}| = k$. A new proof is given. It is based on Daykin's cascade algorithm.

1 Introduction.

Let $\mathcal{C}(n)$ be the set of all vectors $\underset{\sim}{a} = a_1a_2...a_n$ with each coordinate a_i zero or one. For $1 \leq h \leq n$ let $\delta_h\underset{\sim}{a}$ be the vector in $\mathcal{C}(n-1)$ obtained by deleting a_h from $\underset{\sim}{a}$. The *shadow* $\Delta\underset{\sim}{a}$ is the set $\{\delta_1\underset{\sim}{a}, ..., \delta_n\underset{\sim}{a}\}$. The shadow of an arbitrary $\mathcal{A} \subseteq \mathcal{C}(n)$ is $\Delta\mathcal{A} = \bigcup\{\underset{\sim}{a} \in \mathcal{A}\}\Delta\underset{\sim}{a} \subseteq \mathcal{C}(n-1)$. In Theorem 1 below is the best possible lower bound for $|\Delta\mathcal{A}|$ in terms of $|\mathcal{A}|$, where $|.|$ denotes cardinality. If $\underset{\sim}{a}, \underset{\sim}{b} \in \mathcal{C}(n)$ and $\underset{\sim}{a} \neq \underset{\sim}{b}$ then $\alpha(\underset{\sim}{a}, \underset{\sim}{b})$ denotes the first i in $1 \leq i \leq n$ with $a_i \neq b_i$. Also $W\underset{\sim}{a} = a_1 + ... + a_n$.

Definition 1 *(V-order [5]). Let $\underset{\sim}{a}, \underset{\sim}{b} \in \mathcal{C}(n)$ with $\underset{\sim}{a} \neq \underset{\sim}{b}$. Then $\underset{\sim}{a} < \underset{\sim}{b}$ if (i) $W\underset{\sim}{a} < W\underset{\sim}{b}$ or (ii) $W\underset{\sim}{a} = W\underset{\sim}{b}$ and $1 = a_j > b_j = 0$ with $j = \alpha(\underset{\sim}{a}, \underset{\sim}{b})$.*

For $0 \leq k \leq 2^n$ the first k vectors in $\mathcal{C}(n)$ in V-order is called the initial section $IS(k, n)$.

Theorem 1 *(Danh-Daykin [5,6]). If $\mathcal{J}, \mathcal{A} \subseteq \mathcal{C}(n)$ have $|\mathcal{J}| = |\mathcal{A}|$ and $\mathcal{J}$ is an IS then $|\Delta\mathcal{J}| \leq |\Delta\mathcal{A}|$.*

This note is a new proof of Theorem 1, based on Daykin's Cascade Algorithm [3,4]. To set the stage, in Section 2 we state results of Danh-Daykin [5,6] which are easy.

2 Preliminary results.

By $\mathcal{A}+\mathcal{B}$ we denote $\mathcal{A}\cup\mathcal{B}$ and say $\mathcal{A}\cap\mathcal{B}=\emptyset$ the empty set. If $\underset{\sim}{u}=u_1...u_r$ and $\underset{\sim}{v}=v_1...v_s$ are vectors then $\underset{\sim}{u}\underset{\sim}{v}$ is the vector $u_1...u_rv_1...v_s$. If $\mathcal{A}\subseteq\mathcal{C}(n)$ and $\underset{\sim}{e}\in\mathcal{C}(1)=\{0,1\}$ then $\mathcal{A}\underset{\sim}{e}=\{\underset{\sim}{a}\underset{\sim}{e}:\underset{\sim}{a}\in\mathcal{A}\}$, also $\mathcal{A}=A0+B1$ for some $A,B\subseteq\mathcal{C}(n-1)$. We say $\mathcal{A}$ is *part compressed* PC if A,B are IS. By $IS(\underset{\sim}{a})$ we mean $\{\underset{\sim}{x}\in\mathcal{C}(n):\underset{\sim}{x}\le\underset{\sim}{a}\}$ where $\underset{\sim}{a}\in\mathcal{C}(n)$.

Let $\mathcal{J}$ be an IS of $\mathcal{C}(n+1)$. Put $\mathcal{J}=G0+H1$ and $G=G_00+G_11$ and $H=H_00+H_11$. Then
(1) G,H are IS in $\mathcal{C}(n)$, (so G_0,G_1,H_0,H_1 are IS in $\mathcal{C}(n-1)$),
(2) $H\subseteq G$ and $H_0\subseteq G_1$, (so $H_1\subseteq H_0\subseteq G_1\subseteq G_0$),
(3) $\Delta\mathcal{J}=G$, (so $\Delta G=G_0$ and $\Delta H=H_0$),
(4) if $\mathcal{J}=IS(\underset{\sim}{x}10)$ then $G_1=H_0+\underset{\sim}{x}$ but $H_0=G_1$ otherwise.

Theorem 2 *Let D,E be IS of $\mathcal{C}(n)$ with $E\subseteq D$. Put $D=D_00+D_11$ and $E=E_00+E_11$. If $\mathcal{B}$ is the PC family $\mathcal{B}=D0+E1$ then*

$$\Delta\mathcal{B}=\begin{cases}D & \text{if } E_0\subseteq D_1,\\ D_00+E_01 & \text{if } D_1\subseteq E_0.\end{cases}$$

Using induction on dimension we next show that it is sufficient to prove Theorem 1 for PC families. The Theorem is trivial for $n=1,2$. Assume it holds for $n\le m$. Given any $\mathcal{A}\subseteq\mathcal{C}(m+1)$ put $\mathcal{A}=A0+B1$ then

$$\Delta\mathcal{A}=A\cup(\Delta\mathcal{A})0\cup B\cup(\Delta B)1.$$

We may asume $|A|\ge|B|$ by exchanging 0,1 if necessary. Let D,E be the IS of $\mathcal{C}(m)$ with $|D|=|A|$ and $|E|=|B|$ so $E\subseteq D$. By induction $|\Delta D|\le|\Delta A|$ and $|\Delta E|\le|\Delta B|$. If $\mathcal{B}$ is the PC family $\mathcal{B}=D0+E1$ we easily get $|\Delta\mathcal{B}|\le|\Delta\mathcal{A}|$ from Theorem 2.

3 Part compressed families.

Let $\mathcal{B}=D0+E1$ be a PC family in $\mathcal{C}(n+1)$ with $E\subseteq D$. Let $k=|\mathcal{B}|$ and $\mathcal{J}=IS(k,n+1)=G0+H1$.
Case 1. $G\subseteq D$. Here $E\subseteq H$ so (2) gives $E_0\subseteq H_0\subseteq G_1\subseteq D_1$. Then $\Delta\mathcal{J}=G\subseteq D=\Delta\mathcal{B}$ by Theorem 2. This proves the case $G\subseteq D$ of Theorem 1.
Case 2. $|D|=|G|-1$. We claim that $E_0\neq D_1\subseteq E_0$. To get D we remove the last vector $\underset{\sim}{v}$ of G from G. Similarly to get E we add to H the first vector not in H. Let $\underset{\sim}{w}$ be the last vector in $\mathcal{C}(n+1)$ with the properties (i) $\underset{\sim}{v}0\le\underset{\sim}{w}$, and (ii) $w_{n+1}=0$, and (iii) $\underset{\sim}{v}0<\underset{\sim}{u}<\underset{\sim}{w}$ implies $u_{n+1}=0$. Then $\underset{\sim}{w}$ has the form $\underset{\sim}{x}10$, and we can use (4).

Now $\underset{\sim}{w} \notin \mathcal{B}$ and $\underset{\sim}{x} \notin D_1$ but $\underset{\sim}{x}01 \in \mathcal{B}$, so $\underset{\sim}{x} \in E_0$ and the claim is proved.

Case 3. $G \neq D \subseteq G$. The D here is contained in the D of Case 2. So the D_1 here is contained in the D_1 of Case 2. Working similary on E, from Case 2 we conclude that $E_0 \neq D_1 \subseteq E_0$ also in this more general Case 3.

We need only complete the proof of Theorem 1 for Case 3. This can be done by using the push down and $10 \leftarrow 01$ shift of [5], but we omit the routine details. In this note our interest is in cascades.

4 Cascades and valleys.

Please look at Figure 1. The binomial coefficients $\beta = \binom{r}{s}$ cover the plane, so r, s may be negative, although we have ommitted those $\beta = 0$ from Figure 1. Put

$$\uparrow \beta = \binom{r-1}{s-1}, \quad \nearrow \beta = \binom{r-2}{s-1}, \quad \vec{\beta} = \binom{r-1}{s}, \quad \searrow \beta = \binom{r}{s+1},$$

so $\nearrow \beta = \uparrow (\vec{\beta}) = {}^{\to}(\uparrow \beta)$. Also $\beta = (\uparrow \beta) + (\vec{\beta})$ except if $\beta = bom$. This exception requires care, without care we "blow-up" our calculations. Of course $\uparrow \sum \beta = \sum \uparrow \beta$ and so on.

top	$\binom{3}{0}$	$\binom{2}{0}$	$\binom{1}{0}$	|	*bom*	*bom* $= \binom{0}{0}$
– – – – –	– – – –	– – – –	– – – –	|	– – – –	
		$\binom{3}{1}$	$\binom{2}{1}$	|	$\binom{1}{1}$	
			$\binom{3}{2}$	|	$\binom{2}{2}$	
				|	$\binom{3}{3}$	*margin*

Figure 1. *Pascal's Triangle.*

Consider $\mathcal{J} = IS(\underset{\sim}{a})$ where $\underset{\sim}{a} = 0101101 \in \mathcal{C}(7)$, and put

$$p = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} \quad \textit{and} \quad q = \binom{6}{3} + \binom{4}{2} + \binom{1}{0} + \binom{0}{0}.$$

Since $W\underset{\sim}{a} = 4$ we find in $\mathcal{J}$ all $\underset{\sim}{b}$ with $W\underset{\sim}{b} \leq 3$, and these $\underset{\sim}{b}$ are counted by p. The other $\underset{\sim}{b} \in \mathcal{J}$ have $W\underset{\sim}{b} = 4$. There are all such $\underset{\sim}{b}$ starting 1 or 011 or 010111 or $\underset{\sim}{a}$ itself, and the numbers of these are given by the successive coefficients in q. Finally $|\mathcal{J}| = p + q$. It should now be clear how to find $|\mathcal{J}|$ for any $\underset{\sim}{a}$, and conversely, how to find $\underset{\sim}{a}$ given the dimension and $|\mathcal{J}|$.

If we plot the coefficients of p, q on Figure 1 the result looks like a *valley* [5] which we denote by $V(\underset{\sim}{a}) = p + q$. The left side of the valley is from p and has a 45^o slope. The right side is from q and is a *cascade.* In other words the coefficients of q are in adjacent columns, each coefficient not below the one before it. Cascades are well known, particularly in connection with the Kruskal-Katona Theorem [1,2]. Notice that if $\underset{\sim}{a} \in \mathcal{C}(n)$ then the valley and cascade are an n-Valley and a g-Cascade, meaning they start in columns n and $g = n - W\underset{\sim}{a}$ respectively. They can stop anywhere.

We call a cascade *proper* if it does not go into the margin, otherwise *improper*. Using the identity

(5) $$\binom{r+1}{s+1} = \binom{r}{s} + \binom{r-1}{s} + \ldots + \binom{s+1}{s} + \binom{t}{t} \quad for \ r > s \geq 0 \leq t,$$

we can always make an improper cascade proper. Conversely if a proper cascade does not go to the top, the identity will make it improper, if we take $t \leq s$. Given a fixed $g \geq 1$, every positive integer has a unique proper g-Cascade. Given $n \geq 1$, every k in $0 < k < 2^n$ has a unique n-Valley. There is only one n-Valley for 2^n but it goes into the margin. Improper cascades and valleys are unique to within one application of (5). The identity (5) also shows that if a cascade for $m \geq 1$ starts at $\binom{r}{s}$ then $m \leq \binom{r+1}{s+1}$, with strict $<$ when the cascade is proper. Take any $\underset{\sim}{a} \in \mathcal{C}(n)$. It has a valley $V(\underset{\sim}{a})$ and an $IS(\underset{\sim}{a}) = G0 + H1$. We point out

(6) $$^{\rightarrow}V(\underset{\sim}{a}) = V(G),$$

(7) $$\uparrow (proper V(\underset{\sim}{a})) = V(H),$$

(8) $$\nearrow (proper V(\underset{\sim}{a})) = V(H_0),$$

(9) $$\uparrow (proper(^{\rightarrow}V(\underset{\sim}{a})) = V(G_1).$$

In view of (4) it is interesting that (8) and (9) are different iff $V(\underset{\sim}{a})$ ends $\binom{1}{0}$.

5 Daykin's Cascade Algorithm.

Let $g \geq 1$ and M, N be two g-Cascades. We make two new g-Cascades M', N' with (i) $M' + N' = M + N$, and (ii) $M' \geq M$, and (iii) $^{\rightarrow}(M') + {}^{\rightarrow}(N') \leq \vec{M} + \vec{N}$.

Operation 1. (Bigger and smaller cascades). In every column we do the following: If both M and N have a coefficient in the column we give the bigger to M' and the smaller to N'. If only one of M and N have a coefficient in the column we give it to M'.

Operation 2. Make M'= proper M and $N' = N$.

Operation 3. When N does not go to top. Put $M' = M$ and N'=improper N.

Operation 4. (Single coefficient transfer). When the last coefficient of N is at top, so it is 1, but it is not *bom*. Delete this 1 from N to get N'. Make M' the g-Cascade which is M plus one coefficient 1 from top.

Our conditions on M', N' are clearly satisfied. Note that we get equality in (iii) except in Operation 4 with $M' = M + bom$, when there is a difference of 1 in (iii).

We perform the operations in any sensible order. We stop to look around when $N' = 0$, or if M' has a different first coefficient from M. In the latter case we have just used (5) on Operation 2 and M' is just a single coefficient.

$$
\begin{array}{ccccccc}
e & = & \binom{r-1}{s-1} & & & \binom{r-2}{s-1} & \\
i & = & \binom{r}{s} & d & = & \binom{r-1}{s} & \binom{r-2}{s}
\end{array}
$$

Figure 2. *An* $i = d + e$ *triangle has* $\uparrow d = \vec{e}$ *and* $\vec{i} = \vec{d} + \vec{e}$.

6 End of proof of Theorem 1.

We are given a PC family $\mathcal{B} = D0 + E1$. Then we let $\mathcal{J} = G0 + H1$ be the *IS* with $|\mathcal{J}| = |\mathcal{B}|$. By earlier work $E \subseteq D$ and $G \neq D \subseteq G$. Consequently $E_0 \neq D_1 \subseteq E_0$. At each of our steps we will change $\mathcal{B}$ by increasing D but keeping $|\mathcal{B}| = |D| + |E|$ constant. On Figure 1 we plot proper $V(D)$ and $V(E)$. Then we look for the kind of *triangles* in Figure 2. The first such triangle has $e = \binom{n}{-1} = 0$ and $d = \binom{n}{0}$. In a triangle $d \in V(D)$ and $e \in V(E)$. It will turn out later that $i \in V(\mathcal{J})$. The contribution $\uparrow d$ of d to D_1 equals that $\vec{e}$ of e to E_0. Similarly the joint contribution $\vec{d} + \vec{e}$ of d, e to $|\Delta\mathcal{B}| = |D_0| + |E_0|$ equals that $\vec{i}$ of i to $|\Delta\mathcal{J}| = |{}^{\rightarrow}V(\mathcal{J})|$. A triangle is like a link in a zip fastener. We find as many adjacent triangles as there are, and zip them up, removing them from Figure 1. What is left of $V(D)$ and $V(E)$ we call the D-tail and E-tail respectively. Since $|E_0| > |D_1|$ the E-tail is not empty. Let β be its first coefficient. Also let $i' = d' + e'$ be the last removed triangle. Clearly we cannot have β below d'.

Case 1. D-tail empty. Make $\mathcal{J}$-tail= E-tail. We have constructed a proper $V(\mathcal{J})$ with $|\Delta\mathcal{B}| = |D_0| + |E_0| = |\Delta\mathcal{J}|$, so Theorem 1 holds in this case.

Case 2. D-tail not empty. Let $\Theta = \downarrow \beta$ and $\lambda = \vec{\beta}$ and $\mu = \vec{\Theta} = \searrow \beta = \downarrow \lambda$ and $\nu = \downarrow \mu$. Now E-tail$< \Theta$ so (E-tail)$_0 < \vec{\Theta} = \mu$. Suppose the D-tail starts at or below ν. Then (D-tail)$_1 \geq \uparrow \nu = \mu$ contradicting $|E_0| > |D_1|$. The D-tail cannot start at μ for then we would have one more triangle. We conclude that the D-tail starts at or above λ.

Case 2.1. β = E-tail. Make $\mathcal{J}$-tail=β + D-tail. As in Case 1 we have proper $V(\mathcal{J})$ with $|\Delta\mathcal{B}| = |\Delta\mathcal{J}|$.

Case 2.2. $\beta \neq$ E-tail. Suppose that (E-tail)-β starts below λ. Then β must be on the 45° slope and $\beta = d'$, contradicting $|E| \leq |D|$. Hence (E-tail)-β starts like D-tail at or above λ.

All that remains is to apply the algorithm to M=D-tail and N=(E-tail) - β. If we stop with $M' = \mu$ we have another triangle, we zip up and are in Case 1. Otherwise we continue till $N' = 0$, we make M' proper and we are in Case 2.1.

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