On Interactive Communication

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***Abstract*— Almost two decades ago Ahlswede introduced an abstract correlated source $(\mathcal{V}\times\mathcal{W},\mathcal{S})$ with outputs $(v,w)\in\mathcal{S}\subset\mathcal{V}\times\mathcal{W}$, where persons $P_\mathcal{V}$ and $P_\mathcal{W}$ observe v and w, respectively. More recently, Orlitsky considered the minimal number C_m of bits to be transmitted in m rounds to "inform $P_\mathcal{W}$ about v over one channel." He showed that $C_2\le 4C_\infty+3$ and that in general $C_2\not\sim C_\infty$. We give a simple example for $C_3\not\sim C_\infty$. However, for the new model "inform $P_\mathcal{W}$ over two channels," four rounds are optimal for this example—a result we conjecture in general. If both $P_\mathcal{V}$ and $P_\mathcal{W}$ are to be informed over two channels about the other outcome, we determine asymptotically the complexities for all sources. In our last model "inform $P_\mathcal{V}$ and $P_\mathcal{W}$ over one channel" for all sources the total number T_2 of required bits is known asymptotically and T_∞ is bounded from below in terms of average degrees. There are exact results for several classes of regular sources. An attempt is made to discuss the methods of the subject systematically.**

***Index Terms*— Communication complexity, abstract sources, hypergraph covering and coloring, worst case complexity, average degree bound.**

I. Introduction

THE study of channels with several senders and receivers was initiated by Shannon [1] and the first multi-user coding theorem was proved ten years later by Ahlswede in [2]. This led to intensified research activity during the 1970's in an area which is usually called multi-user information theory. On the source coding side, a strong impetus came in 1973 from the paper by Slepian and Wolf [3], which concerns a probabilistic model of correlated sources. In the same year, independently and almost unnoticed, in [4] a hypergraph coloring lemma was presented (see Lemma 1), which yields asymptotically optimal list codes for abstract (purely combinatorial) correlated sources. This work was continued in [5]. There it is shown that hypergraph coloring concepts are at the root of several probabilistic, semi-probabilistic, and nonprobabilistic multi-user source coding problems.

Whereas in [4] correlated sources are modeled as a *sequence* of independent and identically distributed (i.i.d.) pairs of RV's $(X_t,Y_t)_{t=1}^\infty$, in [5] also just one pair (X,Y) is considered.

Again simultaneously and independently this more abstract view was also taken in [6] using a model for information exchange in distributed computing: We are given a function $f:\mathcal{V}\times\mathcal{W}\to\mathcal{Z}$, where $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are usually finite, $\mathcal{V}$ outputs v and $\mathcal{W}$ outputs w. A person $P_\mathcal{V}$ observes v and another person $P_\mathcal{W}$ observes w. They can transmit messages to each other alternately over a binary noiseless channel and their goal is to find out the value $f(v,w)$ with minimal worst case transmission time. We denote this quantity by $C(f;P_\mathcal{V},P_\mathcal{W})$. Specific ingredients here are as follows:

1) No probabilistic assumptions on the source $(\mathcal{V},\mathcal{W},f)$ are made.
2) Correct decoding for all source outputs is required.
3) Both persons send messages according to a protocol. Each message is based on the input known to the transmitter and on his previously received messages.

   When a communicator transmits a message, the other knows when it ends, and when the last message ends, both communicators know that communication has ended.
4) Both persons use the same channel for transmitting their bits.
5) Both persons compute $f(v,w)$.

There are some basic variations of this model. We start with the work of Ahlswede and Csiszár [7], which was performed independently of Yao's work [6].

I. We keep here assumptions 1) and 2) and otherwise assume that

3_I) $P_\mathcal{V}$ sends bits to $P_\mathcal{W}$.

5_I) $P_\mathcal{W}$ computes f.

The one-way communication complexity $C_1(f;P_\mathcal{W})$ is the minimal number of bits to be transmitted from $P_\mathcal{V}$ to $P_\mathcal{W}$ so that $P_\mathcal{W}$ can compute f.

Example 1: Choose

$$(\mathcal{V},\mathcal{W})=(\{1,2,\cdots,\alpha\}^n,\{1,2,\cdots,\alpha\}^n)$$

and consider the parity of the Hamming distance φ_n, that is, for $(x^n,y^n)\in\mathcal{V}\times\mathcal{W}$

$$\varphi_n(x^n,y^n)=\begin{cases}1, & \text{if } d_H(x^n,y^n) \text{ is odd}\\ 0, & \text{if } d_H(x^n,y^n) \text{ is even.}\end{cases}$$

Then

$$C_1(\varphi_n;P_\mathcal{W})=\lceil n\cdot\log_2\alpha\rceil$$

because for any $x^n,x'^n\in\mathcal{V}$ there is a $y^n\in\mathcal{W}$ with $\varphi_n(x^n,y^n)\ne\varphi_n(x'^n,y^n)$ and, therefore, the outputs of $\mathcal{V}$ have to be encoded differently.

II. Problems studied in [7] include the following. Suppose that $(X_t,Y_t)_{t=1}^\infty$ is a discrete memoryless source, where $X^n=X_1\cdots X_n$ (respectively, $Y^n=Y_1\cdots Y_n$) takes values in $\mathcal{X}^n$ (respectively, $\mathcal{Y}^n$), and suppose that $f_n:\mathcal{X}^n\times\mathcal{Y}^n\to\mathbb{N}$ is

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to be computed correctly with a probability tending to 1 as $n \to \infty$ by $P_\mathcal{W}$, who observes Y^n. How many bits must $P_\mathcal{V}$ transmit to $P_\mathcal{W}$?

Example 2: Let X_t and Y_t be independent for $t = 1, 2, \cdots$ and let them take their values with equal probabilities. The so-called 1-Bit Theorem [7], [8] implies that a computation in the sense described requires for the parity function φ_n a one-way transmission rate $\log_2 \alpha$ for $\alpha \geq 3$. Even though one bit of information about the value of f is obtained by $P_\mathcal{W}$, $P_\mathcal{V}$ has to give full information about X^n!

III. An unsolved problem is concerned with the situation, where both $P_\mathcal{V}$ and $P_\mathcal{W}$ can inform a third person, say P_f, about their source outputs. What are the necessary rates of transmission to enable P_f to compute $f_n : \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{N}$ with probability approaching 1 as $n \to \infty$? This problem can be considered in a purely combinatorial setting. An encoding is a product of partitions of $\mathcal{X}^n$ and $\mathcal{Y}^n$, with monochromatic members. This in turn is a special case of a strict coloring [5] of orthogonal hypergraphs. A special case of such hypergraphs is a triple $(\mathcal{V}, \mathcal{W}, S)$ with $S \subset \mathcal{V} \times \mathcal{W}$, that describes the possible values of (v, w).

An exchange of ideas from abstract source coding theory [5] and distributed computing [6] has been very fruitful for the advancement of other models and new methods. In particular, in 1983, during a visit of the first author to the Information System Laboratory at Stanford University, where Yao's ideas were popular, there were stimulating discussions with El Gamal, Pang, Cover, and Orlitsky. The effect was that starting with [9], [10] there was great interest in determining communication complexity for specific functions. This also gave impetus to combinatorial extremal theory [11]–[15] and, as one of the highlights, led to the 4-words inequality [16].

Whereas [17]–[19] concern $C(f; P_\mathcal{V}, P_\mathcal{W})$, a seemingly essential progress in the study of $C(f; P_\mathcal{W})$ (only $P_\mathcal{W}$ computes f) was made in [20].

On the other hand, the Stanford group picked up the idea of abstract source coding in [21]–[23] and combined it with Yao's idea of exchanging messages in several rounds.

(*Cum grano salis* it can be said, that this can be traced back to [1]. See also the discussion in [26].)

It is very interesting to see what has happened in the last decade in this direction, called interactive data communication (see [21]–[25] and [27]–[29]), mainly in several contributions of Orlitsky ([22]–[25]).

IV. We recall first in our terminology the model studied. Given are the (abstract) correlated source $(\mathcal{V}, \mathcal{W}, S)$ and two communicators $P_\mathcal{V}$, an informant, and $P_\mathcal{W}$, a recipient. $P_\mathcal{V}$ knows output $v \in \mathcal{V}$ and $P_\mathcal{W}$ knows output $w \in \mathcal{W}$. Both communicators want the recipient $P_\mathcal{W}$ to learn v without error, whereas the informant $P_\mathcal{V}$ may or may not learn w. They alternately transmit messages with finite sequences of bits over the same binary noiseless channel by a predetermined protocol. The total number of bits are counted. $C_m((\mathcal{V}, \mathcal{W}, \mathcal{S}), P_\mathcal{W})$ is then the minimal number of bits to be exchanged in m rounds in the worst case for a best protocol.

Since hypergraph terminology is more common we write $C_m(\mathcal{H}, P_\mathcal{E})$ instead of $C_m((\mathcal{V}, \mathcal{W}, \mathcal{S}), P_\mathcal{W})$. The main known results about $C_m(\mathcal{H}, P_\mathcal{E})$ are mentioned in the Appendix. After Orlitsky [23] showed that C_2 may be much smaller than C_∞, and Zhang and Xia [29] showed that even C_3 is not optimal, that is, may be smaller than C_∞, we *conjecture* that C_4 is optimal. So far we have the following results:

Already for a nice structure such as the uniform k-regular hypergraph

$$\left([n], \binom{[n]}{k}\right)$$

$C_2 \sim C_3$ and $C_3 > C_4$. Thus we have now a simpler example for "four messages are better than three" than the one of [29]. We actually determine C_2.

In addition, we study a question from [28] about C_2 for product hypergraphs.

V. As in the contexts of [4] and [5], hypergraph coloring and covering play a key role also in establishing achievability results (direct parts) for "interactive communication." Again, as in [5], they often have to be used in combination. Key roles are played by the following three lemmas.

Coloring Lemma 1 [4]: Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with $D_\mathcal{E} = \max_{E \in \mathcal{E}} |E| \leq L$. If for some $t \in \mathbb{N}$

$$|\mathcal{E}| \binom{D_\mathcal{E}}{t} \cdot L^{1-t} < 1 \tag{1.1}$$

then a coloring $\varphi : \mathcal{V} \to \{1, 2, \cdots, L\}$ exists with

$$|\varphi^{-1}(i) \cap E| < t \text{ for all } i = 1, \cdots, L \text{ and all } E \in \mathcal{E}.$$

We give the proof, because it is very simple.

Proof: For every $A \subset \mathcal{V}$ define

$$\mathcal{F}(A) = \{\varphi : \mathcal{V} \to \{1, \cdots, L\} : \varphi \text{ is constant on } A\}.$$

Hence, the set $\mathcal{F}_E^t$ of colorings that are bad on edge E (that is, have the same color on t vertices in E) is given by

$$\mathcal{F}_E^t = \bigcup_{\substack{A \subset E \\ |A| = t}} \mathcal{F}(A)$$

and the set $\mathcal{F}^t$ of all bad colorings equals $\bigcup_{E \in \mathcal{E}} \mathcal{F}_E^t$.

The set $\mathcal{F}$ of all colorings $\varphi : \mathcal{V} \to \{1, \cdots, L\}$ has the cardinality $|\mathcal{F}| = L^{|\mathcal{V}|}$ and we also have

$$\left|\mathcal{F}_E^t\right| \leq \binom{D_\mathcal{E}}{t} \cdot L \cdot L^{|\mathcal{V}| - t}. \tag{1.2}$$

Because $|E| \leq D_\mathcal{E}$ for all $E \in \mathcal{E}$, $\binom{|E|}{t} \leq \binom{D_\mathcal{E}}{t}$, one of L colors is needed to color the vertices in A, and there are $L^{|\mathcal{V}|-t}$ possible colorings of the vertices outside A.

The rough union bound gives now

$$|\mathcal{F}^t| \leq |\mathcal{E}| \binom{D_\mathcal{E}}{t} L \cdot L^{|\mathcal{V}| - t}. \tag{1.3}$$

Therefore

$$\frac{|\mathcal{F}^t|}{|\mathcal{F}|} \leq |\mathcal{E}| \binom{D_\mathcal{E}}{t} L^{1-t} < 1 \text{ (by assumption)}$$

and there exists a good coloring. □

In [5] several other coloring concepts were introduced.

For (L,λ)-colorings of $\mathcal{H}=(\mathcal{V},\mathcal{E})$ in every edge $E\in\mathcal{E}$ at least $(1-\lambda)|E|$ colors occur only once. An $(L,0)$-coloring is called strict coloring or simply L-coloring. Their analysis reduces to coloring the associated graph $(\mathcal{V},\mathcal{E}^*)$, where $\mathcal{E}^*=\{\{v,v'\}: v,v'\in\mathcal{V}$ and $\{v,v'\}\in E$ for some $E\in\mathcal{E}\}$.

The greedy algorithm gives the following simple result.

Coloring Lemma 2: For a graph $(\mathcal{V},\mathcal{E})$ with maximum degree

$$D_{\mathcal{V}}\triangleq\max_{v\in\mathcal{V}}\deg(v)\le D$$

there exists a strict L-coloring, if $L\ge D+1$.

For the results in **IV** we use another lemma.

Covering Lemma 3 [5]: For a hypergraph $\mathcal{H}=(\mathcal{V},\mathcal{E})$ there is a covering $\mathcal{C},\mathcal{C}\subset\mathcal{E}$, of the vertex set $\mathcal{V}$ (that is, $\bigcup_{E\in\mathcal{C}}E=\mathcal{V}$) with

$$|\mathcal{C}|\le\left\lceil|\mathcal{E}|d_{\mathcal{V}}^{-1}\ln|\mathcal{V}|\right\rceil$$

where $d_{\mathcal{V}}=\min_{v\in\mathcal{V}}\deg(v)$ (defined in (2.2)).

To get an idea of the scope of this Covering Lemma we discuss several applications.

Many problems in interactive communication reduce to covering a hypergraph $\mathcal{H}^*=(\mathcal{V}^*,\mathcal{E}^*)$ by a family $\tilde{\mathcal{F}}$ of subhypergraphs of $\mathcal{H}^*$, with minimal size and/or certain required properties in the following sense. "For all $E^*\in\mathcal{E}^*$ there is an $\tilde{\mathcal{H}}=(\mathcal{V}^*,\tilde{\mathcal{E}})\in\tilde{\mathcal{F}}$, such that $E^*\in\tilde{\mathcal{E}}$." It is clear that to such a covering of $\mathcal{H}^*$ by its subhypergraphs corresponds a covering of $\mathcal{H}=(\mathcal{V},\mathcal{E})$ in the sense of [5], if we define $\mathcal{H}$ by $\mathcal{V}=\mathcal{E}^*$, $\mathcal{E}=\{E(\tilde{\mathcal{H}}):\tilde{\mathcal{H}}\in\mathcal{F}^*\}$, and $E(\tilde{\mathcal{H}})=\{E^*\in\mathcal{V}:E^*\in\tilde{\mathcal{E}}\}$, where $\mathcal{F}^*$ is the family of subhypergraphs of $\mathcal{H}^*$ with the required properties.

1) Among the concepts, not already contained in [5], the most basic is "perfect hashing," which comes from computer science (cf. [30]). The concept can be looked at in several ways.

   Strict colorings of hypergraphs often require a large number of colors. In many cases it suffices to work with several functions $f_1,\cdots,f_k:\mathcal{V}\to B$, $B=\{1,2,\cdots,b\}$.

   $f=(f_1,\cdots,f_k)$ is a *perfect hashing* of $\mathcal{H}=(\mathcal{V},\mathcal{E})$, if for every $E\in\mathcal{E}$ there is an f_i with

   $$f_i(v)\ne f_i(v'),\quad \text{for } v,v'\in E; v\ne v'.$$

   This means that $\mathcal{H}$ can be decomposed into hypergraphs $\mathcal{H}_i=(\mathcal{V},\mathcal{E}_i)$, $\bigcup_i\mathcal{E}_i=\mathcal{E}$ such that f_i is a strict coloring of $\mathcal{H}_i$. In other words, a perfect hashing of $\mathcal{H}$ is just a covering of $\mathcal{H}$ by its b-colorable subhypergraphs. Thus the Covering Lemma is applicable. The proof of Theorem 10 i) in Section VII uses the Covering Lemma along these lines in order to get a perfect hashing for $\mathcal{H}=([n],\binom{[n]}{k})$.

   In applications it is relevant that kb is usually much smaller than $\chi(\mathcal{H})$.

2) "The football league problem," of [22] means covering the edges of a complete graph by the edge-sets of bipartite graphs.

   Actually, here the result of [22] is better, because it is constructive.

3) In this last example we show how to get the major part of [28, Lemma 1] from the Covering Lemma:

   For $\mathcal{H}=(\mathcal{V},\mathcal{E})$ with $D_{\mathcal{E}}\ge 31$ and $J=\lceil\theta D_{\mathcal{E}}\log|\mathcal{V}|\rceil$ (for a constant θ), there is a family $\{f_j\}_{j=1}^J$ of functions $f_j:\mathcal{V}\to\{1,2,\cdots,D_{\mathcal{E}}\}$ such that for all E there exists a $j\in\{1,\cdots,J\}$ with

   $$\left|f_j^{-1}(\alpha)\cap E\right|\le\log D_{\mathcal{E}}$$

   for all $\alpha\in\{1,\cdots,D_{\mathcal{E}}\}$.

Indeed, let $\mathcal{V}^*=\mathcal{E}$ and let $\mathcal{H}^*=(\mathcal{V}^*,\mathcal{E}^*)$ be defined such that to each $f:\mathcal{V}\to\{1,2,\cdots,D_{\mathcal{E}}\}$ corresponds an edge

$$E_f=\{E\in\mathcal{V}^*:|f^{-1}(\alpha)\cap E|\le\log D_{\mathcal{E}},\ \text{for all }\alpha\in\{1,2,\cdots,D_{\mathcal{E}}\}\}\in\mathcal{E}^*$$

and conversely. Then the above follows from the Covering Lemma for $\mathcal{H}^*$.

We have made our point that we are dealing with a covering problem. Special properties can sometimes yield better bounds than the general bound in the Covering Lemma.

Finally, we mention that Lovász's Local Lemma has been used by Orlitsky in coloring problems, for instance in [24]. It yields the following improvement of Coloring Lemma 1.

Strengthened Coloring Lemma: Let $\mathcal{H}=(\mathcal{V},\mathcal{E})$ be a hypergraph with

$$D_{\mathcal{E}}=\max_{E\in\mathcal{E}}|E|\le L.$$

If for some $t\in\mathbb{N}$

$$\frac{e\cdot D_{\mathcal{E}}^2D_{\mathcal{V}}}{t!}\le 1 \tag{1.4}$$

then a coloring $\varphi:\mathcal{V}\to\{1,2,\cdots,L\}$ exists with

$$|\varphi^{-1}(i)\cap E|<t,\quad \text{for all } i=1,2,\cdots,L$$

and all $E\in\mathcal{E}$.

Proof: A consequence of the Local Lemma states (see [31, pp. 53–55]):

Let $A_1,\cdots,A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of at least $n-d$ other events A_j. If

$$\max_{1\le i\le n}\Pr(A_i)\le\frac{1}{e(d+1)},\quad\text{then }\Pr\left(\bigwedge_{i=1}^n\overline{A}_i\right)>0. \tag{1.5}$$

Imagine now that a coloring function Φ is choosen according to the uniform distribution on $\mathcal{F}$. Then the event $A_E=\{\Phi\in\mathcal{F}_E^t\}$ has by (1.2)

$$\text{Prob}(A_E)=\frac{|\mathcal{F}_E^t|}{|\mathcal{F}|}\le\binom{D_{\mathcal{E}}}{t}L^{1-t}. \tag{1.6}$$

Instead of the rough union bound leading to (1.3) we use now that by (1.5) for $L=D_{\mathcal{E}}$

$$\text{Prob}\left(\bigwedge_{E\in\mathcal{E}}\overline{A}_E\right)>0$$

(and there exists the desired coloring), if

$$\frac{D_\mathcal{E}}{t!} \le \frac{1}{eD_\mathcal{E} \cdot D_\mathcal{V}}. \tag{1.7}$$

The Covering Lemma can be strengthened in the same way. Since it is not used in this paper, we omit the details and refer to the analogous version for perfect hashing in [24, Lemma 3].

VI. We recall that previous work on interactive communication (see [21]–[25] and [27]–[29]) addresses the case where $P_\mathcal{E}$ is to be informed about the vertex $v \in \mathcal{V}$. It also addresses the case when $P_\mathcal{V}$ and $P_\mathcal{E}$ communicate over the same binary noiseless channel and the total number of bits is counted.

In [5, p. 236] we point out that it makes a difference whether the transmission in different directions runs over different channels (a distinction between the actual and the potential rate!).

Here it is not the sum of the number of bits that matters, but the "capacity" of each channel to cope with the possible bits in each direction. We indicate the two channels by writing now $\overline{\overline{C}}_m(\mathcal{H}, P_\mathcal{E})$ for the m-round communication complexity.

Moreover, we denote by $\overline{\overline{\mathcal{C}}}_m(\mathcal{H}, P_\mathcal{E})$ the region of possible pairs of bit numbers. Our results for this model are almost complete. In particular, we have a nice general lower bound, that is, for instance, tight for the uniform k-regular hypergraph.

VII. To our surprise the study of other models was even more rewarding. As has already been considered in Yao's model, we consider situations where both persons, $P_\mathcal{V}$ and $P_\mathcal{E}$, have to inform each other about their outputs. We write here $C_m(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$ for the communication complexity.

The corresponding region for two channels is denoted by $\overline{\overline{\mathcal{C}}}_m(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$ and its minimal rate sum is denoted by $\overline{\overline{C}}_m(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$.

It turns out that Coloring Lemmas 1 (and its strengthening) and 2 are already sufficient tools for deriving upper bounds.

More specifically, Coloring Lemma 1 alone already gives us

$$\overline{\overline{\mathcal{C}}}_m^{\ell_\mathcal{V}, \ell_\mathcal{E}}(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$$

that is, the respective communication complexities, where E is known to $P_\mathcal{V}$ and v is known to $P_\mathcal{E}$ only as a member of lists of sizes $\ell_\mathcal{V}$ and $\ell_\mathcal{E}$, respectively.

The reduction to list sizes 1 uses Coloring Lemma 2. It essentially states that both persons are to be informed!

List size 1 complexities and complexities for small list sizes are essentially equal. This changes drastically if only $P_\mathcal{E}$ is to be informed. (Compare [4] for a similar phenomenon for zero-error problems in channel coding.) Here the hashing idea becomes relevant.

We obtain complete results for $\overline{\overline{\mathcal{C}}}_m(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$. We actually show that

$$\overline{\overline{\mathcal{C}}}_3(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E}) = \overline{\overline{\mathcal{C}}}_\infty(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E}).$$

We also give a lower bound on $\mathcal{C}_\infty(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$, which we expect to be tight. It uses the idea of decomposing $\mathcal{H}$, which was motivated by the decomposition in [5, p. 225].

Finally, we mention that we have constructions based on Baranyai's Theorem ([32] and [33, pp. 50–56]) and also on Vizing's Theorem [34].

The presentation of our results and proofs proceeds as follows. We begin with the model for two recipients and two channels. This has not been considered in the literature. However, mathematically there is a very close connection with [24]. Nevertheless, for the benefit of the reader we give complete proofs, because they help to build up a certain intuition for the analysis of the other models. Also these proofs in Sections II and III are short and the examples are new by all standards.

We start from first principles by considering first the easier case of list decision in Section II.

The result appears in Theorem 1. It is based on Coloring Lemma 1. In Section III we get via Coloring Lemma 2 the bit numbers regions for exact decisions and $m \ge 3$ rounds (Theorem 2). We also settle the case $m = 2$ (and thus all cases) with Theorem 3.

Next we consider $C_m(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$, that is, the smallest worst case bit number achievable in m rounds for two recipients. For $m = 2$ we give what are essentially exact bounds for two special cases, the general graph (using Vizing's Theorem) and the complete k-uniform hypergraph (using Baranyai's Theorem).

In the same Section IV we then settle the general case with Theorem 6.

Next we show in Section V that

$$C_\infty(\mathcal{H}; P_\mathcal{E}, P_\mathcal{V}) \ge \log \overline{D}_\mathcal{E} + \log \overline{D}_\mathcal{V}$$

(Average-Degree Lemma).

The study of the model with one recipient starts in Section VI for two channels. Again we investigate the complete k-regular hypergraph and derive a lower bound on the bit number $b_\mathcal{E}$ on the channel $\mathcal{E} \to \mathcal{V}$, if the bit number $b_\mathcal{V}$ on the channel $\mathcal{V} \to \mathcal{E}$ is fixed (Theorem 8). This bound is for $m = \infty$. It coincides with the upper bound for $m = 4$ (Theorem 9).

We have not yet succeeded in deriving an analogous result, when only one channel is used. However, we determined C_2 for the same hypergraph (Theorem 10). Whereas the upper bound uses a familiar Covering Lemma from [5], the lower bound is based on new ideas of some independent combinatorial interest. Moreover, we establish $C_2 \sim C_3$ for $k = 0(\log n)$ and demonstrate that $C_3 < C_4$. Thus we have obtained a simpler hypergraph than that of Zhang and Xia [29], for which three "messages" do not suffice.

In Section VIII we study a question of Noar, Orlitsky, and Shor [28] concerning their "amortized complexity" A_2 and take first steps toward a characterization of this two rounds quantity.

Finally, in an Appendix, some earlier basic results of Orlitsky are stated for the orientation of the reader.

II. A CHARACTERIZATION OF $\overline{\overline{\mathcal{C}}}_m^{\ell_\mathcal{V}, \ell_\mathcal{E}}(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$ FOR SMALL LIST SIZES $\ell_\mathcal{V}, \ell_\mathcal{E}$

The essential parameters here are

$$D_\mathcal{V} = \max_{v \in \mathcal{V}} \deg(v) \quad \text{and} \quad D_\mathcal{E} = \max_{E \in \mathcal{E}} \deg(E) \tag{2.1}$$

where

$$\deg(v) = |\{E \in \mathcal{E} : v \in E\}|$$

and

$$\deg(E) = |\{v \in \mathcal{V} : v \in E\}| = |E|. \tag{2.2}$$

We begin with $m = 2$. Clearly, on the channel $\mathcal{V} \to \mathcal{E}$ the number of bits $b_\mathcal{V}$ to be transmitted in the worst case satisfies

$$b_\mathcal{V} \geq \lceil \log D_\mathcal{E} \rceil. \tag{2.3}$$

Analogously, the number of bits $b_\mathcal{E}$ to be transmitted in the worst case over the channel $\mathcal{E} \to \mathcal{V}$ satisfies

$$b_\mathcal{E} \geq \lceil \log D_\mathcal{V} \rceil. \tag{2.4}$$

Moreover, if we require v to be on a list $\mathcal{V}(v, E)$ and E to be on a list $\mathcal{E}(v, E)$, where

$$|\mathcal{V}(v, E)| \leq \ell_\mathcal{V} \quad \text{and} \quad |\mathcal{E}(v, E)| \leq \ell_\mathcal{E} \tag{2.5}$$

then the numbers of bits needed satisfy

$$b_\mathcal{V}(\ell_\mathcal{V}) \geq \left\lceil \log \frac{D_\mathcal{E}}{\ell_\mathcal{V}} \right\rceil \tag{2.6}$$

$$b_\mathcal{E}(\ell_\mathcal{E}) \geq \left\lceil \log \frac{D_\mathcal{V}}{\ell_\mathcal{E}} \right\rceil. \tag{2.7}$$

Also, these bounds hold for any $m \geq 2$.

Next we show that they are essentially optimal.

Actually, this readily follows from Coloring Lemma 1, one of the most basic tools in this area. There are only very few methods of comparable significance for the subject.

We choose $L = D_\mathcal{E}$ and $t = \ell_\mathcal{V}$. The coloring φ of Coloring Lemma 1 serves as encoding function $f_\mathcal{V}$ and the list for $P_\mathcal{E}$ is then

$$\mathcal{V}(v, E) = \{v' : v' \in E, f_\mathcal{V}(v') = f_\mathcal{V}(v)\}.$$

Any list size $\ell_\mathcal{V}$ with

$$\ell_\mathcal{V}! > |\mathcal{E}| D_\mathcal{E} \tag{2.8}$$

is achievable, because $|\mathcal{E}|L < t!$ is sufficient for (1.1).

Since $\ell_\mathcal{V}! > (\frac{\ell_\mathcal{V}}{e})^{\ell_\mathcal{V}}$, we can choose $\ell_\mathcal{V}$ such that

$$\ell_\mathcal{V}(\log \ell_\mathcal{V} - \log e) > \log |\mathcal{E}| D_\mathcal{E}$$

and certainly such that

$$\ell_\mathcal{V} = \lceil \log |\mathcal{E}| D_\mathcal{E} \rceil + 6. \tag{2.9}$$

We need

$$\lceil \log L \rceil = \lceil \log D_\mathcal{E} \rceil \tag{2.10}$$

bits on the channel $\mathcal{V} \to \mathcal{E}$.

Symmetrically, there is an encoding $f_\mathcal{E}$ which requires $\lceil \log D_\mathcal{V} \rceil$ bits on the channel $\mathcal{E} \to \mathcal{V}$ and leaves $P_\mathcal{V}$ with a list

$$\mathcal{E}(v, E) = \{E' : v \in E', f_\mathcal{E}(E') = f_\mathcal{E}(E)\}$$

of size

$$\ell_\mathcal{E} = \lceil \log |\mathcal{V}| D_\mathcal{V} \rceil + 6. \tag{2.11}$$

To see this, apply the former proof to the dual hypergraph

$$(\mathcal{V}^*, \mathcal{E}^*), \mathcal{V}^* = \mathcal{E}, \mathcal{E}^* = \{E_v = \{E : v \in E\} : v \in \mathcal{V}\}. \tag{2.12}$$

We summarize our findings.

Here and later we use the notation $a \sim c$, if $a = c + o(c)$ and

$$(a_1, a_2) \sim (c_1, c_2), \quad \text{if } (a_1, a_2) = (c_1, c_2) + (o(c_1), o(c_2)). \tag{2.13}$$

We say that pairs of numbers of bits $(b_\mathcal{V}, b_\mathcal{E})$ are achievable with list sizes $(\ell_\mathcal{V}, \ell_\mathcal{E})$ iff there are encoding functions $(f_\mathcal{V}, f_\mathcal{E})$ with these list sizes and

$$b_\mathcal{V} \sim \log \|f_\mathcal{V}\| \quad b_\mathcal{E} \sim \log \|f_\mathcal{E}\|. \tag{2.14}$$

Theorem 1: For the list size pairs

$$(\ell_\mathcal{V}, \ell_\mathcal{E}) \sim (\log |\mathcal{V}| D_\mathcal{V}, \log |\mathcal{E}| D_\mathcal{E})$$

the set of achievable pairs of bit numbers in m rounds is given by

$$\bar{\bar{\mathcal{C}}}_m^{\ell_\mathcal{V}, \ell_\mathcal{E}}(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \sim \{(b_\mathcal{V}, b_\mathcal{E}) : b_\mathcal{V} \geq \log D_\mathcal{V}, b_\mathcal{E} \geq \log D_\mathcal{E}\}$$

for $m \geq 2$.

Remark 1: This result can be improved by using the strengthened Coloring Lemma. For this see [24] with an analogous situation. We demonstrate the effect in Section III while improving Theorem 2A to Theorem 2B.

III. A Characterization of $\bar{\bar{\mathcal{C}}}_m(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$: Three Rounds Suffice

Now we use the simple Coloring Lemma 2 in conjunction with the previous result. Suppose that $(f_\mathcal{V}(v), f_\mathcal{E}(E)) = (i, j)$, then both, $P_\mathcal{V}$ and $P_\mathcal{E}$, know that $v \in \mathcal{V}_i = \{v' : f_\mathcal{V}(v') = i\}$ and $E \in \mathcal{E}_j = \{E' : f_\mathcal{E}(E') = j\}$. Thus the hypergraph $(\mathcal{V}, \mathcal{E})$ has been reduced to $\mathcal{H}_{i,j} = (\mathcal{V}_i, \mathcal{E}_j)$. Furthermore, we know that

$$D_{\mathcal{V}_i} \leq \ell_\mathcal{E} = \lceil \log |\mathcal{V}| D_\mathcal{V} \rceil \tag{3.1}$$

$$D_{\mathcal{E}_j} \leq \ell_\mathcal{V} = \lceil \log |\mathcal{E}| D_\mathcal{E} \rceil. \tag{3.2}$$

We introduce the associated graph $\mathcal{G}_{i,j} = (\tilde{\mathcal{V}}_i, \tilde{\mathcal{E}}_j)$, where $\tilde{\mathcal{V}}_i = \mathcal{V}_i$

$$\tilde{\mathcal{E}}_j = \{(v, v') : v, v' \in \mathcal{V}_i \text{ and for some } F \in \tilde{\mathcal{E}}_j \ \{v, v'\} \subset F\}.$$

Notice that

$$\begin{aligned} D_{\tilde{\mathcal{V}}_i} &\leq D_{\mathcal{V}_i}(D_{\mathcal{E}_j} - 1) \leq \ell_\mathcal{E}(\ell_\mathcal{V} - 1) \\ &\leq \lceil \log |\mathcal{V}| D_\mathcal{V} \rceil \cdot \lceil \log |\mathcal{E}| D_\mathcal{E} \rceil. \end{aligned} \tag{3.3}$$

By Coloring Lemma 2 $P_\mathcal{V}$ can inform $P_\mathcal{E}$ about v with $\log(\ell_\mathcal{V} \cdot \ell_\mathcal{E}) + 1$ bits via an encoding function $g_\mathcal{V}$. $P_\mathcal{E}$ can then inform $P_\mathcal{E}$ about E with $\lceil \log \ell_\mathcal{V} \rceil$ bits via an encoding function $g_\mathcal{E}$.

The whole protocol is then to send $f_\mathcal{V}(v)$ over channel $\mathcal{V} \to \mathcal{E}$, then $f_\mathcal{E}(E)$ over channel $\mathcal{E} \to \mathcal{V}$, then $g_\mathcal{V}(v)$ over channel $\mathcal{V} \to \mathcal{E}$, and finally $g_\mathcal{E}(v)$ over channel $\mathcal{E} \to \mathcal{V}$. This uses four rounds.

However, there is a better way! Just follow the order $f_\mathcal{E}(E)$, then $f_\mathcal{V}(v)$ *and* $g_\mathcal{V}(v)$, and then $g_\mathcal{E}(E)$. This requires three rounds. We state the result.

Theorem 2A: For $m \geq 3$

$$\overline{\overline{C}}_m(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \sim \{(b_\mathcal{V}, b_\mathcal{E}) : b_\mathcal{V} \geq \log D_\mathcal{E}, b_\mathcal{E} \geq \log D_\mathcal{V}\}$$

if

$$\log(\log|\mathcal{V}| + \log|\mathcal{E}|) = o(\log\min(D_\mathcal{V}, D_\mathcal{E})). \tag{3.4}$$

What happens in case $m = 2$?

The key parameters are here $\chi(\mathcal{H})$, the chromatic number (vertex coloring), and $\text{ind}\,(\mathcal{H})$, the chromatic index (edge coloring), of $\mathcal{H}$. In a protocol we have either $f_\mathcal{V}$ and then $f_\mathcal{E}$ to be transmitted or $g_\mathcal{E}$ and $g_\mathcal{V}$. In the first case necessarily $\|f_\mathcal{V}\| \geq \chi(\mathcal{H})$ and in the second case necessarily $\|g_\mathcal{E}\| \geq \text{ind}(\mathcal{H})$. This is a simple consequence of conditions 2) and 3) in the Introduction, and an analogous proof appeared in [22, Lemma 2]. Similar inequalities will be used in the sequel without saying. Furthermore, in the first case $\|f_\mathcal{E}(v,\cdot)\| \geq \deg(v)$. So in the worst case

$$\max_{v\in\mathcal{V}} \|f_\mathcal{E}(v,\cdot)\| \geq D_\mathcal{V}. \tag{3.5}$$

On the other hand, a coloring $f_\mathcal{V}$ with $\chi(\mathcal{H})$ colors and a labeling $f_\mathcal{E}(v, E) = j$, if E is the jth set in $\mathcal{E}_v \subset \{E_1, \cdots, E_{|\mathcal{V}|}\}$ achieve the lower bounds.

Theorem 3: For every hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$

$$\overline{\overline{C}}_2(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) = \{(b_\mathcal{V}, b_\mathcal{E}) : b_\mathcal{V} \geq \log\chi(\mathcal{H}), b_\mathcal{E} \geq \log D_\mathcal{V} \text{ or } b_\mathcal{V} \geq \log D_\mathcal{E}, b_\mathcal{E} \geq \log\,\text{ind}\,(\mathcal{H})\}.$$

From Theorems 2 and 3 we conclude that for many hypergraphs

$$\overline{\overline{C}}_2(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \neq \overline{\overline{C}}_3(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \sim \overline{\overline{C}}_\infty(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}).$$

We also notice that condition (3.4) is rather restrictive. Before we present an improved condition in Theorem 2B below, we discuss some examples.

Example 3: $\mathcal{H} = ([n], \binom{[n]}{k})$.

Clearly, $D_\mathcal{E} = k$, $D_\mathcal{V} = \binom{n-1}{k-1}$, $|\mathcal{V}| = n$, $|\mathcal{E}| = \binom{n}{k}$, $\chi(\mathcal{H}) = n$, and Baranyai's Theorem in its generalized form (see [32] and [33, pp. 50–56]) asserts that

$$\text{ind}\,(\mathcal{H}) = \left\lceil \binom{n}{k} \Big/ \left\lfloor \frac{n}{k} \right\rfloor \right\rceil.$$

The two minimal pairs in Theorem 3 are

$$\left(\log n, \log\binom{n-1}{k-1}\right) \quad \text{and} \quad \left(\log k, \log\left\lceil \binom{n}{k} \Big/ \left\lfloor \frac{n}{k} \right\rfloor \right)\right\rceil.$$

If $k \mid n$, then $\text{ind}\,(\mathcal{H}) = \binom{n-1}{k-1}$ and there is only one minimal pair, namely, $(\log k, \log\binom{n-1}{k-1})$.

The optimum is achieved already for two rounds.

Example 4: Consider any graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}), \mathcal{V} = \{v_1, \cdots, v_n\}.$$

By Vizing's Theorem [34], a deep improvement of Shannon's edge coloring result for multigraphs [35]

$$D_\mathcal{V} \leq \text{ind}\,(\mathcal{G}) \leq D_\mathcal{V} + 1. \tag{3.6}$$

Therefore, by Theorem 3, we have

$$(b_\mathcal{V}, b_\mathcal{E}) = (1, \lceil\log\,\text{ind}\,(\mathcal{G})\rceil).$$

One may analyze whether the pair $(\lceil\log\chi(\mathcal{G})\rceil, \lceil\log D_\mathcal{V}\rceil)$ can be smaller in the second component, if $\text{ind}\,(\mathcal{G}) = D_\mathcal{V} + 1$.

Example 5: Consider a k-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{E} \subset \binom{\mathcal{V}}{k}$. Notice that

$$D_\mathcal{V} \leq \text{ind}\,(\mathcal{H}) \leq k \cdot D_\mathcal{V} + 1. \tag{3.7}$$

Example 6: A protocol for the complete graph

$$\mathcal{K}_n = \left([n], \binom{[n]}{2}\right).$$

We present the vertices by binary strings

$$v = (v_1, \cdots, v_m), m = \lceil\log|\mathcal{V}|\rceil$$

and encode the edge $\{v, v'\}$ as

$$(v_1, \cdots, v_m) + (v'_1, \cdots, v'_m) = (w_1, \cdots, w_m)$$

(binary addition mod 2).

$P_\mathcal{V}$, knowing $V = (v_1, \cdots, v_m)$, calculates

$$(w_1, \cdots, w_m) + (v_1, \cdots, v_m) = (v'_1, \cdots, v'_m).$$

Thus he knows also v' and he tells $P_\mathcal{E}$ whether v or v' has the smaller label. Thus $(b_\mathcal{V}, b_\mathcal{E}) = (1, \lceil\log|\mathcal{V}|\rceil)$ is achievable and $|\mathcal{V}| = n = D_\mathcal{V} + 1$.

We see that the pair must be optimal, if $\lceil\log(D_\mathcal{V} + 1)\rceil = \lceil\log D_\mathcal{V}\rceil$. Moreover, the following argument shows that it is always optimal. In the remaining cases we have $|\mathcal{V}| = n = D_\mathcal{V} + 1 = 2^m + 1$ for some $m \in \mathbb{N}$. The protocol provides a tight bound too, because for all $\ell \in \mathbb{N}$

$$\text{ind}\,(\mathcal{K}_{2\ell+1}) \geq 2\ell + 1 (= D_\mathcal{V} + 1)$$

holds. In fact, assume the

$$\binom{2\ell+1}{2} = \ell(2\ell+1)$$

edges of $\mathcal{K}_{2\ell+1}$ are already colored by 2ℓ colors, then there must be

$$\left\lceil \frac{\ell(2\ell+1)}{2\ell} \right\rceil = \ell + 1$$

edges with the same color. However, there are only $2\ell + 1$ vertices. So there must be a pair of edges among them with a common vertex.

Finally, we improve Theorem 2A by using the strengthened Coloring Lemma.

Since $t! \geq (\frac{t}{e})^t$, it suffices to choose

$$2\log D_{\mathcal{E}} + \log D_{\mathcal{V}} + \log e \leq t \log \frac{t}{e}$$

or

$$t = \log \max(D_{\mathcal{E}}, D_{\mathcal{V}}), t \geq e^4. \tag{3.8}$$

Theorem 2B: For $m \geq 3$

$$\overset{=}{C}_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \sim \{(b_{\mathcal{V}}, b_{\mathcal{E}}) : b_{\mathcal{V}} \geq \log D_{\mathcal{E}}, b_{\mathcal{E}} \geq \log D_{\mathcal{V}}\}$$

if

$$\log(\log D_{\mathcal{E}} + \log D_{\mathcal{V}}) = o(\log \min(D_{\mathcal{V}}, D_{\mathcal{E}})). \tag{3.9}$$

Proof: Use the proof for Theorem 2A and replace (3.1) and (3.2) by the sharper bounds (for $D_{\mathcal{E}}, D_{\mathcal{E}} \geq e^4$)

$$D_{\mathcal{V}_i} \leq \ell_{\mathcal{E}} = \lceil \log \max(D_{\mathcal{E}}, D_{\mathcal{V}}) \rceil \tag{3.10}$$
$$D_{\mathcal{E}_j} \leq \ell_{\mathcal{V}} = \lceil \log \max(D_{\mathcal{E}}, D_{\mathcal{V}}) \rceil \tag{3.11}$$

which follow from (3.8). □

Remarks 2: Using a different approach, estimates for probabilities of events with small dependencies were also found in [5]. However, they appear to be not as sharp.

Remark 3: Since

$$\overset{=}{C}_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \leq C_{m-1}(\mathcal{H}; P_{\mathcal{E}}) + \log D_{\mathcal{V}}$$

for $m \geq 4$, the same upper bound (after ignoring the constant term) can be obtained from the upper bound for $C_3(\mathcal{H}; P_{\mathcal{E}})$ in [24, Theorem 1]. However, for $m = 3$ there is no direct implication between the two theorems.

IV. $C_2(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$: Sharp Estimates in Two Special Cases and Asymptotically Sharp Bounds in General

We begin with the special cases, because this way we build an understanding for the general case.

Theorem 4: For any graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$C_2(\mathcal{G}; P_{\mathcal{V}}, P_{\mathcal{E}}) = \lceil \log \operatorname{ind}(\mathcal{G}) \rceil + \varepsilon, \varepsilon \in \{0, 1\}$$

and

$$D_{\mathcal{V}} \leq \operatorname{ind}(\mathcal{G}) \leq D_{\mathcal{V}} + 1.$$

Proof: By Vizing's Theorem there is an edge coloring $f_{\mathcal{E}}:\mathcal{E} \to \mathbb{N}$ with $\|f_{\mathcal{E}}\| - 1 \leq D_{\mathcal{V}}$. We can use the protocol:

For a given (v, E) with $v \in E$, $P_{\mathcal{E}}$ sends $f_{\mathcal{E}}(E)$ with $\lceil \log \operatorname{ind}(\mathcal{G}) \rceil$ bits to $P_{\mathcal{V}}$, who can recover E.

With one bit he informs $P_{\mathcal{E}}$ about his vertex. Then we have shown that

$$C_2(\mathcal{G}; P_{\mathcal{V}}, P_{\mathcal{E}}) \leq \lceil \log \operatorname{ind}(\mathcal{G}) \rceil + 1. \tag{4.1}$$

Now, conversely, even if $P_{\mathcal{E}}$ knows the outcomes v, he has to transmit $\lceil \log D_{\mathcal{V}} \rceil$ bits in the worst case and even if $P_{\mathcal{V}}$ knows the outcome E, he has to transmit one bit in the worst case. Therefore

$$C_2(\mathcal{G}; P_{\mathcal{V}}, P_{\mathcal{E}}) \geq \lceil \log D_{\mathcal{V}} \rceil + 1 \geq \lceil \log(\operatorname{ind}(\mathcal{G}) - 1) \rceil + 1.$$

The upper and lower bounds coincide for $2^r < D_{\mathcal{V}} < 2^{r+1}$.

Example 7: If $\mathcal{G} = ([3], \binom{[3]}{2})$ is a triangle, then $D_{\mathcal{V}} = 2$ and inspection shows that $C_2(\mathcal{G}; P_{\mathcal{V}}, P_{\mathcal{E}}) = 3$ (the upper bound (4.1)).

Remark 4: Again we can use the nice protocol in Example 6. Can this idea be generalized to k larger than 2?

Next we consider again the complete k-regular hypergraph $([n], \binom{[n]}{k})$. The lower bound is derived as before. In any case, $P_{\mathcal{E}}$ has to transmit $\lceil \log \binom{n-1}{k-1} \rceil$ bits (even if he knows v) and $P_{\mathcal{V}}$ has to transmit $\lceil \log n \rceil$ bits (even if he knows E). Formally, this follows from the Average-degree Lemma in Section V. Thus we get

$$C_m\left(\left([n], \binom{[n]}{k}\right); P_{\mathcal{V}}, P_{\mathcal{E}}\right) \geq \lceil \log k \rceil + \left\lceil \log \binom{n-1}{k-1} \right\rceil, \quad m \geq 2.$$

Baranyai's Theorem in its generalized form (see [33, pp. 50–56]) asserts

$$\binom{n-1}{k-1} \leq \operatorname{ind}\left(\mathcal{V}, \binom{\mathcal{V}}{k}\right) = \left\lceil \binom{n}{k} \Big/ \left\lfloor \frac{n}{k} \right\rfloor \right\rceil.$$

Now $P_{\mathcal{E}}$ uses an edge coloring with $\operatorname{ind}([n], \binom{[n]}{k})$ colors and sends the color with $\lceil \log \operatorname{ind}([n], \binom{[n]}{k}) \rceil$ bits. $P_{\mathcal{V}}$ encodes in a binary string the position of $v = v_i$ in E. This requires $\lceil \log k \rceil$ bits.

Theorem 5: For the k-uniform complete hypergraph $\mathcal{H} = ([n], \binom{[n]}{k})$

$$C_2(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) = \lceil \log k \rceil + \left\lceil \log \binom{n-1}{k-1} + \varepsilon \right\rceil, \quad \varepsilon \in \{0, 1\}.$$

Furthermore

$$C_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) = \lceil \log k \rceil + \left\lceil \log \binom{n-1}{k-1} + \varepsilon \right\rceil, \quad \text{for all } m \geq 2.$$

Remark 5: For $m = 2$, if $P_{\mathcal{V}}$ starts sending, then necessarily $b_{\mathcal{V}} = \lceil \log n \rceil$ and $b_{\mathcal{E}} = \lceil \log \binom{n-1}{k-1} \rceil$. This, of course, is not as good as the reverse order.

We turn to $C_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$ for general hypergraphs $\mathcal{H}$. The situation for one channel is quite different from that of two channels studied earlier.

As a strategy, we cannot just add $\max_v b_{\mathcal{V}}(v)$ and $\max_{v,E} b_{\mathcal{E}}(v, E)$, because, as in our earlier work [18], [19], now a small $b_{\mathcal{V}}(v)$ may be coupled with a large $\max_E b_{\mathcal{E}}(v, E)$, and *vice versa.*[1]

On the other hand, if

$$\deg(v) = D_{\mathcal{V}} \text{ for all } v \in \mathcal{V} \tag{4.2}$$

and

$$\deg(E) = D_{\mathcal{E}} \text{ for all } E \in \mathcal{E} \tag{4.3}$$

or these equations "almost" hold, then $\log D_{\mathcal{V}}$ and $\log D_{\mathcal{E}}$ are the crucial parameters again and they can be added! (See again the Average-degree Lemma in Section V.)

[1] We have been informed that this idea appeared already in Orlitsky [23].

Example 8: $|\mathcal{V}| = |\mathcal{E}| = n$, (4.2) and (4.3) hold, and, particularly, $D_{\mathcal{V}} = D_{\mathcal{E}} = k$; $k = k(n)$ with $\lim_{n\to\infty} k(n) = \infty$.

Then (3.9) holds and by Theorem 2B and the previous observation

$$C_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \sim \log D_{\mathcal{E}} + \log D_{\mathcal{V}}.$$

On the other hand, further insight is gained from the next case.

Example 9: Consider the multi-hypergraph $(\mathcal{V}, \mathcal{W}, S)$ with

$$S = \{(v_1, w_j) : 1 \le j \le n\} \cup \{(v_i, w_1) : 1 \le i \le m\}.$$

Then $D_{\mathcal{V}} = n, D_{\mathcal{W}} = m$, and if $m \ge n$, then

$$C_2((\mathcal{V}, \mathcal{W}, S); P_{\mathcal{V}}, P_{\mathcal{W}}) = \lceil \log \max(n, m-1) \rceil + 1.$$

Protocol: If $v \ne v_1$ then necessarily $w = w_1$ and $P_{\mathcal{V}}$ knows this. $P_{\mathcal{W}}$ does not act before $P_{\mathcal{V}}$ acts. $P_{\mathcal{V}}$ sends a string of length $1 + \lceil \log(m-1) \rceil$. He begins with 0. Then $P_{\mathcal{E}}$ knows v. If $v = v_1$, then $P_{\mathcal{V}}$ sends 1 and $P_{\mathcal{E}}$ sends j with $\lceil \log n \rceil$ bits. Thus the result.

Notice that $\overline{\overline{C}}_2 = \lceil \log m \rceil + \lceil \log n \rceil$.

From Example 9 we learn that the coloring numbers $\chi(\mathcal{H}), \text{ind}(\mathcal{H})$ and maximal cross-section numbers $D_{\mathcal{V}}$ and $D_{\mathcal{E}}$ alone do not suffice for a characterization of C_2. We give a somewhat refined analysis.

For $s = 1, 2, \cdots, \lceil \log D_{\mathcal{E}} \rceil$ define

$$\mathcal{E}_s = \{E \in \mathcal{E} : 2^{-s} D_{\mathcal{E}} < |E| \le 2^{-s+1} D_{\mathcal{E}}\} \tag{4.4}$$

and $\mathcal{H}_s = (\mathcal{V}_s, \mathcal{E}_s)$ with $\mathcal{V}_s = \mathcal{V}$.

Let $f_s : \mathcal{E}_s \to \mathbb{N}$ be an edge coloring of $\mathcal{H}_s$ with $\|f_s\| = \text{ind}(\mathcal{H}_s)$.

For a given (v, E), $P_{\mathcal{E}}$ sends s, if $E \in \mathcal{E}_s$, and also $f_s(E)$. $P_{\mathcal{V}}$ sends the position $g(v, s, E)$ of v in E.

How many bits are needed for this? Let s be represented by the string $\alpha(s)$ and let $f_s(E)$ be represented by the string $\beta_s(E)$. Finally, let $g(v, s, E)$ be encoded as $\gamma(v, s, E)$. Thus

$$b_E + b_v(E) = |\alpha(s)| + |\beta_s(E)| + |\gamma(v, s, E)|.$$

Clearly, we have

$$\begin{aligned}
&\lceil \log D_{\mathcal{E}} \rceil + \max_{1 \le s \le D_{\mathcal{E}}} \lceil (\log \text{ind}(\mathcal{H}_s) \rceil - s) \\
&\quad \le \max_{v,E} (b_E + b_v(E)) \\
&\quad \le \lceil \log \lceil \log D_{\mathcal{E}} \rceil \rceil + \max_s \lceil (\log \text{ind}(\mathcal{H}_s) \rceil \\
&\qquad + \lceil \log D_{\mathcal{E}} \rceil - s + 1) \\
&\quad \sim \log D_{\mathcal{E}} + \max_{1 \le s \le D_{\mathcal{E}}} (\log \text{ind}(\mathcal{H}_s) - s + 1)
\end{aligned} \tag{4.5}$$

because all edges in $\mathcal{E}_s$ have up to the factor 2 the same size.

Of course, we can exchange the roles of $\mathcal{V}$ and $\mathcal{E}$.

Theorem 6A: For any hypergraph $\mathcal{H}$ we have (see the bottom of this page).

The bounds differ by at most

$$\max(\lceil \log \log D_{\mathcal{E}} \rceil, \lceil \log \log D_{\mathcal{V}} \rceil) + 1.$$

Now we give an even better protocol and achieve a difference of the upper and lower bound by two bits!

For any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ we weigh subsets of vertices and edges by the degrees

$$D_{\mathcal{U}} = \max_{v \in \mathcal{U}} \deg(v), \ \mathcal{U} \subset \mathcal{V} \tag{4.6}$$

$$D_{\mathcal{F}} = \max_{E \in \mathcal{F}} \deg(E), \ \mathcal{F} \subset \mathcal{E}. \tag{4.7}$$

To a given c-coloring $\varphi : \mathcal{V} \to \{1, 2, \ldots, c\}$ of $\mathcal{V}$, we associate a weight

$$W_\varphi = \sum_{i=1}^{c} D_{\varphi^{-1}(i)} \tag{4.8}$$

and we define the weighted chromatic number $\tilde{\chi}(\mathcal{H})$ as the minimal weight among the vertex colorings of $\mathcal{H}$. Analogously, we define the weighted index $\widetilde{\text{ind}}(\mathcal{H})$ of $\mathcal{H}$.

Now we consider a two-rounds protocol (φ, χ) starting with $P_{\mathcal{V}}$. Obviously φ must be a vertex coloring and also, if $v \in \varphi^{-1}(j)$, then after knowing v $P_{\mathcal{E}}$ has to use a $D_{\varphi^{-1}(j)}$-size code in the second round to inform $P_{\mathcal{V}}$ about his E.

This means that the protocol must use at least W_φ sequences and therefore at least $\log W_\varphi$ bits. We have therefore

$$C_2(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}}) \ge \lceil \log \tilde{\chi}(\mathcal{H}) \rceil. \tag{4.9}$$

On the other hand, we use again our old adaptive coding idea [18], [19].[1]

Let the coloring $\varphi : \mathcal{V} \to \{1, \cdots, c\}$ achieve $\tilde{\chi}(\mathcal{H})$ and set

$$\ell_i = \left\lceil \log \frac{\tilde{\chi}(\mathcal{H})}{D_{\varphi^{-1}(i)}} \right\rceil, \quad \text{for } i = 1, \cdots, c. \tag{4.10}$$

By Kraft's inequality one can code φ with a prefix code $\mathcal{C} = \{u_1, \cdots, u_c\}$ such that u_i has length ℓ_i and $\varphi(v)$ is encoded by u_i, if $\varphi(v) = i$.

The following protocol works:

1) For given $v \in \mathcal{V}$, $P_{\mathcal{V}}$ sends $u_{\varphi(v)}$.
2) After having received $u_{\varphi(v)}$, $P_{\mathcal{E}}$ recovers v and informs $P_{\mathcal{V}}$ about the given E via a block code of a length not exceeding $\lceil \log D_{\varphi^{-1}(i)} \rceil$, if $v \in \varphi^{-1}(i)$.

By (4.10), the total number of bits is always bounded by $\log \tilde{\chi}(\mathcal{H}) + 1$. Introducing

$$\tilde{\sigma}(\mathcal{H}) = \min(\tilde{\chi}(\mathcal{H}), \widetilde{\text{ind}}(\mathcal{H})) \tag{4.11}$$

and exchanging the roles of $\mathcal{V}$ and $\mathcal{E}$ we can draw the following conclusion.

---

i) $C_2(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}}) \le$
$$\min\Big(\max_{1 \le s \le D_{\mathcal{E}}} \lceil \log \text{ind}(\mathcal{V}, \mathcal{E}_s) - s + 1 \rceil + \lceil \log D_{\mathcal{E}} \rceil + \lceil \log \log D_{\mathcal{E}} \rceil, \ \max_{1 \le t \le D_{\mathcal{V}}} \lceil \log \chi(\mathcal{V}_t, \mathcal{E}) - t + 1 \rceil + \lceil \log D_{\mathcal{V}} \rceil + \lceil \log \log D_{\mathcal{V}} \rceil\Big)$$

ii) $C_2(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}}) \ge \min\big(\max_{1 \le s \le D_{\mathcal{E}}} \lceil \log \text{ind}(\mathcal{V}, \mathcal{E}_s) - s \rceil + \lceil \log D_{\mathcal{E}} \rceil, \ \max_{1 \le f \le D_{\mathcal{V}}} \lceil \log \chi(\mathcal{V}_t, \mathcal{E}) - t \rceil + \log D_{\mathcal{V}} \rceil\big)$.

Theorem 6B: For any hypergraph $\mathcal{H}$ we have

$$\lceil \log \tilde{\sigma}(\mathcal{H}) \rceil \le C_2(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E}) \le \lceil \log \tilde{\sigma}(\mathcal{H}) \rceil + 2.$$

V. A Lower Bound on $C_\infty(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$

We know already that under condition (3.9)

$$\bar{\bar{C}}_3(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \sim \log D_\mathcal{V} + \log D_\mathcal{E}. \tag{5.1}$$

Clearly

$$C_3(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \le \bar{\bar{C}}_3(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$$

and Example 9 shows that already $C_2(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$ can be much smaller than $\log D_\mathcal{V} + \log D_\mathcal{E}$. $C_4(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$ can be much smaller in general.

We give here a lower bound on $C_\infty(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$, which we conjecture to be tight. We also believe that it can be achieved in four rounds. The bound uses a decomposition idea.

We start with a purely combinatorial problem. We call a 0–1-matrix a quasi-permutation matrix, if at most one 1 occurs in every row and in every column. Thus a permutation matrix is a square quasi-permutation matrix such that there is exactly one 1 in every row and in every column. A quasi-permutation *decomposition* $\{A_i = i \in I\}$ of M, is a partition of M into submatrices A_i, which are quasi-permutation matrices. Partition here means that every entry of M occurs in exactly one member of $\{A_i : i \in I\}$.

We also introduce the average degrees for $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ by

$$\overline{D}_\mathcal{V} = \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \deg(v), \quad \overline{D}_\mathcal{E} = \frac{1}{|\mathcal{E}|} \sum_{E \in \mathcal{E}} \deg(E) \tag{5.2}$$

and similarly for $M = (\mathcal{V}, \mathcal{W}, S)$ $\overline{D}_\mathcal{V}$ is the average number of 1's in the rows and $\overline{D}_\mathcal{W}$ is the average number of 1's in the columns.

Obviously, we have for the total number of 1's

$$|\mathcal{V}| \cdot \overline{D}_\mathcal{V} = |\mathcal{W}| \cdot \overline{D}_\mathcal{W}. \tag{5.3}$$

Decomposition Lemma: If $\{A_i : i \in I\}$ is a quasi-permutation decomposition of the 0–1-matrix M, then

$$|I| \ge \overline{D}_\mathcal{V} \cdot \overline{D}_\mathcal{W}. \tag{5.4}$$

Proof: Assume that (5.4) does not hold for $\{A_i : i \in I\}$ and that A_i contains r_i rows, c_i columns, and λ_i 1's.

By the definition of a quasi-permutation matrix

$$\lambda_i \le \min(r_i, c_i), \quad \text{for } i \in I. \tag{5.5}$$

Suppose now that M has $|\mathcal{V}|$ rows and $|\mathcal{W}|$ columns, then by the partition property

$$\sum_{i \in I} r_i c_i = |\mathcal{V}||\mathcal{W}| \tag{5.6}$$

and

$$\sum_{i \in I} \lambda_i = |\mathcal{V}|\overline{D}_\mathcal{V} = |\mathcal{W}|\overline{D}_\mathcal{W}. \tag{5.7}$$

Consequently, by (5.7)

$$\begin{aligned}\left(\sum_{i \in I} \frac{1}{|I|}\lambda_i\right)^2 &= \frac{1}{|I|^2}|\mathcal{V}|\overline{D}_\mathcal{V}|\mathcal{W}|\overline{D}_\mathcal{W} \\ &= \left(\sum_{i \in I} \frac{1}{|I|} r_i c_i\right)\left(\frac{1}{|I|}\overline{D}_\mathcal{V}\overline{D}_\mathcal{W}\right) \text{ (by (5.6))} \\ &> \left(\sum_{i \in I} \frac{1}{|I|} r_i c_i\right) \text{ (by assumption)} \\ &\ge \sum_{i \in I} \frac{1}{|I|}\lambda_i^2 \text{ (by (5.5)).}\end{aligned}$$

This contradicts the Cauchy–Schwarz inequality. We readily derive now a basic lower bound.

Average-Degree Lemma: For any multi-hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ (or, equivalently, the 0–1-matrix $M = (\mathcal{V}, \mathcal{W}, S)$)

$$C_\infty(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \ge \log \overline{D}_\mathcal{V} + \log \overline{D}_\mathcal{E}. \tag{5.8}$$

Proof: Each protocol partitions M into quasi-permutation submatrices (just as in Yao's case $f : \mathcal{V} \times \mathcal{W} \to \mathcal{Z}$ it partitions M_f into "monochromatic" submatrices) and the number of bits used in the protocol is bounded by the logarithm of $I(M) \triangleq$ the usual size of a quasi-permutation decomposition of M.

By the Decomposition Lemma

$$\log I(M) \ge \log \overline{D}_\mathcal{V} + \log \overline{D}_\mathcal{W} = \log \overline{D}_\mathcal{V} + \log \overline{D}_\mathcal{E}$$

the desired inequality.

Example 9 shows that (5.8) is not tight. However, it can be used to derive a much better bound. For this we write $M' \prec M$ iff M' is a submatrix of M and $\mathcal{H}' \prec \mathcal{H}$ iff $\mathcal{H}'$ is a subhypergraph of $\mathcal{H}$. Also with $M' = (\mathcal{V}', \mathcal{W}', S')$ we associate

$$\sigma(M') \triangleq \frac{|S'|^2}{|\mathcal{V}'||\mathcal{W}'|} = \overline{D}_{\mathcal{V}'} \cdot \overline{D}_{\mathcal{W}'} \text{ (by (5.3)).}$$

Since obviously

$$C_m(M; P_\mathcal{V}, P_\mathcal{E}) \ge C_m(M'; P_{\mathcal{V}'}, P_{\mathcal{W}'})$$

or in terms of hypergraphs

$$C_m(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \ge C_m(\mathcal{H}'; P_{\mathcal{V}'}, P_{\mathcal{E}'})$$

we get the following consequence of the Average-degree Lemma in hypergraph language.

Theorem 7: For any multi-hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$

$$C_\infty(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \ge \max_{\mathcal{H}' \prec \mathcal{H}} \sigma(\mathcal{H}').$$

VI. A Lower Bound on $\overline{\overline{C}}_\infty(([n], \binom{[n]}{k}), P_\mathcal{E})$

Consider successful protocols with $b_\mathcal{V} = \beta$ and let $\nu_\infty(n, k; \beta)$ be the minimal $b_\mathcal{E}$ among these protocols. Let Ω be a protocol with this property and set $\alpha = \nu_\infty(n, k; \beta)$.

For any (v, E) with $v \in E$

$$\Omega(v, E) = x_1(E) y_1(v, x_1) x_2(E, y_1) \cdots$$

(here x_1 can be the empty word), and we can form the strings

$$x_1(E) x_2(E, y_1) \cdots \quad \text{and} \quad y_1(v, x_1) y_2(v, x_1, x_2)$$

which we extend in any way to strings of length α and β, respectively.

Let $X_v \subset \{0,1\}^\alpha$ be the set of all strings of length α produced this way by $P_\mathcal{E}$ under Ω, if v is given to $P_\mathcal{V}$. Every $v \in \mathcal{V}$ defines then a function $F_v : X_v \to \{0,1\}^\beta$, because $P_\mathcal{V}$'s operations are dependent only on the messages from $P_\mathcal{E}$ and on v.

Clearly, there are in total at most $2^{\beta \cdot 2^\alpha}$ such functions and if $2^{\beta \cdot 2^\alpha} < n$ or, equivalently, if

$$\alpha < \log\log n - \log \beta \tag{6.1}$$

then at least two functions must be the same, that is,

$$F_v \equiv F_{v'}, \quad \text{for some } v, v' \in \mathcal{V}. \tag{6.2}$$

But then $P_\mathcal{E}$ receives the same strings if v or v' are given to $P_\mathcal{V}$. Therefore, for any $E \supset \{v, v'\}$ there is no way for $P_\mathcal{E}$ to separate v and v'. Obviously, in a complete k-uniform hypergraph there is such an edge E. Therefore, (6.1) cannot hold and we have proved the following inequality.

Lemma 1: For all $k \le n$ and $\beta \le \lceil \log n \rceil$

$$\nu_\infty(n, k; \beta) \ge \log\log n - \log \beta. \tag{6.3}$$

This simple fact will yield inductively the main result of this section.

Theorem 8: For $\beta < \frac{1}{\ell} \log n$ and $\ell^2 \le \log n$

$$\nu_\infty(n, 2^\ell; \beta) \ge \log\log n + \ell - 1 - \log \beta. \tag{6.4}$$

Furthermore, for $2^{\ell-1} < k < 2^\ell$

$$\nu_\infty(n, k, \beta) \ge \log\log n + \ell - 2 - \log \beta.$$

A successful strategy for the hypergraph $([n], \binom{[n]}{k})$ must have $\beta \ge \log k$. Therefore, we get the following consequence of Theorem 8.

Corollary 1: For $\lceil \log k \rceil^2 \le \log n$

$$\overline{\overline{C}}_\infty\left(\left([n], \binom{[n]}{k}\right), P_\mathcal{E}\right) \ge \log\log n + 2\log k - 1 - \log\log k.$$

Proof of Theorem 8: We proceed by induction on ℓ. Lemma 1 gives

$$\nu_\infty(n, 2, \beta) \ge \log\log n - \log \beta$$

which settles the case $\ell = 1$. By induction hypothesis it holds for $\ell - 1$.

We call a protocol Ω an (α, β, ℓ) protocol, if $b_\mathcal{E} \le \alpha$, $b_\mathcal{V} \le \beta$, and it is successful (that is, $P_\mathcal{E}$ can separate the vertices in his 2^ℓ-size edges).

Even Case: $P_\mathcal{E}$ sends first.

Suppose that Ω is an (α, β, ℓ) protocol violating (6.4), that is,

$$\alpha < \log\log n + \ell - 1 - \log \beta. \tag{6.5}$$

After having sent his first bit, $P_\mathcal{E}$ divides the protocol Ω into two subprotocols Ω_0 and Ω_1.

Here Ω_i is the protocol, if $i \in \{0, 1\}$ was sent. However, by induction hypothesis and (6.5), both Ω_0 and Ω_1 are not $(\alpha - 1, \beta, \ell - 1)$ protocols. So for $i = 0, 1$ there must be a subset $T_i \subset \mathcal{V}$ of cardinality $2^{\ell-1}$, which is not separated by Ω_i. Define $T^* = T_0 \cup T_1$. Then $|T^*| \le 2^\ell$, but T^* cannot be separated by Ω!

Odd Case: $P_\mathcal{V}$ sends first.

Suppose that $P_\mathcal{V}$ uses in his first round a prefix code $\{w_1, \cdots, w_\gamma\}$ to send his message. Namely, for $v \in \mathcal{V}_i$ he sends w_i of length $\ell_i < \beta$ and $\bigcup_{i=1}^\gamma \mathcal{V}_i = \mathcal{V}$.

This reduces our problem to that of finding an $(\alpha, \beta - \ell_i, \ell)$ protocol in the even case for the hypergraph $\mathcal{H}_i = (\mathcal{V}_i, \binom{\mathcal{V}_i}{2^\ell})$.

If there is an $i \in \{1, \cdots, \gamma_1\}$ with

$$\frac{\log |\mathcal{V}_i|}{\beta - \ell_i} \ge \frac{\log n}{\beta} \tag{6.6}$$

then with the already established (6.4) in the even case

$$\alpha \ge \log\log |\mathcal{V}_i| + \ell - 1 - \log(\beta - \ell_i)$$

as $\beta < \frac{1}{\ell} \log n$ and (6.6) imply $\beta - \ell_i < \frac{1}{\ell} \log |\mathcal{V}_i|$ and $|\mathcal{V}_i| > 2^\ell$.

We conclude that

$$\begin{aligned} \alpha &\ge \log\left(\frac{\beta - \ell_i}{\beta} \log n\right) + \ell - 1 - \log(\beta - \ell_i) \\ &= \log\log n + \ell - 1 - \log \beta \end{aligned}$$

and we are done.

Otherwise

$$\frac{\log |\mathcal{V}_i|}{\beta - \ell_i} < \frac{\log n}{\beta}, \quad \text{for } i = 1, \cdots, \gamma \tag{6.7}$$

which is equivalent to

$$|\mathcal{V}_i| < n^{1 - \frac{\ell_i}{\beta}}, \quad \text{for } i = 1, \cdots, \gamma. \tag{6.8}$$

By summation of both sides

$$n = \sum_{i=1}^{\gamma} |\mathcal{V}_i| < n \sum_{i=1}^{\gamma} n^{-\frac{\ell_i}{\beta}}$$

or

$$1 < \sum_{i=1}^{\gamma} 2^{-\frac{\ell_i}{\beta} \log n}. \tag{6.9}$$

Since $\beta < \log n$, this implies

$$1 < \sum_{i=1}^{\gamma} 2^{-\ell_i}$$

which contradicts Kraft's inequality.

In [28, Theorem 1] it is established that

$$C_4\left(\left([n], \binom{[n]}{k}\right), P_\mathcal{E}\right) \lesssim 2\log k + \log\log n. \tag{6.10}$$

Actually, the protocol used requires always to send $\sim \log k$ bits for $P_\mathcal{V}$ and $\sim \log k + \log\log n$ bits for $P_\mathcal{E}$. It is, therefore, an upper bound also in case of two channels.

In conjunction with Corollary 1 we have therefore a complete characterization.

Theorem 9: For $\lceil \log k \rceil^2 < \log n$ and $m \geq 4$

$$\bar{\bar{C}}_m\left([n], \binom{[n]}{k}, P_\mathcal{E}\right) \sim 2\log k + \log\log n.$$

Remark 6: The pair (v, E) may determine who sends first.

Remark 7: We conjecture that also (6.10) is tight. In fact we tend to believe that four rounds suffice for all $\mathcal{H}$ in the one-channel model with one recipient.

Remark 8: In Section VII we show that for $\mathcal{H}(n,k) = ([n], \binom{[n]}{k})$

$$\begin{aligned} C_2(\mathcal{H}(n,k), P_\mathcal{E}) &\simeq C_3(\mathcal{H}(n,k), P_\mathcal{E}) \\ &> C_4(\mathcal{H}(n,k), P_\mathcal{E}). \end{aligned}$$

It was shown first by Zhang and Xia [29] that three rounds may not suffice. The present example is simpler.

Remark 9: With similar ideas we obtained also tight bounds for

$$\bar{\bar{C}}_m\left(\binom{[n]}{L}, \binom{[n]}{K}, P_\mathcal{E}\right)$$

where $K > L$ and an edge is a K-element subset with its L-element subsets as its vertices.

VII. Characterization of $C_2(([n], \binom{[n]}{k}), P_\mathcal{E})$

We use here the abbreviations

$$\mathcal{H}(n,k) = \left([n], \binom{[n]}{k}\right)$$

and

$$\mu_m(n,k) = C_m(\mathcal{H}(n,k), P_\mathcal{E}). \tag{7.1}$$

The following upper bound i) on $\mu_2(n,k)$ is due to Orlitsky and appears in [22, Theorem 4]. It is included here, because we want to demonstrate that it is a simple consequence of the Covering Lemma, and also in order to show that our deeper lower bound is "almost optimal."

Theorem 10:

i) $\mu_2(n,k) \leq (3\log k + \log\log n)$ for $k \geq k_0$ (suitable).

ii) For all $\varepsilon > 0$

$$\mu_2(n,k) \geq (1-\varepsilon)(3\log k + \log\log n)$$

if $k = 0(\log n)$ and n, k are large.

A. The Upper Bound

We make use of a consequence of the Covering Lemma stated in Section I. It is formulated for every hypergraph $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ with $\mathcal{E}' \subset \binom{[n]}{k}$, but will be used here only for $\mathcal{H}(n,k)$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{H}')$ be the automorphism group of $\mathcal{H}'$, that is, the maximal subgroup of the group Σ_n of all permutations on $\mathcal{V}' = [n]$, for whose elements σ

$$\sigma E = \{\sigma v : v \in E\} \in \mathcal{E}, \quad \text{for all } E \in \mathcal{E}. \tag{7.2}$$

$\mathcal{A}(\mathcal{H}')$ is transitive, if for any $E_1, E_2 \in \mathcal{E}$ a $\sigma \in \mathcal{A}(\mathcal{H}')$ exists with $\sigma E_1 = E_2$.

Corollary 2: For a hypergraph $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ with transitive automorphism group $\mathcal{A}(\mathcal{H}')$ and any subhypergraph $\mathcal{H}'' = (\mathcal{V}', \mathcal{E}''), \mathcal{E}'' \subset \mathcal{E}'$, there is a subset $\mathcal{B}'' \subset \mathcal{A}(\mathcal{H}')$ with the following properties:

i) For every $E \in \mathcal{E}'$ there is a $\sigma \in \mathcal{B}''$ with

$$E \in \sigma\mathcal{E}'' = \{\sigma E'' : E'' \in \mathcal{E}''\}.$$

ii) $|\mathcal{B}''| \leq \lceil \frac{|\mathcal{E}'|}{|\mathcal{E}''|} \ln |\mathcal{E}'| \rceil$.

Proof: Choose in the Covering Lemma

$$\mathcal{V} \triangleq \mathcal{E}', \mathcal{E} \triangleq \{\sigma\mathcal{E}'' : \sigma \in \mathcal{A}(\mathcal{H}')\}.$$

Thus

$$|\mathcal{V}| = |\mathcal{E}'|, |\mathcal{E}| = |\mathcal{A}(\mathcal{H}')|, d_\mathcal{V} = D_\mathcal{V} = \frac{|\mathcal{A}(\mathcal{H}')||\mathcal{E}''|}{|\mathcal{E}'|}$$

and the result follows.

Proof of i) in Theorem 10: Choose

$$\mathcal{H}' = (\mathcal{V}', \mathcal{E}') = \mathcal{H}(n,k)$$

and partition $\mathcal{V}'$ into k^2 parts $U_1, \cdots, U_{k^2}$ of sizes $\lfloor \frac{n}{k^2} \rfloor$ or $\lceil \frac{n}{k^2} \rceil$ and define $\mathcal{H}'' = (\mathcal{V}', \mathcal{E}'')$ with

$$\mathcal{E}'' = \{E'' \in \mathcal{E}' : |E'' \cap U_j| \leq 1 \text{ for } j = 1, \cdots, k^2\}.$$

Let $\mathcal{B}''$ be the set of permutations with the properties i), ii) in Corollary 2.

Since

$$\begin{aligned} \frac{|\mathcal{E}'|}{|\mathcal{E}''|} &\leq \frac{\binom{n}{k}}{\binom{k^2}{k}\left\lfloor \frac{n}{k^2} \right\rfloor^k} \leq \frac{1}{\prod_{i=0}^{k-1}\left(1 - \frac{i}{k^2}\right)} \\ &= \exp\left\{-\sum_{i=1}^{k-1} \ln\left(1 - \frac{i}{k^2}\right)\right\} \\ &\leq \exp\left\{\sum_{i=1}^{k-1} \frac{i}{k^2} + \theta\left(\frac{i}{k^2}\right)^2\right\} \\ &\leq \exp\left\{\frac{k(k-1)}{2k^2} + \theta\frac{1}{k}\right\} \leq c \end{aligned}$$

for suitable constants θ and c, we get

$$|\mathcal{B}''| \leq c \cdot \ln\binom{n}{k}. \tag{7.3}$$

Protocol: $P_\mathcal{E}$ tells $P_\mathcal{V}$ the σ with $E \in \sigma\mathcal{E}''$ and then $P_\mathcal{V}$ tells $P_\mathcal{E}$ the index j with $v \in \sigma U_j$. This requires exactly $\lceil \log |\mathcal{B}''| \rceil + 2\log k$ bits and not more than $\log\log n + 3\log k$ bits for $k \geq k_0(c)$. This establishes i).

B. The Lower Bound

Since there are two rounds, if $P_\mathcal{V}$ starts in a protocol he has to send $\lceil \log n \rceil$ bits so that $P_\mathcal{E}$ knows v. We can assume therefore that always $P_\mathcal{E}$ starts. We split the proof into two parts.

Lemma 2: For a successful strategy $\Omega = (f, g)$, the set of edges with "few answers"

$$\mathcal{E}_1 \triangleq \{E \in \mathcal{E} : |\{g(v, f(E)) : v \in \mathcal{V}\}| \leq k^{2-\delta}\}$$

requires either "many questions." that is,

$$\ln |\{f(E) : E \in \mathcal{E}_1\}| > \frac{\theta_1}{8} k^\delta \tag{7.4}$$

for some constant $\theta_1 \in (0, 1)$, or else there is a set

$$F = \{v_1, \cdots, v_{k_1}\} \subset \mathcal{V}, k_1 = \lfloor \frac{k}{2} \rfloor$$

such that for all $E \in \mathcal{E}_1$ there are $v_i, v_j \in F$ with

$$g(v_i, f(E)) = g(v_j, f(E)). \tag{7.5}$$

Proof: Let us write $\ell \triangleq k^{2-\delta}$. For each fixed $f(E), E \in \mathcal{E}_1$, there are at most $\binom{\ell}{k_1} \lceil \frac{n}{\ell} \rceil^{k_1}$ edges E' in $(\mathcal{V}, \binom{\mathcal{V}}{k_1})$ such that

$$g(v, f(E)) \neq g(v', f(E)), \quad \text{for all } v, v' \in E' (v \neq v'). \tag{7.6}$$

Therefore, the number of $E' \in \binom{\mathcal{V}}{k_1}$ satisfying (7.6) for some $E \in \mathcal{E}_1$ cannot exceed

$$\binom{\ell}{k_1} \left\lceil \frac{n}{\ell} \right\rceil^{k_1} \cdot |\{f(E) : E \in \mathcal{E}_1\}|. \tag{7.7}$$

We evaluate and upper-bound this number

$$\begin{aligned} \binom{\ell}{k_1} \left\lceil \frac{n}{\ell} \right\rceil^{k_1} &\sim \binom{\ell}{k_1} \left(\frac{n}{\ell}\right)^{k_1} \\ &= \frac{\ell^{k_1}}{k_1!} \left(\frac{n}{\ell}\right)^{k_1} \prod_{i=0}^{k_1-1} \left(1 - \frac{i}{\ell}\right) \\ &= \binom{n}{k_1} \frac{\prod_{i=0}^{k_1-1} \left(1 - \frac{i}{\ell}\right)}{\prod_{i=0}^{k_1-1} \left(1 - \frac{i}{n}\right)} \\ &\sim \binom{n}{k_1} \exp\left\{ -\frac{k_1(k_1 - 1)}{2\ell} + \frac{k_1(k_1 - 1)}{2n} \right\} \\ &\leq \binom{n}{k_1} \exp\left\{ -\frac{k_1 \theta_1}{\ell} \right\} \end{aligned}$$

if n and k are large enough, $\frac{n}{\ell} \to \infty$, $(n \to \infty)$, and $\theta_1 \in (0, 1)$ is a constant.

Further

$$|\{f(E) : E \in \mathcal{E}_1\}| = \exp\{\ln |\{f(E) : E \in \mathcal{E}_1\}|\}$$

and thus the upper bound

$$\binom{n}{k_1} \exp\left\{ \ln |\{f(E) : E \in \mathcal{E}_1\}| - \frac{\theta_1}{8} k^\delta \right\}.$$

Therefore, there is the claimed F unless (7.4) holds.

Lemma 3: Suppose that for a successful strategy $\Omega = (f, g)$ for all possible pairs $(x^{\alpha_1}, y^{\beta_2})$

$$\beta_2 \leq \lceil 3 \log k + \log \log n \rceil = \log \gamma$$

say, then for the set $\mathcal{X} = \{x^{\alpha_1} = f(E) : E \in \mathcal{E}_2 = \mathcal{E} \setminus \mathcal{E}_1\}$

$$|\mathcal{X}| \geq (k_2 \log n / \log \gamma)(1 - \varepsilon') \tag{7.8}$$

for $k_2 = \lfloor \frac{k_1}{2} \rfloor = \lfloor \frac{\lfloor \frac{k}{2} \rfloor}{2} \rfloor$, for all $\varepsilon' \in (0, 1)$, and all large n, k.

Proof: Assume that (7.8) does not hold. Partition $\mathcal{X}$ into $\mathcal{X}_1, \cdots, \mathcal{X}_{k_2}$ such that for $i = 1, \cdots, k_2$

$$|\mathcal{X}_i| \leq \left\lceil \frac{|\mathcal{X}|}{k_2} \right\rceil < \frac{\log n}{\log \gamma}. \tag{7.9}$$

Equivalent is

$$\gamma^{|\mathcal{X}_i|} < n = |\mathcal{V}|, \quad \text{for } i = 1, 2, \cdots, k_2. \tag{7.10}$$

Since our assumption implies that $g(\cdot, E)$ takes at most γ values for all $E \in \mathcal{E}$, for every $i \in \{1, \cdots, k_2\}$ one can find a pair $\{u_i, u_i'\} \subset \mathcal{V}$ such that for all $x^{\alpha_1} \in \mathcal{X}_i$

$$g(u_i, x^{\alpha_1}) = g(u_i', x^{\alpha_1}).$$

Let

$$E^* = \left(\bigcup_{i=1}^{k_2} \{u_i, u_i'\} \right) \cup \{v_1, \cdots, v_{k_1}\}$$

for the above pairs and $\{v_1, \cdots, v_{k_1}\}$ as in Lemma 2. Then $|E^*| \leq k$, so that an $E \in \mathcal{E}$ exists with $E^* \subset E$.

However, when $E \in \mathcal{E}_1$, then by (7.5) $P_\mathcal{E}$ cannot distinguish between some v_i and v_j and when $E \in \mathcal{E}_2$, there exists an i such that $f(E) \in \mathcal{X}_i$ and, therefore, $P_\mathcal{E}$ cannot distinguish between u_i and u_i'. This contradiction proves the theorem.

Remark 10: The strategy yielding our upper bound can be viewed as a perfect hashing: there are strict colorings for every $(\mathcal{V}, \sigma\mathcal{E}'')$.

Corollary 3: $\mu_3(n, k) \geq \mu_2(n, k)$ if $k = 0(\log n)$ and n, k are large.

Proof: Write the possible sequences under a strategy Ω as $(y^{\beta_1}, x^{\alpha_2}, y^{\beta_3})$ with $y^{\beta_1} = f(v)$. Now, if $|\{y^{\beta_1} = f(v) : v \in \mathcal{V}\}| \leq n^{\frac{1}{2}}$, then for some y^{β_1} $\mathcal{V}' = \{v \in \mathcal{V} : f(v) = y^{\beta_1}\}$ has cardinality at least $n^{\frac{1}{2}}$.

When y^{β_1} has been sent we are in the case of two rounds for the hypergraph $\mathcal{H}' = (\mathcal{V}', \binom{\mathcal{V}'}{k})$ and have the lower bound $(1 - \varepsilon)(3 \log k + \log \log n - 1)$.

We can assume therefore that there are at least $n^{\frac{1}{2}}$ possible strings y^{β_1} and one must have a length of at least $\frac{1}{2} \log n \gg 3 \log k + \log \log n$ by our assumption.

Remark 11: Is it true that $\mu_3(n, k) = \mu_2(n, k)$?

VIII. On Cartesian Products of Hypergraphs

One can come to the present hypergraphs as follows.

Consider a discrete memoryless correlated source $(X^n, Y^n)_{n=1}^\infty$, where $X^n = X_1 \cdots X_n$, $Y^n = Y_1 \cdots Y_n$.

Person $P_\mathcal{X}$ observes the output $X^n = x^n$ and person $P_\mathcal{Y}$ observes $Y^n = y^n$. They exchange knowledge with zero probability of error. How many bits do they have to exchange in the worst case until $P_\mathcal{Y}$ knows x^n?

We construct the multi-hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$, where $\mathcal{E} = \{E_y : y \in \mathcal{Y}\}$ and

$$E_y = \{x \in \mathcal{X} : \text{Prob}(X = x, Y = y) > 0\} \tag{8.1}$$

and its nth Cartesian product $\mathcal{H}^n = (\mathcal{X}^n, \mathcal{E}^n)$, where

$$\mathcal{X}^n = \prod_1^n \mathcal{X}$$

and

$$= \prod_1^n \mathcal{E} = \{E_{y^n} = \textstyle\prod_{t=1}^n E_{y_t} : y^n = (y_1, \cdots, y_n) \in \mathcal{Y}^n\}.$$

Clearly

$$\deg(x^n) = |\{E^n \in \mathcal{E}^n : x^n \in E^n\}| = \prod_{t=1}^n \deg(x_t) \tag{8.2}$$

and

$$\deg(E_{y^n}) = |E_{y^n}| = \prod_{t=1}^n |E_{y^t}|. \tag{8.3}$$

$$D_{\mathcal{X}^n} = \max_{x^n} \deg(x^n) = \prod_{t=1}^n D_{\mathcal{X}} = (D_{\mathcal{X}})^n$$
$$D_{\mathcal{E}^n} = \max_{E^n} \deg(E^n) = \prod_{t=1}^n D_{\mathcal{E}} = (D_{\mathcal{E}})^n.$$

For $C_m(\mathcal{H}^n, P_{\mathcal{E}^n})$ we define

$$A_m = \lim_{n\to\infty} \frac{1}{n} C_m(\mathcal{H}^n, P_{\mathcal{E}^n}). \tag{8.4}$$

Then obviously

$$A_2 \geq A_3 \geq A_4 \geq \cdots \geq A_\infty \geq \log D_{\mathcal{E}}.$$

It was shown in [28] that $A_4 = A_\infty = \log D_{\mathcal{E}}$ and that

$$A_2 \leq 2 \log D_{\mathcal{E}}. \tag{8.5}$$

We mention in the Appendix that for "balanced" hypergraphs [24]

$$C_3 \leq \log D_{\mathcal{E}} + 3 \log\log \max(D_{\mathcal{E}}, D_{\mathcal{Y}}) + 11. \tag{8.6}$$

For our product hypergraph $\mathcal{H}^n$ this implies

$$C_3(\mathcal{H}^n; P_{\mathcal{E}^n}) \leq n \log D_{\mathcal{E}} + 3 \log n + 3 \log\log \max(D_{\mathcal{E}}, D_{\mathcal{Y}}) + 11$$

and thus

$$A_3 \leq \log D_{\mathcal{E}}. \tag{8.7}$$

Remark 12: In [28] only $A_4 \leq \log D_{\mathcal{E}}$ was proved.

Our main concern is to determine A_2. We succeeded to derive bounds for special classes of hypergraphs.

For $\mathcal{H}^n = (\mathcal{X}^n, \mathcal{E}^n)$ as above let $\eta_n(\nu)$ be the maximal number of edges in $\mathcal{H}^n$ which can be properly colored with a vertex coloring $\varphi : \mathcal{X}^n \to \{1, 2, \cdots, \nu\}$.

When $\mathcal{H}^n$ has a transitive automorphism group, then by Corollary 2 in Section VII one can use the set of properly colored edges $\mathcal{E}'$ to cover $\mathcal{E}^n$ with at most

$$n(\ln|\mathcal{E}|) \cdot \frac{|\mathcal{E}|^n}{\eta_n(\nu)} + 1 \text{"copies."}$$

Thus similarly to Theorem 10

$$C_2(\mathcal{H}^n; P_{\mathcal{E}^n}) \leq \min_\nu \left[\log \nu + \log n + \log\log|\mathcal{E}| + n \log \frac{|\mathcal{E}|}{(\eta_n(\nu))^{\frac{1}{n}}}\right] \times (1 + o(1)). \tag{8.8}$$

Suppose, on the other hand, that $P_{\mathcal{E}^n}$ decomposes $\mathcal{E}^n$ into t parts, informs $P_{\mathcal{Y}}$ that the part E_{y^n} is in, and then $P_{\mathcal{Y}}$ uses at most ν colors for the vertices.

To guarantee that $P_{\mathcal{E}^n}$ knows x^n at the end of the communication for all $E^n \in \mathcal{E}^n$ there has to be one of the parts in which E^n is properly colored. Therefore

$$t\, \eta_n(\nu) \geq |\mathcal{E}|^n \tag{8.9}$$

or

$$\frac{1}{n} \log t \geq \log|\mathcal{E}| - \frac{1}{n} \log \eta_n(\nu). \tag{8.10}$$

Therefore

$$\frac{1}{n} C_2(\mathcal{X}^n; P_{\mathcal{E}^n}) \geq \frac{1}{n} \min\left\{\log t + \log \nu : \frac{1}{n} \log t \geq \log|\mathcal{E}| - \frac{1}{n} \log \eta_n(\nu)\right\}$$

and this and (8.8) imply the following result.

Lemma 4: If the product hypergraph $\mathcal{H}^n = (\mathcal{X}^n, \mathcal{Y}^n)$ has a transitive automorphism group, then

$$A_2 = \lim_{n\to\infty} \min_\nu \left(\frac{1}{n} \log \nu + \log|\mathcal{E}| - \frac{1}{n} \log \eta_n(\nu)\right). \tag{8.11}$$

Here the limit exists, since

$$\eta_{n_1+n_2}(\nu_1 \times \nu_2) \geq \eta_{n_1}(\nu_1)\eta_{n_2}(\nu_2).$$

Example 10 (See Also [36]):

$$\mathcal{H} = (\mathcal{X}, \mathcal{E}) = (\{0, 1, 2\}, \{e_1, e_2, e_3\})$$

where $e_i = \{i, i + 1 \bmod 3\}$ for $i = 0, 1, 2$.

Obviously, choosing $\nu = 3$ in (8.11), we see that

$$A_2 \leq \log 3 < 2 \log D_{\mathcal{E}} = 2 \log 2 = 2.$$

Actually, we can do even better and show that

$$A_2 \leq \log \sqrt{6}. \tag{8.12}$$

Consider the 4-coloring of $\mathcal{H}^2$

$$\mathcal{X}_1^2 = \{(0,1), (1,0)\}, \quad \mathcal{X}_2^2 = \{(1,2), (2,1)\}$$
$$\mathcal{X}_3^2 = \{(2,0), (0,2)\}, \quad \mathcal{X}_4^2 = \{(0,0), (1,1), (2,2)\}. \tag{8.13}$$

It is proper for all edges

$$\mathcal{E}^2 \setminus \{\{0,1\} \times \{0,1\}, \{1,2\} \times \{1,2\}, \{2,0\} \times \{2,0\}\}.$$

Therefore, (8.11) implies (8.12).

Example 11: For graphs $\mathcal{H} = (\mathcal{X}, \mathcal{E})$, $|E| = 2$ for all $E \in \mathcal{E}$, (8.5) is also not tight. It suffices to consider $\mathcal{H} = ([\ell], \binom{[\ell]}{2})$. Moreover, as adding vertices does not decrease the complexity, we assume $3 \mid \ell$.

Partition $[\ell]$ into the parts $\mathcal{L}_0, \mathcal{L}_1$, and $\mathcal{L}_2$, and let $g(v) = i$ iff

$$v \in \mathcal{L}_i = \{m : 1 \le m \le \ell, m \equiv i \bmod 3\} \quad (i = 0, 1, 2).$$

On $\mathcal{H}^2$ define $g_2(v_1, v_2) = (g(v_1), g(v_2))$. Replacing (v_1, v_2) in (8.13) by $g_2(v_1, v_2)$ one obtains a 4-coloring of $\mathcal{H}^2$. Write the edges of $\mathcal{H}^2$ in the form

$$\{v_1, v_1'\} \times \{v_2, v_2'\}.$$

As before, we can see that an edge is properly colored if $g(v_i) \ne g(v_i')$ for $i = 1, 2$ and

$$\{g(v_1), g(v_1')\} \ne \{g(v_2), g(v_2')\}.$$

Therefore, the number of properly colored edges is at least

$$6 \cdot \left(\frac{\ell}{3}\right)^4 = \frac{2}{27}\ell^4$$

and, consequently, by definition

$$\eta_2(4) \ge \frac{2\ell^4}{27}.$$

However

$$\frac{\left|\binom{[\ell]}{2}\right|}{\eta_2(4)} \le \frac{\left(\frac{\ell(\ell-1)}{2}\right)^2}{\frac{2\ell^4}{27}} < \frac{27}{8}$$

and thus by (8.11)

$$A_2 \le \frac{1}{2}\log\left(4 \cdot \frac{27}{8}\right) = \frac{1}{2}\log\frac{7}{2} = 1.877\cdots < 2. \qquad (8.14)$$

Expression (8.14) can be improved.

Using Hamming codes, we show next that (8.12) is not tight either for the triangle.

Our bound (8.12) was derived with the 4-coloring of $\mathcal{H}^2$, say $f : \mathcal{X}^2 \to \{1, 2, 3, 4\}$ described in (8.13) and its n-fold product

$$f_n = (f, \cdots, f) : (\mathcal{X}^2)^n \to \{1, 2, 3, 4\}^n.$$

We call this the *first* coloring.

Now we introduce a second coloring. For this we extend the sets $\mathcal{X}_1^2, \mathcal{X}_2^2, \mathcal{X}_3^2$ to the sets $\mathcal{Y}_1^2 = \mathcal{X}_1^2 \cup \{y_1\}$, $\mathcal{Y}_2^2 = \mathcal{X}_2^2 \cup \{y_2\}$, $\mathcal{Y}_3^2 = \mathcal{X}_3^2 \cup \{y_3\}$ where y_1, y_2, y_3 are new symbols. We also set $\mathcal{Y}_4^2 = \mathcal{X}_4^2$. Thus the four new sets have all three elements and the set

$$\mathcal{Z}^n = \prod_{t=1}^{n} \mathcal{Z}_i \text{ with } \mathcal{Z}_i \in \{\mathcal{Y}_j^2 : 1 \le j \le 4\}$$

has 3^n elements.

For this *fixed* $\mathcal{Z}^n$ let $g_i : \mathcal{Z}_i \to \{0, 1, 2\}$ be bijective.

Next we color the vertices in $\{0, 1, 2\}^n$ according to the cosets of an $[n, k]$ (where $k \triangleq n - r$, $n \triangleq \frac{3^r - 1}{2}$) ternary Hamming code $H_{n,k}$:

Two elements have the same color iff they are in the same coset. Let h_n be this coloring. Finally, we color the elements in $\mathcal{Z}^n$ through

$$G_{\mathcal{Z}^n} = h_n(g_1, \cdots, g_n). \qquad (8.15)$$

This is a coloring with 3^r colors. Combination with the coloring f_n gives a $4^n \times 3^n$ coloring F_n.

In order to derive a new bound on A_2 we need an estimate for the number of good edges under F_n. Denote the edges of the $2n$th product of triangles as $U^n = U_1 \times \cdots \times U_n$, where U_i is a two-dimensional edge (a "square").

Notice that under our first coloring, difficulties arise only at the U_i's equal to $E_j = e_j \times e_j$ $(j = 0, 1, 2)$.

Considering that any two elements in (the fixed) $\mathcal{Z}^n$ with Hamming distance smaller than 2 have different second color, because the Hamming code has minimal distance 3, it is not hard to see that all $U^n = U_1 \times \cdots \times U_n$ with

$$\left|\{U_j : 1 \le i \le n\} \cap \left(\bigcup_{i=0}^{2} E_i\right)\right| < 3 \qquad (8.16)$$

are well-colored.

Let now $\nu = 4^n \cdot 3^r$, then

$$\eta_{2n}(\nu) \ge \sum_{j=0}^{2} \binom{n}{j} 6^{n-j} 3^j.$$

Analogous to the previous derivation for the product of triangles, we get now for $n = \frac{3^r - 1}{2}$, $r = 2, 3, \cdots$

$$\begin{aligned} A_2 &\le \frac{1}{2n}\log\frac{3^{2n} \cdot 4^n \cdot 3^r}{\sum_{j=0}^{2}\binom{n}{j}6^{n-j}3^j} \\ &= \frac{1}{2n}\log\frac{6^n \cdot 3^r}{\sum_{j=0}^{2}\binom{n}{j}\left(\frac{1}{2}\right)^j} \\ &= \log\sqrt{6} + \frac{1}{2n}\log\frac{2n+1}{1 + \frac{n}{2} + \frac{n(n-1)}{8}} \\ &= \log\sqrt{6} + \gamma_n, \quad \text{say.} \end{aligned}$$

Choosing a large value for $n = \frac{3^r - 1}{2}$, γ_n becomes negative, which results in a bound better than $\log\sqrt{6}$. Choosing n very large makes this bound less good, because of the factor $\frac{1}{2n}$.

Remark 13: It should be possible to improve the bound with other linear codes.

An obvious lower bound on A_2 for graphs is $\log D_{\mathcal{E}} = 1$. The simplest open question is to determine A_2 for the triangle, for which we know that

$$1 \le A_2 < \log\sqrt{6}.$$

Appendix
Known Results for $C_m(\mathcal{H}, P_{\mathcal{E}})$

We feel that our sketch is helpful for the orientation of the reader. The shorthand C_m is used for $C_m(\mathcal{H}, P_{\mathcal{E}})$. It was shown in [22] that for every hypergraph

$$C_1 = \lceil \log \chi(\mathcal{H}) \rceil \qquad (A1)$$

and

$$C_\infty \geq \lceil \log C_1 \rceil + 1 \tag{A2}$$

if $\chi(\mathcal{H}_1) > 1$, that is, $\mathcal{H}$ is nontrivial. Such a bound (without the extra bit) has appeared for communication complexity already in [37].

The complete hypergraph $\mathcal{H} = ([n], \binom{[n]}{2})$ (League Problem in [22]) is an interesting example. It was shown in [22] that

$$C_2 = \cdots = C_\infty = \lceil \log \log n \rceil + 1, C_1 = \lceil \log n \rceil. \tag{A3}$$

The idea generalizes to all graphs ($D_\mathcal{E} = 2$)

$$C_\infty = \lceil \log \lceil \log \chi(\mathcal{H}) \rceil \rceil + 1 = \lceil \log C_1 \rceil + 1. \tag{A4}$$

For every hypergraph an obvious bound is

$$C_\infty \geq \log D_\mathcal{E}. \tag{A5}$$

Using a perfect hashing $f = (f_1, \cdots, f_k)$ with c colors, that is, the k functions $f_i : \mathcal{V} \to \{1, \cdots, c\}$ are such that for every $E \in \mathcal{E}$ some f_j is bijective on E, one obtains

$$C_2 \leq \lceil \log k \rceil + \lceil \log c \rceil.$$

Actually, knowing E $P_\mathcal{E}$ encodes i and $P_\mathcal{V}$ encodes $f_i(v)$.

By random choice it is shown in [22] that

$$C_2 \leq \log \log |\mathcal{V}| + 3 \log D_\mathcal{E}.$$

In the same argument $\mathcal{V}$ can be replaced by $\{1, \cdots, \chi\}$

$$C_2 \leq \log \log \chi + 3 \log D_\mathcal{E}.$$

This and the relations (A1), (A2), and (A5) imply

$$C_2 \leq 4C_\infty + 3. \tag{A6}$$

For communication complexity it was noticed in [37] that there can be an exponential gap between $C_1(f)$ and $C_2(f)$.

In [38] it was shown that for every m there is a function g with an exponential gap between $C_m(g)$ and $C_{m+1}(g)$. Expression (A6) shows that this does not happen for interactive communication.

How large can the gaps between C_m and C_∞ be?

The main result of [23] is, that for the "generalized league problem"

$$C_2 \geq (2 - \varepsilon) C_\infty \tag{A7}$$

can happen for every $\varepsilon > 0$.

Therefore, two rounds ("messages") are not optimal.

It was shown in [29] that three rounds are not optimal either! In the present paper we have shown that already a simply structured hypergraph such as the k-uniform hypergraph $([n], \binom{[n]}{k})$ has this property. Moreover, in this case $C_2 = C_3$.

For balanced hypergraphs ($D_\mathcal{V} = D_\mathcal{E}$) it is an immediate consequence of Coloring Lemma 2 that (see [24])

$$C_1 \leq 2 \log D_\mathcal{E} + 1 \leq 2C_\infty + 1. \tag{A8}$$

A clever construction in projective planes gives an example of a balanced hypergraph ([24] with J. Kahn) with

$$C_1 \geq 2C_\infty - 6. \tag{A9}$$

Finally, the hashing idea and Coloring Lemma 1 in conjunction with Lovász's Local Lemma give in the balanced case [24]

$$C_3 \leq \log D_\mathcal{E} + 3 \log \log D_\mathcal{E} + 11 \tag{A10}$$

or by (A5)

$$C_3 \leq C_\infty + 3 \log \log C_\infty + 11.$$

Presently, there is no example in the balanced case (only an existence proof in [24]) for

$$C_2 \geq (2 - \varepsilon) C_\infty \geq c \quad \text{(any const.)} \tag{A11}$$

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