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NOTE

COUNTEREXAMPLE TO THE FRANKL–PACH CONJECTURE FOR UNIFORM, DENSE FAMILIES

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$\mathbb{N}$ denotes the set of positive integers and for ℓ, $n \in \mathbb{N}$, $\ell \le n$ we set

$$2^{[n]} = \{F : F \subset [1,n]\}, \quad \binom{[n]}{\ell} = \left\{F \in 2^{[n]} : |F| = \ell\right\}.$$

A family $\mathcal{F} \subset 2^{[n]}$ is called ℓ-dense, $\ell \in \mathbb{N}$, if there exists an ℓ-element subset $D \in \binom{[n]}{\ell}$ with $\mathcal{F}(D) = \{F \cap D : F \in \mathcal{F}\}$ satisfying

$$|\mathcal{F}(D)| = 2^{\ell}. \tag{1}$$

A well-known result of Sauer [1], Shelah–Perles [2], and Vapnik–Ĉervonenkis [3] says that any $\mathcal{F} \subset 2^{[n]}$ is ℓ-dense, if

$$|\mathcal{F}| > \sum_{i<\ell} \binom{n}{i}. \tag{2}$$

Frankl–Pach [4] proved that any ℓ-uniform $\mathcal{F}$, that is $\mathcal{F} \subset \binom{[n]}{\ell}$, is ℓ-dense, if

$$|\mathcal{F}| > \binom{n}{\ell - 1}, \tag{3}$$

and they *conjectured* that for every ℓ-uniform, but not ℓ-dense, $\mathcal{F}$ with $n > 2\ell$ necessarily

$$|\mathcal{F}| \le \binom{n-1}{\ell - 1}. \tag{4}$$

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It was pointed out in [4], [5], and also by Erdős [6] that the truth of this conjecture would mean a sharpening of the Erdős–Ko–Rado Theorem [7]. While (4) holds for $\ell=2$, it is unfortunately false for $\ell\geq 3$ and $n\geq 2\ell$.

Example. Let $\mathcal{F}=\mathcal{F}_1\dot{\cup}\mathcal{F}_2\dot{\cup}\mathcal{F}_3\dot{\cup}\mathcal{F}_4\subset\binom{[n]}{\ell}$, where

$$\mathcal{F}_1=\left\{F\in\binom{[n]}{\ell}:1\in F,2\notin F\right\},\quad \mathcal{F}_2=\left\{F\in\binom{[n]}{\ell}:1,2,3\in F\right\},$$
$$\mathcal{F}_3=\left\{F\in\binom{[n]}{\ell}:1,2,4\in F,3\notin F\right\},\quad \mathcal{F}_4=\left\{F\in\binom{[n]}{\ell}:1\notin F,2,3,\in F\right\}.$$

It remains to be seen that for no $D\in\binom{[n]}{\ell}$ (1) holds.

Obviously, a candidate $D=\{d_1,\dots,d_\ell\}$ must satisfy $D\in\mathcal{F}$.

It is convenient to use also the sequence representation $(f_1,\dots,f_\ell)$ for $D\cap F$, where

$$f_t=\begin{cases}1 & \text{if } d_t\in D\cap F\\ 0 & \text{if } d_t\notin D\cap F.\end{cases}$$

Now let $\mathcal{F}_1=\mathcal{F}_1^1\dot{\cup}\mathcal{F}_1^2$, where

$$\mathcal{F}_1^1=\{F\in\mathcal{F}_1:3\in F\}\quad\text{and}\quad \mathcal{F}_1^2=\mathcal{F}_1\setminus\mathcal{F}_1^1.$$

Notice that $D\notin\mathcal{F}_1^1$, because $(0,0,\dots,0)\notin\mathcal{F}(D)$.

Also, $D\notin\mathcal{F}_1^2$, because $(0,1,1,\dots,1)\notin\mathcal{F}(D)$. Hence $D\notin\mathcal{F}_1$.

Furthermore, $D\notin\mathcal{F}_2\cup\mathcal{F}_3$, because $(0,0,\dots,0)\notin\mathcal{F}(D)$.

Finally, let $\mathcal{F}_4=\mathcal{F}_4^1\dot{\cup}\mathcal{F}_4^2$, where $\mathcal{F}_4^1=\{F\in\mathcal{F}_4:4\in F\}$ and $\mathcal{F}_4^2=\mathcal{F}_4\setminus\mathcal{F}_4^1$.

Now we have $D\notin\mathcal{F}_4^1$, because $(1,0,0,\dots,0)\notin\mathcal{F}(D)$, and $D\notin\mathcal{F}_4^2$, because $(1,0,1,1,\dots,1)\notin\mathcal{F}(D)$. Thus $\mathcal{F}$ is not ℓ-dense, however,

$$|\mathcal{F}|=|\mathcal{F}_1|+|\mathcal{F}_2|+|\mathcal{F}_3|+|\mathcal{F}_4|=$$
$$=\binom{n-2}{\ell-1}+\binom{n-3}{\ell-3}+\binom{n-4}{\ell-3}+\binom{n-3}{\ell-2}=\binom{n-1}{\ell-1}+\binom{n-4}{\ell-3}>\binom{n-1}{\ell-1}.$$

Remark. Frankl–Watanabe [5] even conjectured that for every k-uniform, but not ℓ-dense, $\mathcal{F}$ for $k>\ell>2$ necessarily

$$|\mathcal{F}|\leq\binom{n-k+\ell-1}{\ell-1}\quad\text{for}\quad n>n_0(k). \tag{5}$$

Of course, our example can be used to disprove this for every k, ℓ, n; $n>k>\ell>2$.

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