IDENTITIES FOR COMBINATORIAL EXTREMAL THEORY

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1. *Introduction*

Let Ω be the set $\{1, 2, \ldots, n\}$, and let $\varnothing$ be the empty set. Let $\mathscr{G}$ be the family of all non-empty sets of subsets of Ω. For $\mathscr{A} \in \mathscr{G}$ and $X \subseteq \Omega$, put

$$Z_{\mathscr{A}}(X) = \begin{cases} \varnothing & \text{if there is no } A \in \mathscr{A} \text{ with } A \subseteq X, \\ \bigcap_{A \in \mathscr{A},\, A \subseteq X} A & \text{otherwise.} \end{cases}$$

An important discovery is the following.

THEOREM 1 (Ahlswede and Zhang [1]). *If* $\varnothing \notin \mathscr{A} \in \mathscr{G}$, *then*

$$\sum \frac{|Z_{\mathscr{A}}(X)|}{|X| \binom{n}{|X|}} = 1, \tag{1}$$

where summation is over non-empty $X \subseteq \Omega$.

This theorem has a dual, Theorem 1D below. For $\mathscr{A} \in \mathscr{G}$ and $X \subseteq \Omega$, put

$$Z^*_{\mathscr{A}}(X) = \begin{cases} \Omega & \text{if there is no } A \in \mathscr{A} \text{ with } A \supseteq X, \\ \bigcup_{A \in \mathscr{A},\, A \supseteq X} A & \text{otherwise.} \end{cases}$$

Also, let f be given by $f(0) = 0$ and $f(m) = 1 + (1/2) + \ldots + (1/m)$ for integer $m \geqslant 1$.

THEOREM 1D (Daykin and Thu [2]). *If* $\Omega \notin \mathscr{A} \in \mathscr{G}$, *then*

$$\sum \frac{|Z^*_{\mathscr{A}}(X)|}{(n - |X|) \binom{n}{|X|}} = nf(n-1), \tag{1D}$$

where summation is over $X \subseteq \Omega$, $X \neq \Omega$.

In this note we present the results of a search for similar identities.

2. *A method of investigation*

To illustrate our method, let $y_1, y_2, \ldots, y_n$ be real variables. To study (1) for $\mathscr{A} \in \mathscr{G}$, where $\varnothing \notin \mathscr{A}$, we look for solutions of the identity $(*)$ below, in which summation is over non-empty $X \subseteq \Omega$:

$$\sum |Z_{\mathscr{A}}(X)|\, y_{|X|} = 1. \tag{$*$}$$

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We write out $(*)$ for the example $\mathscr{A} = \{A\}$ with $A = \{1, 2, \ldots, a\}$ and $1 \leqslant a \leqslant n$. This gives the equation

$$\binom{n-a}{0} y_a + \binom{n-a}{1} y_{a+1} + \ldots + \binom{n-a}{n-a} y_n = 1/a.$$

Taking $a = 1, 2, \ldots, n$ in turn, we obtain a triangular set of simultaneous equations with unique solution $y_k = 1/k\binom{n}{k}$, proving that the multipliers in (1) are unique.

In general, we choose a new function instead of $Z_{\mathscr{A}}(X)$, and a trial identity similar to $(*)$. To determine $y_1, y_2, \ldots, y_n$ for the trial identity, we write out particular cases of $\mathscr{A}$, and solve the equations. Finally, we test to see if the trial identity holds in general. In this paper we present our discoveries. Details of the work involved will appear in [**4**].

3. *The function T*

For $\mathscr{A} \in \mathscr{G}$ and $X \subseteq \Omega$, put

$$T_{\mathscr{A}}(X) = \begin{cases} \varnothing & \text{if there is no } A \in \mathscr{A} \text{ with } A \supseteq X, \\ \bigcup\limits_{A \in \mathscr{A},\, A \supseteq X} A & \text{otherwise.} \end{cases}$$

Usually, empty unions are $\varnothing$. Notice that T follows this convention, but Z^* does not.

Let $X \subseteq \Omega$ and $\mathscr{A}, \mathscr{B} \in \mathscr{G}$. As usual, $\mathscr{A} \vee \mathscr{B} = \{A \cup B\colon A \in \mathscr{A}, B \in \mathscr{B}\}$ and $\mathscr{A} \wedge \mathscr{B} = \{A \cap B\colon A \in \mathscr{A}, B \in \mathscr{B}\}$. Then we have all of the following.

$$\text{If } \Omega \in \mathscr{A}, \text{ then } T_{\mathscr{A}} \equiv \Omega. \quad \text{If } \mathscr{A} = \{\varnothing\}, \text{ then } T_{\mathscr{A}} \equiv \varnothing. \tag{2}$$

Now

$$T_{\mathscr{A} \cup \mathscr{B}}(X) = T_{\mathscr{A}}(X) \cup T_{\mathscr{B}}(X) \quad \text{and} \quad T_{\mathscr{A} \wedge \mathscr{B}}(X) = T_{\mathscr{A}}(X) \cap T_{\mathscr{B}}(X), \tag{3}$$

so

$$|T_{\mathscr{A} \cup \mathscr{B}}(X)| = |T_{\mathscr{A}}(X)| + |T_{\mathscr{B}}(X)| - |T_{\mathscr{A} \wedge \mathscr{B}}(X)|. \tag{4}$$

We omit the proofs of (2) and (3) because they are easy and similar to the proofs in [**2**]. For $\mathscr{A} \in \mathscr{G}$, put

$$P(\mathscr{A}) = \sum (-1)^{|X|} |T_{\mathscr{A}}(X)| \quad \text{and} \quad Q(\mathscr{A}) = \sum (-1)^{|X|} |T_{\mathscr{A}}(X)| f(|X|),$$

where summation is over all $X \subseteq \Omega$ and f is as defined in Section 1.

THEOREM 2. *For all $\mathscr{A} \in \mathscr{G}$, we have $P(\mathscr{A}) = 0$.*

Proof. *Case* 1. $\mathscr{A} = \{\varnothing\}$. Here $T_{\mathscr{A}} \equiv \varnothing$ by (2), and $P(\mathscr{A}) = 0$.

Case 2. $\mathscr{A} = \{A\}$, where $A \neq \varnothing$. Then $T_{\mathscr{A}}(X)$ is A or $\varnothing$ as $X \subseteq A$ or $X \nsubseteq A$, respectively. Hence

$$P(\mathscr{A}) = \sum_{X \subseteq A} (-1)^{|X|} |A| = |A| \sum_{X \subseteq A} (-1)^{|X|} = 0.$$

Case 3. $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ with $m > 1$. Let $\mathscr{E} = \{A_1, A_2, \ldots, A_{m-1}\}$ and $\mathscr{F} = \{A_m\}$. By using (4), we obtain

$$P(\mathscr{A}) = P(\mathscr{E}) + P(\mathscr{F}) - P(\mathscr{E} \wedge \mathscr{F}).$$

Then $|\mathscr{E}|, |\mathscr{F}|, |\mathscr{E} \wedge \mathscr{F}| < m$, so by the induction hypothesis, $P(\mathscr{A})$ is $0 + 0 - 0$.

THEOREM 3. *If* $\mathscr{S} \in \mathscr{G}$ *and* $1 \in A$ *for every* $A \in \mathscr{S}$, *then* $Q(\mathscr{S}) = -1$.

Proof. *Case* 1. $\mathscr{S} = \{A\}$, $A \neq \varnothing$ because $1 \in A$. Then $T_{\mathscr{S}}(X)$ is A or $\varnothing$ as $X \subseteq A$ or $X \nsubseteq A$, respectively. Hence if $a = |A|$, then

$$\begin{aligned} Q(\mathscr{S}) &= \sum_{X \subseteq A} (-1)^{|X|} |A| f(|X|) \\ &= |A| \sum_{X \subseteq A} (-1)^{|X|} f(|X|) \\ &= a \sum_{0 \leqslant k \leqslant a} (-1)^k \binom{a}{k} f(k) = -1. \end{aligned}$$

For the last step above, we replace $\binom{a}{k}$ by $\binom{a-1}{k} + \binom{a-1}{k-1}$ and $f(k)$ by $f(k-1) + 1/k$. Then the coefficient of $(-1)^k/k$ in the sum reduces to $\binom{a-1}{k-1}$, which is $\binom{a}{k} k/a$. So the sum is $a(((1-1)^a - 1)/a) = -1$.

Case 2. $\mathscr{S} = \{A_1, A_2, \ldots, A_m\}$ with $m > 1$ and $1 \in A_i$ for all $1 \leqslant i \leqslant m$. Let $\mathscr{E} = \{A_1, \ldots, A_{m-1}\}$ and $\mathscr{F} = \{A_m\}$. Again using (4), we have

$$Q(\mathscr{S}) = Q(\mathscr{E}) + Q(\mathscr{F}) - Q(\mathscr{E} \wedge \mathscr{F}).$$

Then $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{E} \wedge \mathscr{F}$ have cardinalities less than m, and $1 \in A$ for each $A \in \mathscr{E}, \mathscr{F}, \mathscr{E} \wedge \mathscr{F}$. We can apply induction on m to obtain $Q(\mathscr{S}) = (-1) + (-1) - (-1) = -1$.

Next, we present Theorem 4. The method of proof is the same as above.

THEOREM 4. *Let* $\mathscr{B}$ *be a set of disjoint subsets of* Ω, *and* $\varnothing \notin \mathscr{B}$. *Then* $Q(\mathscr{B}) = -|\mathscr{B}|$.

4. *Identities involving chains or disjoint families*

By changing slightly the definition of T, we obtain the function H below, and some interesting results. The proofs are easy. For $\mathscr{A} \in \mathscr{G}$ and $X \subseteq \Omega$, put

$$H_{\mathscr{A}}(X) = \begin{cases} \varnothing & \text{if there is no } A \in \mathscr{A} \text{ with } A \subseteq X, \\ \bigcup\limits_{A \in \mathscr{A},\, A \subseteq X} A & \text{otherwise.} \end{cases}$$

For $\mathscr{A} \in \mathscr{G}$, put $W(\mathscr{A}) = \sum |H_{\mathscr{A}}(X)| / \left(|X| \binom{n}{|X|}\right)$, where summation is over non-empty $X \subseteq \Omega$.

THEOREM 5. *Let* $\mathscr{B}$ *be a set of disjoint subsets of* Ω, *and* $\varnothing \notin \mathscr{B}$. *Then* $W(\mathscr{B}) = |\mathscr{B}|$.

THEOREM 6. *Let* $\mathscr{C}$ *be the chain* $\varnothing \neq A_1 \subset A_2 \subset \ldots \subset A_m$ *of subsets of* Ω. *For* $1 \leqslant i \leqslant m$, *write* $a_i = |A_i|$. Then

$$W(\mathscr{C}) = m - \sum_{1 \leqslant i \leqslant m-1} a_i / a_{i+1}.$$

Proof. *Case* 1. $m = 1$. Then the functions $Z_{\mathscr{C}}$ and $H_{\mathscr{C}}$ are the same, and $W(\mathscr{C}) = 1$, by Theorem 1.

Case 2. $m > 1$. Let $\mathscr{E} = \{A_1, A_2, \ldots, A_{m-1}\}$ and $\mathscr{F} = \{A_m\}$.

If $A_m \subseteq X$, then $H_{\mathscr{C}}(X) = A_m$ and $H_{\mathscr{E}}(X) = A_{m-1}$. If $A_m \nsubseteq X$, then $H_{\mathscr{C}}(X) = H_{\mathscr{E}}(X)$. Hence $W(\mathscr{C}) = W(\mathscr{F}) + \Delta$, where $W(\mathscr{F}) = 1$, by Case 1, and

$$\begin{aligned}\Delta &= \sum_{\varnothing \neq X \subseteq \Omega,\, A_m \nsubseteq X} \frac{|H_{\mathscr{E}}(X)|}{|X|\binom{n}{|X|}} \\ &= \sum_{\varnothing \neq X \subseteq \Omega} \frac{|H_{\mathscr{E}}(X)|}{|X|\binom{n}{|X|}} - \sum_{A_m \subseteq X \subseteq \Omega} \frac{|H_{\mathscr{E}}(X)|}{|X|\binom{n}{|X|}} \\ &= W(\mathscr{E}) - \sum_{A_m \subseteq X \subseteq \Omega} \frac{|A_{m-1}|}{|X|\binom{n}{|X|}} \\ &= W(\mathscr{E}) - \frac{a_{m-1}}{a_m} W(\mathscr{F}).\end{aligned}$$

We apply induction on m to end the proof.

COMMENT. For each identity in this paper, we obtain the dual by taking complements in the same way as for the dual (1D) of the identity (1) in [**2**].

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