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The Complete Intersection Theorem for Systems of Finite Sets

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1. HISTORICAL BACKGROUND AND THE NEW THEOREM

We are concerned here with one of the oldest problems in combinatorial extremal theory.

It is readily described after we have made a few conventions. $\mathbb{N}$ denotes the set of positive integers and, for $i, j \in \mathbb{N}$, $i < j$, the set $\{i, i+1, \ldots, j\}$ is abbreviated as $[i, j]$. For $k, n \in \mathbb{N}$, $k \leqslant n$, we set

$$2^{[n]} = \{F : F \subset [1, n]\}, \qquad \binom{[n]}{k} = \{F \in 2^{[n]} : |F| = k\}. \tag{1.1}$$

A system of sets $\mathscr{A} \subset \binom{[n]}{k}$ is called *t-intersecting* if

$$|A_1 \cap A_2| \geqslant t \qquad \text{for all } A_1, A_2 \in \mathscr{A}, \tag{1.2}$$

and $I(n, k, t)$ denotes the set of all such systems.

The investigation of the function

$$M(n, k, t) = \max_{\mathscr{A} \in I(n,k,t)} |\mathscr{A}|, \qquad 1 \leqslant t \leqslant k \leqslant n, \tag{1.3}$$

and the structure of maximal systems was initiated by Erdös, Ko and Rado. According to reference [7], they had already proved the following theorem by the year 1938, although it was only published in 1961 in their famous paper [9].

THEOREM EKR. *For $1 \leqslant t \leqslant k$ and $n \geqslant n_o(k, t)$ (suitable)*

$$M(n, k, t) = \binom{n-t}{k-t}. \tag{1.4}$$

Clearly, the system

$$\mathscr{A}(n, k, t) = \left\{A \in \binom{[n]}{k} : [1, t] \subset A\right\} \tag{1.5}$$

has cardinality $\binom{n-t}{k-t}$ and is therefore optimal for $n \geqslant n_o(k, t)$.

The smallest $n_o(k, t)$ for which this is the case has been determined by Frankl [10] for $t \geqslant 15$ and subsequently by Wilson [13] for all t:

$$n_o(k, t) = (k - t + 1)(t + 1). \tag{1.6}$$

Moreover, for $n > (k - t + 1)(t + 1)$, there is—up to obvious permutations on the ground set $[1, n]$—only one optimal system.

In the present paper we settle all of the remaining cases

$$n < (k - t + 1)(t + 1).$$

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In particular, we prove the so-called $4m$-*Conjecture* (Erdös, Ko and Rado, 1938; see [7, page 56] and the survey [6])

$$M(4m, 2m, 2) = \left|\left\{F \in \binom{[4m]}{2m} : F \cap [1, 2m] \geq m + 1\right\}\right|. \tag{1.7}$$

Thus, obviously,

$$M(4m, 2m, 2) = \frac{1}{2}\left(\binom{4m}{2m} - \binom{2m}{m}^2\right). \tag{1.8}$$

The previously best upper bound on $M(4m, 2m, 2)$ is due to Calderbank and Frankl [5].

There is a natural extension of the $4m$-Conjecture in terms of the systems

$$\mathcal{F}_i = \left\{F \in \binom{[n]}{k} : |F \cap [1, t+2i]| \geq t + i\right\} \qquad \text{for } 0 \leq i \leq \frac{n-t}{2} \tag{1.9}$$

to all possible parameters.

General Conjecture [10]. For $1 \leq t \leq k \leq n$,

$$M(n, k, t) = \max_{0 \leq i \leq \frac{n-t}{2}} |\mathcal{F}_i|. \tag{1.10}$$

Notice that for $n = 4m$, $k = 2m$, $t = 2$ the maximum is assumed for $i = m - 1$ and so the $4m$-Conjecture is covered. For $n \geq (k - t + 1)(t + 1)$ the maximum is assumed for $i = 0$. A further step towards proving the General Conjecture was taken in [10] for $t \geq 15$, where the cases $0{\cdot}8(k - t + 1)(t + 1) < n < (k - t + 1)(t + 1)$ are settled. Here $\mathcal{F}_1$ is (up to permutations) the only optimal system. Some other cases have been settled in [12].

Our main result establishes the validity of the General Conjecture and provides an even more specific answer concerning uniqueness.

Theorem. *For* $1 \leq t \leq k \leq n$:
(i) *With* $(k - t + 1)(2 + \frac{t-1}{r+1}) < n < (k - t + 1)(2 + \frac{t-1}{r})$ *for some* $r \in \mathbb{N} \cup \{0\}$, *we have*

$$M(n, k, t) = |\mathcal{F}_r|$$

and $\mathcal{F}_r$ *is—up to permutations—the unique optimum.* (*By convention,* $\frac{t-1}{r} = \infty$ *for* $r = 0$.)
(ii) *With* $(k - t + 1)(2 + \frac{t-1}{r+1}) = n$ *for* $r \in \mathbb{N} \cup \{0\}$
we have

$$M(n, k, t) = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$$

and an optimal system equals—up to permutations—either $\mathcal{F}_r$ *or* $\mathcal{F}_{r+1}$.

Convention. In the sequel, we write $\mathcal{A}_1 \equiv \mathcal{A}_2$ if the systems $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal up to permutations.

Remarks. (1) It suffices to treat the cases

$$n > 2k - t, \tag{1.11}$$

because for $n \leq 2k - t$ the whole system $\binom{[n]}{k}$ is t-intersecting.

(2) Our method of proof follows ideas in our work in Number Theory (see [2], [3], [4]). They led to the concept of generating sets in Section 2 and related pushing techniques in later sections, which go considerably beyond known techniques in Combinatorial Extremal Theory (see [11]).

(3) The t-intersecting systems in $I(n, k, t)$ can be understood as systems with a *diameter* less than $2k-2t$ in the Hamming distance. Instead of Intersection Theorems one can then speak of Diametric Theorems—a concept of geometrical meaning. This and relations to Isoperimetric Theorems have been discussed in [1].

(4) In [8], Erdös mentions the $4m$-Conjecture as the last open problem from [9].

2. Generating Sets and Their Properties

We begin with well-known notions.

Definition 2.1. For $A_1 = \{i_1, i_2, \dots, i_s\} \in \binom{[n]}{s}$, $i_1 < i_2 < \cdots < i_s$, and $A_2 = \{j_1, j_2, \dots, j_s\} \in \binom{[n]}{s}$, $j_1 < j_2 < \cdots < j_s$, we write

$$A_1 \prec A_2 \text{ iff } i_l \leqslant j_l \qquad \text{for all } 1 \leqslant l \leqslant s;$$

that is, A_1 can be obtained from A_2 by *left-pushing*. Furthermore, let $\mathscr{L}(A_2)$ be the set of all sets obtained this way from A_2. Also, set $\mathscr{L}(\mathscr{A}) = \bigcup_{A \in \mathscr{A}} \mathscr{L}(A)$.

Definition 2.2. $\mathscr{A} \subset 2^{[n]}$ is said to be *left compressed* iff $\mathscr{A} = \mathscr{L}(\mathscr{A})$.

Definition 2.3. We denote by $LI(n, k, t) \subset I(n, k, t)$ the set of all left compressed systems from $I(n, k, t)$.

It is well-known and easily follows with the shifting technique of [9] that

$$M(n, k, t) = \max_{\mathscr{A} \in I(n, k, t)} |\mathscr{A}| = \max_{\mathscr{A} \in LI(n, k, t)} |\mathscr{A}|. \tag{2.1}$$

Definition 2.4. For any $B \in 2^{[n]}$ we define the upset $\mathscr{U}(B) = \{B' \in 2^{[n]} : B \subset B'\}$. More generally, for $\mathscr{B} \subset 2^{[n]}$ we define the upset

$$\mathscr{U}(\mathscr{B}) = \bigcup_{B \in \mathscr{B}} \mathscr{U}(B).$$

Now we introduce new concepts.

Definition 2.5. For any $\mathscr{B} \subset \binom{[n]}{k}$ a set $g(\mathscr{B}) \subset \bigcup_{i \leqslant k} \binom{[n]}{i}$ is called a *generating set of* $\mathscr{B}$, if $\mathscr{U}(g(\mathscr{B})) \cap \binom{[n]}{k} = \mathscr{B}$. Furthermore, $G(\mathscr{B})$ is the set of all generating sets of $\mathscr{B}$. ($G(\mathscr{B}) \neq \varnothing$, because $\mathscr{B} \in G(\mathscr{B})$).

A first auxiliary result is readily established.

Lemma 1. *For $\mathscr{A} \in I(n, k, t)$, $n > 2k - t$,*

$$|E_1 \cap E_2| \geqslant t \qquad \textit{for } E_1, E_2 \in g(\mathscr{A}) \in G(\mathscr{A}).$$

PROOF. Since $2k-t<n$, $|E_1 \cap E_2|<t$ would allow k-element extensions of E_1 and E_2 which are not t-intersecting.

Next, we introduce further basic concepts.

DEFINITION 2.6. For $B=\{b_1, b_2, \ldots, b_{|B|}\} \subset [1, n]$, $b_1 \cap b_2 < \cdots < b_{|B|}$, write the largest element $b_{|B|}$ as $s^+(B)$. Also, for $\mathscr{B} \subset 2^{[n]}$ set

$$s^+(\mathscr{B}) = \max_{B \in \mathscr{B}} s^+(B).$$

DEFINITION 2.7. Let $\mathscr{A} \subset \binom{[n]}{k}$ be left compressed, i.e. $\mathscr{A} = \mathscr{L}(\mathscr{A})$. For any generating set $g(\mathscr{A}) \in G(\mathscr{A})$, consider $\mathscr{L}(g(\mathscr{A}))$ and introduce its set of minimal (in the sense of set-theoretical inclusion) elements $\mathscr{L}_*(g(\mathscr{A}))$.

Furthermore, define $G_*(\mathscr{A}) = \{g(\mathscr{A}) \in G(\mathscr{A}) : \mathscr{L}_*(g(\mathscr{A})) = g(\mathscr{A})\}$. (Notice that $\mathscr{A} \in G_*(\mathscr{A})$).

We continue with simple properties.

LEMMA 2. *For a left compressed* $\mathscr{A} \subset \binom{[n]}{k}$ *and any* $g(\mathscr{A}) \in G(\mathscr{A})$:
(i) $\mathscr{L}_*(g(\mathscr{A})) \in G(\mathscr{A})$;
(ii) $s^+(\mathscr{L}_*(g(\mathscr{A})) \leqslant s^+(g(\mathscr{A})))$;
(iii) *for* $A \in \mathscr{L}_*(g(\mathscr{A}))$ *and* $B \prec A$, *we have either* $B \in \mathscr{L}_*(g(\mathscr{A}))$ *or there exists a* $B' \in \mathscr{L}_*(g(\mathscr{A}))$ *with* $B' \subset B$.

The next important properties immediately follow from the definition of $G_*(\mathscr{A})$ and the left-compressedness of $\mathscr{A}$.

LEMMA 3. *For a left compressed* $\mathscr{A} \subset \binom{[n]}{k}$ *and* $g(\mathscr{A}) \in G_*(\mathscr{A})$, $\mathscr{A}$ *is a disjoint union*

$$\mathscr{A} = \bigcup_{E \in g(\mathscr{A})} \mathscr{D}(E),$$

where

$$\mathscr{D}(E) = \left\{ B \in \binom{[n]}{k} : B = E \cup B_1,\ B_1 \subset [s^+(E)+1, n],\ |B_1| = k - |E| \right\}.$$

LEMMA 4. *For a left compressed* $\mathscr{A} \subset \binom{[n]}{k}$ *and* $g(\mathscr{A}) \in G_*(\mathscr{A})$, *choose* $E \in g(\mathscr{A})$ *such that* $s^+(E) = s^+(g(\mathscr{A}))$ *and consider the set of elements of* $\mathscr{A}$ *which are only generated by* E; *that is,*

$$\mathscr{A}_E = (\mathscr{U}(E) \setminus \mathscr{U}(g(\mathscr{A}) \setminus \{E\})) \cap \binom{[n]}{k}.$$

Then

$$\mathscr{A}_e = \mathscr{D}(E) \qquad \text{and} \qquad |\mathscr{A}_E| = \binom{n - s^+(E)}{k - |E|}.$$

LEMMA 5. *Let* $\mathscr{A} \in LI(n, k, t)$, $g(\mathscr{A}) \in G_*(\mathscr{A})$, *and let* $E_1, E_2 \in g(\mathscr{A})$ *have the properties*

$$i \notin E_1 \cup E_2, \qquad j \in E_1 \cap E_2$$

for some $i, j \in [1, n]$ *with* $j < j$. *Then*

$$|E_1 \cap E_2| \geqslant t + 1.$$

Finally, we use the following convention.

Definition 2.8. For $\mathscr{A} \in LI(n, k, t)$, we set

$$s_{\min}(G(\mathscr{A})) = \min_{g(A) \in G(\mathscr{A})} s^+(g(\mathscr{A})).$$

3. The Main Two Auxiliary Results

Lemma 6. *Let* $n > 2k - t$, $\mathscr{A} \in LI(n, k, t)$, *with* $|A| = M(n, k, t)$, *and let*

$$n > \frac{(k - t + 1)(t + 2r + 1)}{r + 1} = (k - t + 1)\left(2 + \frac{t - 1}{r + 1}\right) \tag{3.1}$$

for some $r \in \mathbb{N} \cup \{0\}$. *Then*

$$s_{\min}(G(\mathscr{A})) \leqslant t + 2r. \tag{3.2}$$

Proof. We can assume that $n \geqslant 2k - t + 2$, because in the case $n = 2k - t + 1$ we have, from (3.1), $r \geqslant k - t + 1$, and hence (3.2) trivially holds.

By Lemma 2 we have, for some $g(\mathscr{A}) \in G_*(\mathscr{A})$,

$$s^+(g(\mathscr{A})) = s_{\min}(G(\mathscr{A})).$$

Now assume that, in the opposite to (3.2),

$$s^+(g(\mathscr{A})) = t + 2r + \delta \qquad \text{for some } \delta > 0. \tag{3.3}$$

We shall show that under the assumptions (3.1) and (3.3) there exists an $A' \in I(n, k, t)$ with $|\mathscr{A}'| > |\mathscr{A}| = M(n, k, t)$, which is a contradiction.

For this, we start with the partition $g(\mathscr{A}) = g_0(\mathscr{A}) \cup g_1(\mathscr{A})$, where $g_0(\mathscr{A}) = \{B \in g(\mathscr{A}) : s^+(B) = t + 2r + \delta\}$ and $g_1(\mathscr{A}) = g(\mathscr{A}) \setminus g_0(\mathscr{A})$.

Obviously, for every $B_1 \in g_0(\mathscr{A})$ and $B_2 \in g_1(\mathscr{A})$ we have

$$|(B_1 \setminus \{t + 2r + \delta\}) \cap B_2| \geqslant t. \tag{3.4}$$

The elements in $g_0(\mathscr{A})$ have an important property, which follows immediately from Lemma 5:

(P) For any $E_1, E_2 \in g_0(\mathscr{A})$ with $|E_1 \cap E_2| = t$, necessarily $|E_1| + |E_2| = 2t + 2r + \delta$.

Now, we partition $g_0(\mathscr{A})$ according to the cardinalities of its members:

$$g_0(\mathscr{A}) = \dot{\bigcup_{t<i<t+2r+\delta}} \mathscr{R}_i, \qquad \mathscr{R}_i = g_0(\mathscr{A}) \cap \binom{[n]}{i}.$$

(Here we have used that in case $\mathscr{R}_t \neq \varnothing$, by Lemma 1 and left-compressedness, necessarily $\mathscr{R}_t = \{[1, t]\} = g(\mathscr{A})$ and (3.2) holds. Furthermore, if $\mathscr{R}_{t+2r+\delta} \neq \varnothing$, then $\mathscr{R}_{t+2r+\delta} = \{[1, t + 2r + \delta]\} = g(\mathscr{A})$, and by optimality of $\mathscr{A}$ necessarily $r = \delta = 0$ and again (3.2) holds.)

Of course, some of the $\mathscr{R}_i$'s can be empty.

Next, we omit the element $t+2r+\delta$; that is, we consider

$$\mathcal{R}_i' = \{E \subset [1, t+2r+\delta-1] : E \cup \{t+2r+\delta\} \in \mathcal{R}_i\}.$$

So $|\mathcal{R}_i| = |\mathcal{R}_i'|$ and, for $E' \in \mathcal{R}_i'$, $|E'| = i-1$ ($i \in \mathbb{N}$). From property (P) we know that, for any $E_1' \in \mathcal{R}_i'$, $E_2' \in \mathcal{R}_j'$ with $i+j \neq 2t+2r+\delta$,

$$|E_1' \cap E_2'| \geqslant t. \tag{3.5}$$

We shall prove that (under conditions (3.1) and (3.3)) all $\mathcal{R}_i$'s are empty.

First we notice that the equation $|E_1| + |E_2| = 2t+2r+\delta$, for $E_1, E_2 \in g_0(\mathcal{A})$, implies that

$$|E_j| > k - (n-t-2r-\delta) \qquad \text{for } j = 1, 2,$$

because otherwise, for instance, $|E_1| \leqslant k-(n-t-2r-\delta)$ implies that $|E_2| = 2t+2r+\delta-|E_1| \geqslant n-k+t$, which, together with $|E_2| \leqslant k$, contradicts $n > 2k-t$.

Hence, if for all $i, i > k-(n-t-2r-\delta)$, we have $\mathcal{R}_i = \varnothing$, then $\mathcal{H} = (g(\mathcal{A}) \setminus g_0(\mathcal{A})) \cup (\cup_{t<i<t+2r+\delta} \mathcal{R}_i') \in I(n, t)$, the set of all (unrestricted) t-intersecting systems in $2^{[n]}$, $|\mathcal{U}(\mathcal{H}) \cap \binom{[n]}{k}| \geqslant |\mathcal{A}|$, and $s^+(\mathcal{H}) < s^+(g(\mathcal{A}))$, which is a contradiction.

Suppose, then, that for some $i, k-(n-t-2r-\delta) < i < t+2r+\delta$, $\mathcal{R}_i \neq \varnothing$ or, equivalently, $\mathcal{R}_i' \neq \varnothing$. We distinguish two cases: (a) $i \neq (2t+2r+\delta)/2$ and (b) $i = (2t+2r+\delta)/2$.

Case (*a*). We consider the sets

$$f_1 = g_1(\mathcal{A}) \cup (g_0(\mathcal{A}) \setminus (\mathcal{R}_i \cup \mathcal{R}_{2t+2r+\delta-i})) \cup \mathcal{R}_i',$$
$$f_2 = g_1(\mathcal{A}) \cup (g_0(\mathcal{A}) \setminus (\mathcal{R}_i \cup \mathcal{R}_{2t+2r+\delta-i})) \cup \mathcal{R}_{2t+2r+\delta-i}'.$$

We know already (see property (P) and (3.5)) that

$$f_1, f_2 \in I(n, t)$$

and hence

$$\mathcal{B}_i = \mathcal{U}(f_i) \cap \binom{[n]}{k} \in I(n, k, t) \qquad \text{for } i = 1, 2.$$

The desired contradiction will take the form

$$\max_{i=1,2} |\mathcal{B}_i| > |\mathcal{A}|. \tag{3.6}$$

We consider the set $\mathcal{A} \setminus \mathcal{B}_1$.

From the construction of f_1 and $\mathcal{R}_i'$, it follows that $\mathcal{A} \setminus \mathcal{B}_1$ consists of those elements of $\binom{[n]}{k}$ which are extensions only of the elements from $\mathcal{R}_{2t+2r+\delta-i}$. We determine their number:

$$|\mathcal{A} \setminus \mathcal{B}_1| = \left| (\mathcal{U}(\mathcal{R}_{2t+2r+\delta-i}) \setminus \mathcal{U}(g(\mathcal{A}) \setminus \mathcal{R}_{2t+2r+\delta-i}))) \cap \binom{[n]}{k} \right|$$

and, by Lemma 4,

$$|\mathcal{A} \setminus \mathcal{B}_1| = |\mathcal{R}_{2t+2r+\delta-i}| \cdot \binom{n-t-2r-\delta}{k-2t-2r-\delta+i}. \tag{3.7}$$

Symmetrically, we consider the set $\mathcal{B}_1 \setminus \mathcal{A}$.

Let B_1 be any element of $\mathcal{R}_i'$, so $|B_1| = i-1$. Clearly, $B_1 \notin g(\mathcal{A})$, because

$$B_1 \cup \{t+2r+\delta\} \in g(\mathcal{A}) \qquad \text{and} \qquad g(\mathcal{A}) \in G_*(\mathcal{A}).$$

Therefore, for every $A \in \binom{[n]}{k}$ in the form $A = B_1 \cup B_2$, where $|B_2| = k-i+1$ and

$B_2 \subset [t+2r+\delta+1, n]$, we have $A \in \mathcal{B}_1$ and $A \notin \mathcal{A}$, because $s^+(g(\mathcal{A})) = t+2r+\delta$. We also notice that, for $B_1, B_1' \in \mathcal{R}_i'$, $B_1 \neq B_1'$, we have

$$B_1 \cup B_2 \neq B_1' \cup B_2' \qquad \text{for all } B_2, B_2' \subset [t+2r+\delta+1, n].$$

Therefore

$$|\mathcal{B}_1 \setminus \mathcal{A}| \geq |\mathcal{R}_i'| \binom{n-t-2r-\delta}{k-i+1} = |\mathcal{R}_i| \binom{n-t-2r-\delta}{k-i+1}. \tag{3.8}$$

Analogously, we have

$$|\mathcal{A} \setminus \mathcal{B}_2| = |\mathcal{R}_i| \binom{n-t-2r-\delta}{k-i}, \tag{3.9}$$

$$|\mathcal{B}_2 \setminus \mathcal{A}| \geq |\mathcal{R}_{2t+2r+\delta-i}| \binom{n-t-2r-\delta}{k-2t-2r-\delta+i+1}. \tag{3.10}$$

Actually, it is easy to show that there are equalities in (3.8) and in (3.10). However, that is not needed here.

Now (3.7)–(3.10) enable us to state the *negation* of (3.6) in the form

$$\left.\begin{aligned} &|\mathcal{R}_i| \binom{n-t-2r-\delta}{k-i+1} \leq |\mathcal{R}_{2t+2r+\delta-i}| \binom{n-t-2r-\delta}{k-2t-2r-\delta+i} \\ \text{and} \quad &|\mathcal{R}_{2t+2r+\delta-i}| \binom{n-t-2r-\delta}{k-2t-2r-\delta+i+1} \leq |\mathcal{R}_i| \binom{n-t-2r-\delta}{k-i} \end{aligned}\right\}. \tag{3.11}$$

Since, by assumption, $\mathcal{R}_i \neq \emptyset$, we can also assume that $\mathcal{R}_{2t+2r+\delta-i} \neq \emptyset$, because otherwise the first inequality in (3.11) is false.

Furthermore, (3.11) implies that

$$(n+t-k-i)(n-t-2r-\delta-k+i) \leq (k-i+1)(k-2t-2r-\delta+i+1).$$

However, this is false, because $n \geq 2k-t+2$ and, consequently, $n+t-k-i > k-i+1$ as well as $n-t-2r-\delta-k+i > k-2t-2r-\delta+i+1$. Hence, (3.6) holds, in contradiction to the optimality of $\mathcal{A}$. Therefore, necessarily, $\mathcal{R}_i = \emptyset$ for all $i \neq (2t+2r+\delta)/2$. (In this case (a) we did not use the condition (3.1), and not even the condition $\delta > 0$).

Case (*b*). $i = (2t+2r+\delta)/2$. Here, necessarily, $2 \mid \delta$. We consider the set $\mathcal{R}_{t+r+\delta/2}'$ and recall that for $B \in \mathcal{R}_{t+r+\delta/2}'$ $|B| = t+r+\delta/2-1$ and $B \subset [1, t+2r+\delta-1]$.

By the pigeon-hole principle, there exists an $i \in [1, t+2r+\delta-1]$ and a $\mathcal{T} \subset \mathcal{R}_{t+r+\delta/2}'$ such that $i \notin B$ for all $B \in \mathcal{T}$ and

$$|\mathcal{T}| \geq \frac{r+\delta/2}{t+2r+\delta-1} |\mathcal{R}_{t+r+\delta/2}'| = \frac{r+\delta/2}{t+2r+\delta-1} |\mathcal{R}_{t+r+\delta/2}|. \tag{3.12}$$

By Lemma 5, we have $|B_1 \cap B_2| \geq t$ for all $B_1, B_2 \in \mathcal{T}$, and since by case (a) $\mathcal{R}_i = \phi$ for $i \neq t+r+\delta/2$, we obtain

$$f' = (g(\mathcal{A}) \setminus \mathcal{R}_{t+r+\delta/2}) \cup \mathcal{T} \in I(n, t).$$

We show now that under condition (3.1) we have

$$\left| \mathcal{U}(f') \cap \binom{[n]}{k} \right| > |\mathcal{A}|. \tag{3.13}$$

Indeed, let us write $\mathcal{A} = \mathcal{U}(g(\mathcal{A})) \cap \binom{[n]}{k} = \mathcal{D}_1 \dot\cup \mathcal{D}_2$, where $\mathcal{D}_1 = \mathcal{U}(g(\mathcal{A}) \setminus \mathcal{R}_{t+r+\delta/2}) \cap \binom{[n]}{k}$, $\mathcal{D}_2 = (\mathcal{U}(\mathcal{R}_{t+r+\delta/2}) \setminus \mathcal{U}(g(\mathcal{A}) \setminus \mathcal{R}_{t+r+\delta/2})) \cap \binom{[n]}{k}$ and $\mathcal{U}(f') \cap \binom{[n]}{k} = \mathcal{D}_1 \dot\cup \mathcal{D}_3$, where $\mathcal{D}_3 = (\mathcal{U}(\mathcal{T}) \setminus \mathcal{U}(g(\mathcal{A}) \setminus \mathcal{R}_{t+r+\delta/2})) \cap \binom{[n]}{k}$. In this terminology, the following is equivalent to (3.13):

$$|\mathcal{D}_3| > |\mathcal{D}_2|\,. \tag{3.14}$$

We know (see Lemma 4) that

$$|\mathcal{D}_2| = |\mathcal{R}_{t+r+\delta/2}| \binom{n-t-2r-\delta}{k-t-r-\delta/2} \tag{3.15}$$

and estimate now $|\mathcal{D}_3|$ from below.

Consider any $B \in \mathcal{T}$, $|B| = t + r + \delta/2 - 1$. Clearly, $B \notin g(\mathcal{A})$, because

$$B \cup \{t + 2r + \delta\} \in g(\mathcal{A}) \qquad \text{and} \qquad g(\mathcal{A}) \in G_*(\mathcal{A}).$$

Hence, for every $A \in \binom{[n]}{k}$ in the form $A = B \cup C_1$, where $|C_1| = k - |B|$ and $C_1 \subset [t + 2r + \delta, n]$, we have $A \in \mathcal{D}_3$.

We also notice that, for all $B_1, B_2 \in \mathcal{T}$, $B_1 \neq B_2$, one has $B_1 \cup C_1 \neq B_2 \cup C_2$ for all $C_1, C_2 \subset [t + 2r + \delta, n]$.

Therefore

$$|\mathcal{D}_3| \geqslant |\mathcal{T}| \binom{n-t-2r-\delta+1}{k-t-r-\delta/2+1}. \tag{3.16}$$

In the light of (3.12) and (3.14)–(3.16), it is sufficient for (3.13) that

$$\frac{r+\delta/2}{t+2r+\delta-1} \begin{pmatrix} n-t-2r-\delta+1 \\ k-t-r-\dfrac{\delta}{2}+1 \end{pmatrix} > \begin{pmatrix} n-t-2r-\delta \\ k-t-r-\dfrac{\delta}{2} \end{pmatrix} \tag{3.17}$$

or, equivalently, that

$$\left(r + \frac{\delta}{2}\right)(n - t - 2r - \delta + 1) > \left(k - t - r - \frac{\delta}{2} + 1\right)(t + 2r + \delta - 1), \tag{3.18}$$

which in turn is equivalent to

$$n > \frac{(k-t+1)(t+2r+\delta-1)}{r+\delta/2}.$$

This is true by (3.1), because δ is even and

$$\frac{t+2r+1}{r+1} \geqslant \frac{t+2r+\delta-1}{r+\delta/2} \qquad \text{for } \delta \geqslant 2.$$

Hence $\delta = 0$ and the lemma is proved. □

Corollary (for the $4m$ − Conjecture). *Let* $\mathcal{A} \in LI(4m, 2m, 2)$ *and* $|\mathcal{A}| = M(4m, 2m, 2)$. *Then*

$$s_{\min}(G(\mathcal{A})) \leqslant 2m.$$

Proof. Just choose $r = m - 1$ and notice that (3.1) in Lemma 6 holds:

$$4m > \frac{(2m-1)(2m+1)}{m}.$$ □

Finally, we present the second auxiliary result.

LEMMA 7. *For any (not necessarily maximal) $\mathscr{A} \in I(n, k, t)$, consider any generating set $g(\mathscr{A}) \in G(\mathscr{A})$. Furthermore, for the complemented system $\mathscr{A} \in I(n, n-k, n-2k+t)$, let $f(\bar{\mathscr{A}})$* be any generating set from $G(\bar{\mathscr{A}})$. Then, for all $A \in g(\mathscr{A})$ *and* $B \in f(\bar{\mathscr{A}})$, we have

$$|A \cup B| \geqslant n - k + t. \tag{3.19}$$

PROOF. Assume that (3.19) does not hold:

$$|A \cup B| \leqslant n - k + t - 1.$$

Choose any $F \supset A \cup B$ with $|F| = n - k + t - 1$. We know that

$$\mathscr{U}(A) \cap \binom{[n]}{k} \subset \mathscr{A} \qquad \text{and} \qquad \mathscr{U}(B) \cap \binom{[n]}{n-k} \subset \bar{\mathscr{A}}.$$

Since $n > 2k - t$, $t > 0$, and $n - k + t - 1 \geqslant \max(k, n-k)$, we can choose $A^* \in \mathscr{U}(A) \cap \binom{[n]}{k}$ and $B^* \in \mathscr{U}(B) \cap \binom{[n]}{n-k}$ in such a way that $A^* \subset F$ and $B^* \subset F$.

Consider $\overline{B^*} \in \mathscr{A}$ and observe that $|A^* \cap \overline{B^*}| \leqslant (n-k+t-1) - (n-k) = t-1$, contradicting $\mathscr{A} \in I(n, k, t)$. □

4. PROOF OF THE $4m$-CONJECTURE

We treat this famous case here separately, even though it is covered by the proof of the Theorem.

Let $\mathscr{A} \in LI(4m, 2m, 2)$, let $|\mathscr{A}| = M(4m, 2m, 2)$ and let $g(\mathscr{A}) \in G(\mathscr{A})$ such that $s^+(g(\mathscr{A})) = s_{\min}(G(\mathscr{A}))$.

From the corollary we know that $s^+(g(\mathscr{A})) \leqslant 2m$, that is, for $B \in g(\mathscr{A})$ necessarily $B \subset [1, 2m]$. We consider the complemented $\mathscr{A}$:

$$\bar{\mathscr{A}} = \{A \subset [1, 4m] : [1, 4m] \setminus A \in \mathscr{A}\}.$$

Clearly, $\bar{\mathscr{A}} \in I(4m, 2m, 2)$, $|\bar{\mathscr{A}}| = M(4m, 2m, 2)$, $\mathscr{A} \cap \bar{\mathscr{A}} = \varnothing$ and $\bar{\mathscr{A}}$ is right-compressed.

From the left–right symmetry and the corollary we conclude that there exists a generating set $f(\bar{\mathscr{A}})$ such that for every $B \in f(\bar{\mathscr{A}})$ necessarily $B \subset [2m+1, 4m]$.

Now, if for all $B_1 \in g(\mathscr{A})$ we have $|B_1| \geqslant m + 1$ then, necessarily, $\mathscr{A} = \mathscr{F}_{m-1}$ and $|\bar{\mathscr{A}}| = |\mathscr{A}| = |\mathscr{F}_{m-1}|$.

The same conclusion holds if, for all $B_2 \in f(\bar{\mathscr{A}})$, we have $|B_2| \geqslant m + 1$. However, if neither of these two cases occurs then, for some $B_1 \in g(\mathscr{A})$ and $B_2 \in f(\bar{\mathscr{A}})$, we have $|B_1|, |B_2| \leqslant m$.

Finally, for every $A \in \binom{[4m]}{2m}$ of the form $A \supset B_1 \cup B_2$, necessarily $A \in \mathscr{A} \cap \bar{\mathscr{A}}$, a contradiction to $\mathscr{A} \cap \bar{\mathscr{A}} = \varnothing$.

REMARK 5. Instead of the last step of the proof, one can argue that $|B_1 \cup B_2| \leqslant 2m < 2m + 2 = n - k + t$ is in contradiction with Lemma 7.

5. PROOF OF THE THEOREM

CASE (i):

$$(k - t + 1)\left(2 + \frac{t-1}{r+1}\right) < n < (k - t + 1)\left(2 + \frac{t-1}{r}\right). \tag{5.1}$$

Choose $\mathscr{A} \in LI(n, k, t)$ with $|\mathscr{A}| = M(n, k, t)$ and let $g(\mathscr{A}) \in G(\mathscr{A})$ with $s^+(g(\mathscr{A})) = s_{\min}(G(\mathscr{A}))$.

We know from Lemma 6 that $s^{+}(g(\mathscr{A})) \leqslant t+2r$ and we also know that $\bar{\mathscr{A}}$, the complemented $\mathscr{A}$, is right compressed, an element of $I(n, n-k, n-2k+t)$ and satisfies $|\bar{\mathscr{A}}| = M(n, n-k, n-2k+t)$. An easy calculation leads from (5.1) to

$$(k-t+1)\Big(2+\frac{n-2k+t-1}{k-t-r+1}\Big) < n < (k-t+1)\Big(2+\frac{n-2k+t-1}{k-t-r}\Big) \qquad \text{for } k > t+r \quad (5.2)$$

and to

$$(k-t+1)(n-2k+t+1) < n \qquad \text{for } k = t+r. \qquad (5.3)$$

Since $n > 2k-t$, (5.1) implies that always $k \geqslant t+r$.

Therefore, for $k' = n-k$ and $t' = n-2k+t$ we obtain the *dual relations*

$$(k'-t'+1)\Big(2+\frac{t'-1}{r'+1}\Big) < n < (k'-t'+1)\Big(2+\frac{t'-1}{r'}\Big) \qquad \text{for } r' = k-t-r$$

and

$$(k'-t'+1)(t'+1) < n \qquad \text{for } k = t+r.$$

Using the dual version (with respect to right compressed sets) of Lemma 6, we obtain an $f(\bar{\mathscr{A}}) \in \mathrm{G}(\bar{\mathscr{A}})$ with the property:

every $B \in f(\bar{\mathscr{A}})$ satisfies $B \subset [n-(t'+2r')+1, n]$
$= [t+2r+1, n]$, because $t' = n-2k+t$ and $r' = k-t-r$.

Now, if for all $B_1 \in g(\mathscr{A})$ we have $|B_1| \geqslant t+r$, then clearly $\mathscr{A} = \mathscr{F}_r$, $\bar{\mathscr{A}} \equiv \mathscr{F}_{r'}$ (for $k' = n-k$, $t' = n-2k+t$) and the optimal sets are unique up to permutations.

The same sets are obtained if, for all $B_2 \in f(\bar{\mathscr{A}})$, the inequality $|B_2| \geqslant t'+r'$ holds.

If none of the cases occurs, then for some $B_1^* \in g(\mathscr{A})$, $B_2^* \in f(\bar{\mathscr{A}})$, $|B_1^*| \leqslant t+r-1$ and $|B_2^*| \leqslant t'+r'-1 = n-k-r-1$ and, by Lemma 7, we will have a contradiction, because $|B_1^* \cup B_2^*| \leqslant n-k+t-2$.

Case (ii):

$$n = (k-t+1)\Big(2+\frac{t-1}{r+1}\Big). \qquad (5.4)$$

Let $\mathscr{A} \in LI(n, k, t)$ and let $|\mathscr{A}| = M(n, k, t)$. Since $n > (k-t+1)(2+\frac{t-1}{r+2})$, we conclude with Lemma 6 that

$$s_{\min}(G(\mathscr{A})) \leqslant t+2r+2.$$

Now choose $g(\mathscr{A}) \in G(\mathscr{A})$ with $s^{+}(g(\mathscr{A})) = s_{\min}(G(\mathscr{A}))$.

We again consider $\bar{\mathscr{A}}$, which is again right compressed and of maximal cardinality within $I(n, n-k, n-2k+t)$.

From (5.4), we derive

$$n = (k-t+1)\Big(2+\frac{t-1}{r+1}\Big) = (k-t+1)\Big(2+\frac{n-2k+t-1}{k-t-r}\Big) = (k'-t'+1)\Big(2+\frac{t'-1}{r'}\Big),$$

where $k' = n-k$, $t' = n-2k+t$ and $r' = k-t-r$.

Since $n = (k' - t' + 1)(2 + \frac{t'-1}{r'}) > (k' - t' + 1)(2 + \frac{t'-1}{r'+1})$, from the dual version (with respect to right compressed sets) of Lemma 6 one has an $f(\bar{\mathscr{A}}) \in G(\bar{\mathscr{A}})$ the elements B_1 of which satisfy $B_1 \subset [n - t' - 2r' + 1, n] = [t + 2r + 1, n]$.

Now, if for all $B_2 \in g(\mathscr{A})$, $|B_2| \geq t + r + 1$, then uniquely $\mathscr{A} = \mathscr{F}_{r+1}$ and $\bar{\mathscr{A}} \equiv \mathscr{F}_{r'-1}$ (for $k' = n - k$, $t' = n - 2k + t$, $r' = k - t - r$).

Also, if for all $B_1 \in f(\bar{\mathscr{A}})$, $|B_1| \geq t' + r'$, then uniquely $\bar{\mathscr{A}} \equiv \mathscr{F}_{r'}$ and $\mathscr{A} \equiv \mathscr{F}_r$. Otherwise, if for some $B_2 \in g(\mathscr{A})$ and $B_1 \in f(\bar{\mathscr{A}})$, $|B_2| \leq t + r$ and $|B_1| \leq t' + r' - 1$, then $|B_1 \cup B_2| \leq t + r + t' + r' - 1 = n - k + t - 1$.

This again contradicts Lemma 7.

This finishes the proof, if we allow only left compressed systems as competitors. We thank K. Engel for asking for an argument for not left compressed competitors. We follow an idea of [10] to prove the uniqueness, stated in the theorem.

We use the well-known exchange operation S_{ij}, $i < j$, defined for any family $\mathscr{A} \subset 2^{[n]}$ as follows: for $A \in \mathscr{A}$,

$$S_{ij}(A) = \begin{cases} \{i\} \cup (A \setminus \{j\}) & \text{if } i \notin A, j \in A, \{i\} \cup (A \setminus \{j\}) \notin \mathscr{A}, \\ A & \text{otherwise,} \end{cases}$$

$$S_{ij}(\mathscr{A}) = \{S_{ij}(A) : A \in \mathscr{A}\}.$$

Proposition. *Suppose that $\mathscr{A} \in I(n, k, t)$ and that $\mathscr{A}$ gets transformed by finitely many exchange operations to the set $\mathscr{F}_r$ (see (1.9)) for some $0 \leq r \leq (n-t)/2$. Then, necessarily, $\mathscr{A} \equiv \mathscr{F}_r$, provided that*

$$\begin{aligned} &n \geq 2k - t + 2 && \text{for } t \geq 2, \\ &n = 2k - t + 1 && \text{for } t \geq 2 \text{ and } k = t + r \text{ or } k = t + r + 1, \\ &n \geq 2k + 1 && \text{for } t = 1 \text{ and } r = 0 \text{ or } r = 1. \end{aligned} \tag{$*$}$$

Proof. Without loss of generality, we can assume that

$$S_{ij}(\mathscr{A}) = \mathscr{F}_r. \tag{5.5}$$

It is clear that, if $i, j \in [1, t + 2r]$ (or $i, j \notin [1, t + 2r]$), then $\mathscr{A} = \mathscr{F}_r$ and the Proposition is true.

Assume, then, that $i = t + 2r$ and $j = n$. Let

$$\mathscr{A}_1 = \{A \in \mathscr{A} : j \in A, i \notin A, ((A \setminus \{j\}) \cup \{i\}) \notin \mathscr{A}\},$$
$$\mathscr{A}_2 = \{A \in \mathscr{A} : j \notin A, i \in A, ((A \setminus \{i\}) \cup \{j\}) \notin \mathscr{A}\}.$$

Clearly, if $\mathscr{A}_1 = \varnothing$, then $\mathscr{A} = \mathscr{F}_r$, and if $\mathscr{A}_2 = \varnothing$, then $\mathscr{A}$ is obtained from $\mathscr{F}_r$ by exchanging the co-ordinates $i = t + 2r$ and $j = n$, so the proposition is true. Suppose, then, that $A_1 \neq \varnothing$, $\mathscr{A}_2 \neq \varnothing$, and let us show that $\mathscr{A} \notin I(n, k, t)$ (under conditions $(*)$).

We consider

$$\mathscr{H} = \left\{ H \in \binom{[n] \setminus \{i, j\}}{k-1} : |H \cap [1, t + 2r - 1]| = t + r - 1 \right\}.$$

We observe that, for any $B \in \mathscr{A}_1 \dot{\cup} \mathscr{A}_2$, $|B \cap [1, t + 2r - 1]| = t + r - 1$ holds.

This fact follows from (5.5). Moreover, from the same assumption (5.5) we have the following: for every $H \in \mathscr{H}$ either $H \cup \{j\} \in \mathscr{A}_1$ or $H \cup \{i\} \in \mathscr{A}_2$.

Now we form a graph $G(V, E)$ as follows:

$$V = \mathscr{H} \text{ and } e(H_1, H_2) \in E \qquad \text{iff } (H_1 \cap H_2) = t - 1.$$

It can be verified that the graph $G(V, E)$ is connected iff the conditions $(*)$ hold.

Hence, under conditions $(*)$, if $\mathscr{A}_1 \neq \emptyset$ and $\mathscr{A}_2 \neq \emptyset$, then there exist $B_1 \in \mathscr{A}_1$ and $B_2 \in \mathscr{A}_2$ with $|B_1 \cap B_2| = t - 1$, which contradicts $\mathscr{A} \in I(n, k, t)$. □

PROOF OF THE UNIQUENESS. Let $n > 2k - t$, $\mathscr{A} \in I(n, k, t)$ and $|\mathscr{A}| = M(n, k, t)$, and—after finitely many exchange operations S_{ij}, $i < j$—let $\mathscr{A}$ be transformed to the left-compressed set $\mathscr{A}'$, $\mathscr{A}' \in LI(n, k, t)$, $|\mathscr{A}'| = |\mathscr{A}| = M(n, k, t)$.

We already know that $\mathscr{A}' = \mathscr{F}_r$ for $r \in \mathbb{N} \cup \{0\}$, where r is defined by the conditions in the theorem. It can be easily verified that these r's satisfy the conditions $(*)$ stated in the proposition, and hence that $\mathscr{A} \equiv \mathscr{F}_r$. □

REFERENCES

1. R. Ahlswede and G. Katona, Contributions to the geometry of Hamming spaces, *Discr. Math.*, **17,** (1977), 1–22.
2. R. Ahlswede and L. H. Khachatrian, Maximal sets of numbers not containing $k+1$ pairwise coprime integers, *Acta Arithmetica LXX,* **1,** (1995), 77–100.
3. R. Ahlswede and L. H. Khachatrian, Sets of integers and quasi-integers with pairwise common divisor, *Acta Arithmetica LXXIV,* **2,** (1996), 141–153.
4. R. Ahlswede and L. H. Khachatrian, Sets of integers with pairwise common divisor and a factor from a specified set of primes, *Acta Arithmetica LXXV,* **3,** (1996), 259–276.
5. A. R. Calderbank and P. Frankl, Improved upper bounds concerning the Erdös–Ko–Rado Theorem, *Combin. Prob. Comput.,* **1,** (1992), 115–122.
6. M. Deza and P. Frankl, "Erdös–Ko–Rado Theorem—22 years later, *SIAM J. Algebra Discr. Math.,* **4** (4) (1983), 419–431.
7. P. Erdös, My joint work with Richard Rado, in: *Surveys in Combinatorics, Lond. Math. Soc. Lect. Note Ser.,* **123** (1987), 53–80.
8. P. Erdös, Some of my favourite unsolved problems, in: *A Tribute to Paul Erdös,* A. Baker, B. Bollobás, and A. Hajnal (eds), Cambridge University Press, Cambridge, 1990.
9. P. Erdös, Chao Ko and R. Rado, Intersection theorems for systems of finite sets, *Q. J. Math. Oxford,* **12** (1961), 313–320.
10. P. Frankl, The Erdös–Ko–Rado Theorem is true for $n = ckt$, *Coll. Soc. Math. J. Bolyai,* **11** (1978), 365–375.
11. P. Frankl, The shifting technique in extremal set theory, in: *Surveys in Combinatorics, Lond. Math. Soc. Lect. Note Ser.,* **123** (1987), 81–110.
12. P. Frankl and Z. Füredi, Beyond the Erdös–Ko–Rado Theorem, *J. Combin. Theory, Ser. A,* **56** (1991), 182–194.
13. R. M. Wilson, The exact bound on the Erdös–Ko–Rado Theorem, *Combinatorica,* **4** (1984), 247–257.

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