Common Randomness in Information Theory and Cryptography—Part II: CR Capacity

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Abstract—The common randomness (CR) capacity of a two-terminal model is defined as the maximum rate of common randomness that the terminals can generate using resources specified by the given model. We determine CR capacity for several models, including those whose statistics depend on unknown parameters. The CR capacity is shown to be achievable robustly, by common randomness of nearly uniform distribution no matter what the unknown parameters are. Our CR capacity results are relevant for the problem of identification capacity and also yield a new result on the regular (transmission) capacity of arbitrarily varying channels with feedback.

Index Terms—Arbitrarily varying channel, common randomness, correlated sources, feedback, identification capacity, randomization.

I. INTRODUCTION

Suppose two terminals, called Terminal $X$ and Terminal $Y$, have resources such as access to side information and communication links that allow them to observe and (perhaps cooperatively) generate certain random variables (RV's). The permissible rules for this are specified by the particular model, but it is always assumed that the terminals have unrestricted computational power, thus the RV's that can be generated and observed at a terminal at a given time include, as a minimum, all functions of the RV's previously observed there. Common randomness (CR) of $X$ and $Y$ means, intuitively, an RV generated by them and observable to both, perhaps with a small probability of error.

An RV generated by a terminal is not necessarily observable there, e.g., when Terminal $X$ inputs an RV $X$ into a noisy channel to Terminal $Y$, he thereby generates an output $Y$ observable only at $Y$. If Terminal $X$ suitably encodes the RV $X$ he wants to transmit, enabling $Y$ to decode, then $Y$ will represent CR. If noiseless feedback from $Y$ to $X$ is available then the output $Y$ will always represent CR.

In Part I [7] we were interested in CR under an additional secrecy constraint, with the motivation that the generated CR will be used as an encryption key. In this paper we do not require secrecy, and just study the maximum amount of CR afforded by a given model, the amount measured by entropy. The most convenient form of CR is uniform common randomness (UCR), i.e., CR represented by a uniformly (or nearly uniformly) distributed RV. For the type of models we will consider, the maximum attainable amount of CR and UCR will be asymptotically the same.

As a very simple example, suppose that there is a discrete memoryless channel (DMC) from Terminal $X$ to Terminal $Y$, Terminal $X$ can randomize (i.e., can generate RV's with arbitrary distributions), and can input into the DMC any random sequence $X^n$ he has generated (of given "large" length $n$). Terminal $Y$ can observe the output $Y^n$ but the terminals have no other resources, It is intuitively clear that in this case $X$ has to choose $X^n$ to be uniformly distributed on the $\approx \exp(nC)$ codewords of an optimum code; then $Y$ can decode, and the achieved $nC$ amount of CR is best possible. If noiseless feedback is available, it is better for $X$ to send independent repetitions of an RV $X$ that produces maximum output entropy $H(Y)$. As now $X$ can observe $Y^n$, in this way CR of amount $nH(Y)$ results, clearly the largest possible. Notice that here, too, the optimum could (almost) be attained by a nearly uniform RV, obtained by applying a compression code to $Y^n$.

As a combinatorial example, let $G$ be a bipartite graph with vertex sets $X$ and $Y$ (we continue the practice of Part I that the symbols of the terminals also denote sets assigned to them). Nature selects an edge $(x, y) \in G$ at random, Terminal $X$ observes $x$, Terminal $Y$ observes $y$. The terminals can communicate over a noiseless channel, but at most $b$ binary digits may be transmitted, in any number of rounds. No other resources are available (above the minimum described in the first paragraph), in particular, neither terminal can randomize. Then, clearly, $\log |G|$ is an upper bound to CR, which can be attained if and only if (iff) the communication complexity $C_{\infty}(H, P_x, P_y)$ does not exceed $b$ (with the notation of [17]); here $H$ denotes the hypergraph with vertex set $V = X$ and edge set $E$ consisting of the sets $\{x: (x, y) \in G\}$, $y \in Y$. It may be an interesting study in communication complexity to determine the maximum amount of CR when $b < C_{\infty}(H, P_x, P_y)$, i.e., the maximum entropy of a function on $G$ that may be computed at both terminals with communication of at most $b$ bits.

One obvious motivation of our interest in CR is that if the two terminals have access to the outcome of the same random experiment, this knowledge may be used to implement correlated random protocols, perhaps leading to much faster algorithms than deterministic ones or those using independent randomization only. In information theory, in particular for arbitrarily varying channels (AVC's), correlated random codes may greatly outperform deterministic (or randomized) codes; indeed, they may be necessary to attain positive capacity [8].
For example, for the additive Gaussian AVC with power constraints, (average error) capacity for deterministic codes equals random code capacity only if the sender’s power exceeds the jammer’s, otherwise, the deterministic code capacity is 0 [11].

An even more striking CR application appears in the theory of identification capacity [4]. It was shown in [5] that for any kind of channel, if sender and receiver can build up $nR$ bits of UCR, this can be used to construct ID codes for $\approx 2^{nR}$ messages, with small probability of misidentification and misrejection, provided that the channel capacity is positive. The asymptotic optimality of this construction was also established in [5], for DMC’s with no feedback and with complete feedback. Similar results for multuser channels were obtained in [6].

One feature of this paper is that we also study “robust common randomness.” This concept refers to models whose statistical properties are not completely specified but depend on certain parameters (“state”) out of control of the two terminals and at least partially unknown to them. Then the distribution of the RV representing CR will depend on the actual state, and the minimum of its entropy (for all possible states) may be called the amount of robust CR. Most desirable is to have robust UCR, i.e., such RV representing CR whose distribution is nearly uniform, no matter what the actual state is. Again, for the type of models we will consider, the maximum attainable amount of robust CR and robust UCR will be asymptotically the same.

We will study robust CR for AVC’s and the results allow us to determine identification capacity for various AVC models. Quite remarkably, we also obtain a new result on regular (transmission) capacity, namely, that the average error capacity of an AVC with complete feedback always equals the random code capacity of this AVC.

The problem of robust uniform randomness is of interest even if it is not required that distant terminals have access to it. The problem is that, if several probability distributions (PD’s) are given on a set $V$, how large can be the number of values of a function $f$ on $V$ whose distribution is nearly uniform under each of the given PD’s. Here we state a simple combinatorial lemma, similar in spirit to the hypergraph coloring lemmas of [3]. It says that if the given PD’s on $V$ are uniform distributions on the edges $E \in \mathcal{E}$ of a hypergraph $(V, \mathcal{E})$ then the maximum number of values of an $f$ with the required property is not much smaller than the smallest edge size $|E|$. We believe that this lemma will help the reader to develop intuition, as it helped us to arrive at the results on robust VCR in Section III.

Throughout, logarithms and exponents are to the base 2. Natural logarithms are denoted by $\ln$.

Lemma 1.1 Balanced Coloring: Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $|\mathcal{E}| = N$ edges, each of size $|E| \geq d$. Then for any $0 < \varepsilon < \frac{1}{2}$ and $k < \frac{d\varepsilon}{\ln(2N)}$ there exists an $\varepsilon$-balanced vertex coloring with $k$ colors, i.e., a function $f: V \rightarrow \{1, \ldots, k\}$, such that

$$\frac{|f^{-1}(i) \cap E|}{|E|} - \frac{1}{k} \leq \frac{\varepsilon}{k},$$

for every $1 \leq i \leq k$ and $E \in \mathcal{E}$. (1.1)

Proof: Let $\{Z(v), v \in V\}$ be a family of independent and identically distributed (i.i.d.) RV’s such that $\Pr\{Z(v) = i\} = \frac{1}{k}, i = 1, \ldots, k$, and let $Z_i(v) = 1$ if $Z(v) = i$, and 0 otherwise. Then for the random coloring $f(v) = Z(v)$ we have

$$|f^{-1}(i) \cap E| = \sum_{v \in E} Z_i(v)$$

and the standard large deviation bound for the binomial distribution gives, for every fixed $1 \leq i \leq 1$ and $E \in \mathcal{E}$, that

$$\Pr\left\{|f^{-1}(i) \cap E| < \frac{1 - \varepsilon}{k} |E|\right\} \leq \exp\left\{-|E|D\left(\frac{1 - \varepsilon}{k} \left| \frac{1}{k}\right|\right)\right\}.$$

$$\Pr\left\{|f^{-1}(i) \cap E| > \frac{1 + \varepsilon}{k} |E|\right\} \leq \exp\left\{-|E|D\left(\frac{1 + \varepsilon}{k} \left| \frac{1}{k}\right|\right)\right\}$$

where

$$D(\frac{1 + \varepsilon}{k} \left| \frac{1}{k}\right|) = \frac{\varepsilon^2}{k \ln 2}$$

is a convex function of $\varepsilon$ in the interval $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$, with minimum equal to 0 attained at $\varepsilon = 0$. It follows that the probability that (1.1) does not hold for the random coloring $f(v) = Z(v)$ is upper-bounded by $N \cdot 2 \exp(-\varepsilon^2/k \ln 2)$. Under the hypothesis of Lemma 1.1, this bound is less than 1, and the assertion follows.

II. PRELIMINARIES

A key concept studied in this paper for various models is what we call CR capacity. In this section, we first formally describe one model to be considered, and define achievable CR rates and CR capacity for that model. Then we indicate the changes needed for other models, including those where the underlying statistics depend on unknown parameters. (For all models considered, alternative definitions of capacities— or capacity functions—lead to the same values.) As one of our reasons for studying CR capacity is its relationship to ID capacity, at the end of this section we sketch how the latter concept can be defined for the type of models we are interested in, as a straightforward extension of the definition of ID capacity of a DMC without or with feedback, cf. [4] and [5]. A general definition of transmission capacity is also included.

In Section III we will establish some general results, including the achievability of CR capacity with UCR, i.e., with nearly uniformly distributed RV’s. For models where the statistics depend on unknown parameters, this UCR result holds in a robust sense. Then the result of Ahlswede and Dueck [5] referred to in the Introduction affords the conclusion that for the type of models considered in this paper, CR
capacity is always a lower bound to ID capacity, whenever
the transmission capacity is positive.

Our results on the CR capacity of particular models will be
stated and proved in Sections IV and V.

As in Part I, we use the terminology of [16], and refer to
it for notations not defined here.

One of the stimuli for this investigation came from [18],
where first basic observations are made and first results are
established for the binary-symmetric case of the model we
now describe.

A. Model i): Two-Source with One-Way Communication

Given a discrete memoryless multiple source (DMMS) with
two components, with alphabets $X$, $Y$ and generic variables
$X$, $Y$; the $n$-length source outputs are observable at Terminals
$X$, $Y$, respectively. Moreover, $X$ can send information to
$Y$ over a noiseless channel of capacity $R$, namely, he can
noiselessly transmit any function $f(X^n)$ of $X^n$ to $Y$, subject
to the rate constraint

$$\frac{1}{n} \log |f| \leq R. \quad (2.1)$$

Other resources are not available to the terminals. We will
say that a pair of RV’s $(K, L)$ is permissible if $K$ and $L$ are
functions of the data available at $X$ respectively $Y$, i.e.,

$$K = K(X^n) \quad L = L(Y^n, f(X^n)). \quad (2.2)$$

A permissible pair $(K, L)$ represents $\epsilon$-common randomness if

$$Pr\{K \neq L\} < \epsilon. \quad (2.3)$$

As $K$ and $L$ represent the same CR, intuition dictates that the
entropy rates $(1/n)H(K)$ and $(1/n)H(L)$ be arbitrarily close if $\epsilon$
is small, independently of $n$. In order to ensure this, via
Fano’s inequality, we impose the technical condition that $K$
and $L$ take values in the same set $\mathcal{K}$ whose cardinality satisfies

$$|\mathcal{K}| \leq \exp(cn) \quad (2.4)$$

for some $c$ not depending on $n$.

For Model i), we adopt the following definition that, with
suitable interpretation, will be appropriate also for other models,

**Definition 2.1**: A number $H$ is an achievable CR rate if for
some constant $c$ and every $\epsilon > 0$, $\delta > 0$, for all
sufficiently large $n$ there exists a permissible pair of RV’s
$(K, L)$ satisfying (2.3) and (2.4), such that

$$\frac{1}{n} H(K) > H - \delta. \quad (2.5)$$

The largest achievable CR rate is the CR capacity.

Remark: In Part I [7] we considered the related concept of
key capacity, where also a secrecy requirement was im-
posed on the CR. There, nearly uniform distribution was
also required in the sense that the entropy rate $(1/n)H(K)$
be close to $(1/n) \log |\mathcal{K}|$. As stated before, CR of nearly
uniform distribution or UCR is desirable also in the present
context. It turns out, however, that the CR capacity in the
sense of Definition 2.1 can always be attained with nearly
uniformly distributed RV’s, even in the stronger sense of
variation distance, i.e., with $K$ satisfying

$$\sum_{k \in \mathcal{K}} \left| Pr\{K = k\} - \frac{1}{|\mathcal{K}|} \right| < \epsilon.$$

Actually, it will be seen in Theorem 3.2 that a still stronger
kind of near uniformity can be attained, with the variation
distance above going to 0 exponentially as $n \rightarrow \infty$. Of course,
then also $H(K)$ will be exponentially close to $\log |\mathcal{K}|$.

For orientation notice that the CR capacity for Model i)
never exceeds $H(X)$. If $H(X|Y) < R$ then an $f$ satisfying
(2.1) can be chosen to let $Y$ recover $X^n$ from $f(X^n)$ and $Y^n$
with small probability of error (Slepian and Wolf [15]). Thus
in this case the CR capacity equals $H(X)$.

The question of how large CR rate can be attained in the extreme case $R = 0$ of Model i), when no communication is
permitted between $X$ and $Y$, was asked by the second author in
1970. It was answered by Gács and Körner [12] who showed
that it was equal to the largest entropy of a common function
of $X$ and $Y$, hence always 0 if $X$ and $Y$ had indecomposable
joint distribution. This paper was one of the starting points
of multiuser information theory, at least for the Hungarian
research group, It will turn out that the (by now) standard
"multiuser" techniques permit to determine the CR capacity
both for Model i) and its extensions considered in Section IV.

In the model described above, randomization was not permitted.
As in Part I, we will always regard randomization (at either
terminal) as generating an RV at the very start, and let
further actions depend on this RV, but already in a determin-
istic way. Thus Model i) with randomization at $X$ means that
an RV $M = M_Y$ of arbitrary distribution, but independent of
$X^n, Y^n$ may be generated at $X$; then the information sent to
$Y$ may be $f(X^n, M)$ (still subject to (2.1)), and Definition
2.1 applies with the understanding of permissible pairs as

$$K = K(X^n, M), \quad L = L(Y^n, f(X^n, M)).$$

Randomization at $Y$ might also be permitted, then $Y$ could generate an RV
$M_2$ (independent of $X^n, Y^n, M_1$), and let $L$ be a function of $M_2$, too. Notice that whereas randomization at $X$ may increase
the CR capacity of Model i), randomization at $Y$ cannot.

A variant of Model i) is when the given channel from $X$ to $Y$
is not noiseless but a DMC, say with the same wordlength $n$ as
the observed source output. The input is selected by Terminal
$X$ as a function of $X^n$ (or of $X^n$ and $M$ if randomization
is permitted) and Terminal $Y$ observes the output, say $Z^n$.
Then the change required in the definition of permissible pairs
$(K, L)$ is that now $L = L(Y^n, Z^n)$.

A somewhat different model is Model ii),

B. Model ii): DMC with Active Feedback

Given a DMC $\{W: X \rightarrow Y\}$, Terminal $X$ selects the
inputs, Terminal $Y$ observes the outputs, and $Y$ can send
back information to $X$ over a noiseless channel of capacity
$B$. We assume that $X$ is permitted to randomize but $Y$ is not.

Formally, the terminals’ permissible actions are as follows.
Initially, $X$ generates a randomization RV $M = M_1$, then
he inputs $X_i = f_1(M)$ into the DMC. The output $Y_i$ is
observed by $Y$ who then noiselessly sends $X$ a message
Then $X'$ sends $X_2 = f_2(M, g_1(Y_1))$ over the DMC, $Y$ observes the output $Y_2$ and sends back $g_2(Y_1, Y_2)$. Next $X'$ sends $X_3 = f_3(M, g_1(Y_1), g_2(Y_1, Y_2))$ and $Y$ sends back $g_3(Y_1, Y_2, Y_3)$, etc., through $n$ rounds.

The individual feedback messages may be arbitrary, but $g = (g_1, \ldots, g_n)$ is supposed to satisfy the global rate constraint

$$\frac{1}{n} \log \|g\| \leq R.$$  

For example, $g_1, \ldots, g_n$ may be binary words of variable length with prefix property, then (2.6) will mean that their total length is $\leq nR$.

In this model, the permissible pairs $(K, L)$ (which will represent $\varepsilon$-common randomness if they satisfy (2.3)) are of the form

$$K = K(M, g_1, \ldots, g_n) \quad L = L(Y^n).$$  

With this understanding, Definition 2.1 of the achievable CR rates and CR capacity applies to the present model. Notice that some of the messages $g_k$ may be empty, indeed it is permissible that Terminal $Y$ sends only one message to $X'$ after having received the whole $Y^n$ (of course, then the input $X^n$ must be a function of $M$ alone). We will show that the CR capacity for Model ii) is always attainable that way.

In another version of Model ii) also Terminal $Y$ is permitted to randomize, which formally means that he, too, generates a randomization RV $M_Y$ at the start (independent of $M_X$), and then $g_1, \ldots, g_n$ as well as $L$ may depend also on $M_Y$. Still another version would be when neither terminal is allowed to randomize, but that will not be considered here.

Just as Model i) could be modified replacing the noiseless channel from $Y$ to $X$ by a DMC, the same is possible also for Model ii). Actually, several such variants of Model ii) could be considered, one of them is when the $i$th input of the backward channel is a function $g(Y_1, \ldots, Y_i)$ of the first $i$ outputs of the forward channel, and Terminal $X$ observes the corresponding output $Z_i$ before selecting the input $X_{i+1}$ to the forward channel. Then the permissible pairs $(K, L)$ are defined by letting $K = K(M, Z^n)$, while $L = L(Y^n)$ as before.

Remark: Our terminology "active feedback" refers to the freedom of Terminal $Y$ to select the inputs of the backward channel. It differs from the terminology of [16], where "active feedback" means that $Y$ is allowed to randomize by passive feedback: we mean that the inputs of the backward channel are equal to the outputs of the forward channel. In particular, noiseless passive feedback (also called complete feedback) means that the outputs of the DMC $\{W: X \to Y\}$ are observable not only to Terminal $Y$ but also to Terminal $X$. The variant of Model ii) with complete feedback has been hinted at in the Introduction as a simple example for which the problem of CR capacity is trivial. The variant with noisy passive feedback deserves interest, but will not be considered in this paper.

A more complex version of the two-source model is Model iii).

C. Model iii): Two-Source with Two-Way Noiseless Communication

Given a DMMS as in Model i), suppose that after Terminal $Y$ received the message sent by $X$ over a noiseless channel of capacity $R_1$, he in turn can send $X$ a message over a noiseless channel of capacity $R_2$. This can be any function $g$ of $Y^n$ and the received $f(X^n)$ (or $f(X^n, M_X)$), subject to the rate constraint

$$\frac{1}{n} \log \|g\| \leq R_2.$$  

If $Y$ is permitted to randomize, $g$ may also depend on $Y$'s randomization RV $M_Y$, chosen at the start, independently of $(M_X, X^n, Y^n)$.

Now $(K, L)$ is a permissible pair of RV's if $K = K(X^n, g)$ or $K = K(X^n, M_Y, g)$ and $L = L(Y^n, f)$ or $L = L(Y^n, M_Y, f)$, depending on whether randomization is permitted or not. With this understanding of the permissible pairs, Definition 2.1 applies as before.

It is obvious how to extend the model to permit several rounds of communication between $X$ and $Y$, each transmission subject to a rate constraint. Alternatively, the transmissions may not be constrained individually only their total rate is. The CR capacity can always be defined as in Definition 2.1, letting the permissible $(K, L)$ pairs be functions of the data that become available at the corresponding terminals after having executed a protocol allowable by the particular model.

D. Models with Robust CR

The simplest model of this kind is that when both terminals can observe the output of an arbitrary varying source (AVS) but have no other resources whatsoever. An AVS with alphabet $X$ and state set $S$ (both finite) is determined by a family $\{P(\cdot|s), s \in S\}$ of P.D.'s on $X$. The distribution of the $n$-length source output $X^n$ depends on the state sequence $\mathbf{s} \in S^n$, and equals

$$P(\mathbf{s}) = P(\cdot|s_1) \times \cdots \times P(\cdot|s_n), \quad \mathbf{s} = (s_1, \ldots, s_n).$$  

In this model, any function $K = K(X^n)$ represents CR, thus the largest CR, for any fixed blocklength $n$, is represented by $K = X^n$. In the definition of achievable CR rates, the condition (2.5) is now required to hold independently of the underlying statistics, i.e., for all $\mathbf{s} \in S^n$. Thus the CR capacity for this model equals $H_{\text{min}} = \min_{s \in S} H(P(\cdot|s))$. It is nontrivial, but will be shown in Theorem 3.1 that this CR capacity can be attained with robust UCR, i.e., that $K = K(X^n)$ satisfying (2.6) for all possible $\mathbf{s} \in S^n$ can be given, such that

$$\frac{1}{n} \log |K| > H_{\text{min}} - \delta.$$  

We will consider various AVC models in this paper. An AVC with input alphabet $X$, output alphabet $Y$, and state set $S$, each finite, is determined by a family $W = \{W(\cdot|s), s \in S\}$ of channels $W(\cdot|s): X \to Y$. Terminal $X$ selects the inputs, Terminal $Y$ observes the outputs, and the state
sequence \( s \in S^n \) governing the \( n \)-length transmission may be arbitrary. Several different models are possible according to the availability of information to Terminal \( \mathcal{X} \) about the states and the previous outputs when selecting the input \( X_i \), and whether or not randomization is allowed.

We now formally describe two models, both with randomization permitted at \( \mathcal{X} \), thus Terminal \( \mathcal{X} \) first generates a randomization RV \( M_\mathcal{X} = M \). In the “no-feedback” model, Terminal \( \mathcal{X} \) selects the input sequence \( X^n \) as a function of \( M \). In the “complete-feedback” model, the inputs \( X_1, \ldots, X_n \) are selected successively as \( X_i = f_i(M, Y_1, \ldots, Y_{i-1}) \), where \( Y_1, \ldots, Y_{i-1} \) are the previous outputs ("seen" by Terminal \( \mathcal{X} \) through a noiseless feedback channel from \( \mathcal{Y} \) to \( \mathcal{X} \)). In both models, the joint distribution of \( M \) and the output sequence \( Y^n \), when the state sequence is \( s = (s_1, \ldots, s_n) \), is given by

\[
\Pr\{M = m, Y^n = y\} = \Pr\{M = m\} \prod_{i=1}^n W(y_i|x_i, s_i). \tag{2.10}
\]

Here \( x_i \) denotes the \( i \)-th input symbol when \( M = m \) (in the no-feedback model) or when \( M = m, Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1} \) (in the complete-feedback model). The permissible pairs \( K, L \) are of form \( K = K(M) \), \( L = L(Y^n) \) in the no-feedback case, and formally, \( K \) should be replaced by \( K = K(M, Y^n) \) in the complete-feedback case; for the latter model, however, \( K = L = L(Y^n) \) may be taken, without restricting generality.

Both the “no-feedback” and “complete-feedback” AVC models can be modified by letting Terminal \( \mathcal{X} \) know the state sequence \( s \). Then the inputs \( X_1, \ldots, X_n \) and the RV \( K \) may depend also on \( s \). Also, the AVC analog of Model ii), i.e., AVC with active feedback could be considered.

We will not attempt to give a general formal definition of the class of models we are interested in, but all our models involve the blocklength \( n \) of observable source RV’s or permissible channel transmissions (or both). For such models, Definition 2.1 always makes sense if we specify, for every \( n \), the class of permissible pairs of RV’s that may be generated by the terminals as functions of the data available to them. We now sketch how ID codes and ID capacity can be defined for arbitrary models of this kind, as a straightforward extension of the corresponding definitions for channels without or with feedback [4], [5].

Suppose one of \( N \) contingencies \( k \in \{1, \ldots, N\} \) takes place, Terminal \( \mathcal{X} \) knows this \( k \), and the goal is to let Terminal \( \mathcal{Y} \) reliably decide, for any \( 1 \leq j \leq N \) he may choose, whether or not \( k = j \). To this end, the terminals perform a protocol permissible by the given model, for some blocklength \( n \), with the understanding that the actions of Terminal \( \mathcal{X} \), but not those of Terminal \( \mathcal{Y} \), may explicitly depend on \( k \). For example, for Model ii), the functions \( f_i(M, g(Y_1), \ldots, g_{i-1}(Y_1, \ldots, Y_i-1)) \) specifying the channel inputs \( X_i \), will depend on \( k \), whereas for the feedback messages \( g_i(Y_1, \ldots, Y_i) \) no such dependence is allowed, except for implicit dependence through the \( Y_i \)’s. Let \( U \) denote all information available at Terminal \( \mathcal{Y} \) after having performed the protocol, e.g., for Model ii), \( U = Y^n \). Then, if \( \mathcal{Y} \) wants to decide whether or not \( k = j \), he decides “yes” if \( U \in D_j \) and “no” if \( U \notin D_j \), where \( D_j, 1 \leq j \leq N \) are certain subsets of \( U \), the range of \( U \).

**Definition 2.2:** A protocol as above together with a family \( \{D_j, 1 \leq j \leq N\} \) of subsets of \( U \) is called an \( (N, n, \varepsilon) \) ID code for the given model if for each distinct \( k, j \) in \( \{1, \ldots, N\} \)

\[
P_j(D_j) \leq \varepsilon \quad P_k(D_j) \leq \varepsilon. \tag{2.11}
\]

Here \( P_k \) denotes the distribution of \( U \) when contingency \( k \) has taken place. The ID capacity of the given model is the supremum of the numbers \( R \) such that for every \( \varepsilon > 0 \) and sufficiently large \( n \) there exists an \( (N, n, \varepsilon) \) ID code with

\[
N \geq \exp(\exp(nR)).
\]

For models whose statistics depend on unknown parameters ("state"), Definition 2.2 applies with the obvious modification. Namely, as then the distributions \( P_k \) also depend on the state, we require (2.11) to hold robustly, i.e., for all possible states. In particular, for an AVC without feedback (with \( \mathcal{X} \) permitted to randomize) an \( (N, n, \varepsilon) \) ID code is defined by a family \( \{Q_j, 1 \leq j \leq N\} \) of PD’s on \( X^n \), \( Q_j \) representing the distribution of the input sequence when contingency \( j \) takes place, together with a family \( \{D_j, 1 \leq j \leq N\} \) of subsets of \( Y^n \), such that for each distinct \( k, j \) in \( \{1, \ldots, N\} \) and all \( s \in S^n \)

\[
\sum_{x \in X^n} Q_j(x)W^n(D_j|x, s) \leq \varepsilon
\]

\[
\sum_{x \in X^n} Q_k(x)W^n(D_j|x, s) \leq \varepsilon. \tag{2.12}
\]

It is important to emphasize that the sets \( D_j \) in Definition 2.2 need not be disjoint. If they were, Terminal \( \mathcal{Y} \) could infer \( k \) (as that \( j \) for which \( U \in D_j \)) with probability of error less than \( \varepsilon \), thus the ID code would become a transmission code. Whereas for ID codes \( N \) can grow doubly exponentially with the blocklength \( n \), for transmission codes only exponential growth is possible.

As a straightforward generalization of the concept of channel capacity, we can define the transmission capacity of a general model as the supremum of numbers \( R \) such that for every \( \varepsilon > 0 \) and sufficiently large \( n \) there exists an \( (N, n, \varepsilon) \) transmission code. Notice that for transmission codes, i.e., when the sets \( D_j, 1 \leq j \leq N \), are disjoint, it suffices to impose the first inequality in (2.11). More exactly, the transmission capacity defined in this way is that for the “maximum-error” criterion, whereas transmission capacity for the “average-error” criterion is obtained if the transmission codes are required to satisfy only

\[
\frac{1}{N} \sum_{j=1}^{N} P_j(D_j) \leq \varepsilon \tag{2.13}
\]

a weaker condition than (2.11). Just as for standard channel capacity, these two concepts of transmission capacity coincide.
for models with uniquely determined statistics, but transmission capacity for average error can be larger than that for maximum error when the statistics depend on unknown states.

Remark: For models with randomization allowed at Terminal \( \mathcal{X} \), transmission capacity (for average error) is always a lower bound to CR capacity. Indeed, a trivial way of generating CR is that Terminal \( \mathcal{X} \) generates an RV uniformly distributed on \( \{1, \ldots, N\} \) and then transmits it to Terminal \( \mathcal{Y} \) with probability of error less than \( \varepsilon \). From the point of view of CR capacity, the interesting models are those for which this trivial scheme is not optimal.

III. SOME GENERAL RESULTS

Lemma 3.1: Let \( \mathcal{P} \) be any family of \( N \) PD’s \( P = \{p(v), v \in V\} \) on a finite set \( V \), let \( 0 < \varepsilon \leq \frac{1}{2} \) and let \( d > 0 \) be such that for every \( P \in \mathcal{P} \) the set

\[
E(P, d) = \left\{ v : p(v) \leq \frac{1}{d} \right\}
\]  

(3.1)

has \( P \)-probability

\[
P(E(P, d)) \geq 1 - \varepsilon. \tag{3.2}
\]

Then for \( k \leq (2e/3 \log (2N))d \), there exists \( f : V \to \{1, \ldots, k\} \) such that for every \( 1 \leq i \leq k \) and \( P \in \mathcal{P} \) the conditional \( P \)-probability of \( f(v) = i \) on the condition \( v \in E(P, d) \) differs from \( 1/k \) by less than \( \varepsilon/k \), i.e.,

\[
\left| P(f^{-1}(i) \cap E(P, d)) - \frac{1}{k} \right| < \frac{\varepsilon}{d}, \quad 1 \leq i \leq k, \quad P \in \mathcal{P}. \tag{3.3}
\]

In particular, the variation distance of the distribution of \( f \) from the uniform distribution on \( \{1, \ldots, k\} \) is less than \( 3\varepsilon \), i.e.,

\[
\sum_{i=1}^{k} \left| P(f^{-1}(i)) - \frac{1}{k} \right| < 3\varepsilon \tag{3.4}
\]

for each of the PD’s \( P \in \mathcal{P} \).

Proof: Similar to that of Lemma 1.1 but requires a little more calculation. See the Appendix.

Consider now the problem of robust uniform randomness obtained by encoding the \( n \)-length output \( X^n \) of an AVS, where the distribution of \( X^n \) depending on the state sequence \( s \in S^n \) is given by (2.9). We are interested in mappings \( f : \mathcal{X}^n \to \mathcal{M} \) of possibly large rate \((1/n) \log |\mathcal{M}| \) for which \( f(X^n) \) represents robust \( \varepsilon \)-uniform randomness, i.e., the variation distance of the distribution of \( f(X^n) \) from the uniform distribution on \( \mathcal{M} \) is less than \( \varepsilon \), no matter what is the state sequence \( s \in S^n \).

Theorem 3.1: Let an AVS be given by a set of PD’s \( \{P(\cdot | s), s \in S\} \) on \( \mathcal{X} \), such that

\[
H_{\min} = \min_{s \in S} H(P(\cdot | s)) > 0.
\]

Then, for every \( 0 < \varepsilon < \frac{1}{2} \) and every \( n \), there exists a mapping \( f : \mathcal{X}^n \to \mathcal{M} \) of rate

\[
\frac{1}{n} \log |\mathcal{M}| > H_{\min} - \delta(\varepsilon, n) \tag{3.5}
\]

such that \( f(X^n) \) represents robust \( \varepsilon \)-uniform randomness, where

\[
\delta(\varepsilon, n) = \sqrt{\frac{2 \ln 3/e}{n} \log |\mathcal{X}| + \frac{2 \log 1/e}{n} + \frac{\log (2|\mathcal{S}|)}{n} + \frac{\log n}{n}} \tag{3.6}
\]

if \(|\mathcal{X}| > 3\), and \(|\mathcal{X}| \) should be replaced by \( 3 \) if \(|\mathcal{X}| = 2 \), the \( 0((\log n)/n) \) term in (3.6) does not depend on \( \varepsilon \) and the AVS, not even on \( \mathcal{X} \) and \( S \).

Remarks: One feature of Theorem 3.1 that will be used in Theorem 3.2 below is that it brings out explicitly the dependence of \( \delta(\varepsilon, n) \) on \( \mathcal{X} \) and \( S \). For a fixed AVS, Theorem 3.1 shows that robust \( \varepsilon \)-uniform randomness for (arbitrarily small but) constant \( \varepsilon \) can be attained by mappings of rate approaching \( H_{\min} \), with speed \((n^{-1/2}) \), and the rate will approach \( H_{\min} \) even if \( \varepsilon = \varepsilon_n \to 0 \), providing it goes to 0 slower than exponentially. Moreover, robust \( \varepsilon \)-uniform randomness with rate \((1/n) \log |\mathcal{M}| > H_{\min} - \delta \) with an arbitrarily small but constant \( \delta > 0 \) is attainable even with \( \varepsilon \) going to 0 exponentially.

Proof: Apply Lemma 3.1 to the family of PD’s \( P(\cdot | s), s \in S^n \), on \( V = \mathcal{X}^n \), with \( \varepsilon \) replaced by \( \varepsilon/\beta \) (in order to get \( \varepsilon \)-uniform rather than \( 3\varepsilon \)-uniform randomness, cf. (3.4)). Then

\[
N = |S|^n,
\]

and we will choose the number \( d \) in (3.1) as

\[
d = \exp \left[ n(H_{\min} - \xi) \right]
\]

(3.7)

with \( \xi > 0 \) such that (3.2) (with \( \varepsilon \) replaced by \( \varepsilon/\beta \)) is fulfilled for each \( P = P(\cdot | s) \). As shown in the Appendix

\[
\xi = \sqrt{\frac{2 \ln 3/e}{n} \log |\mathcal{X}|} \tag{3.8}
\]

is an adequate choice, with the understanding (as also in the remainder of the proof) that \(|\mathcal{X}| \) should be replaced by \( 3 \) if \(|\mathcal{X}| = 2 \). Then Lemma 3.1 gives that for

\[
|\mathcal{M}| \leq \frac{(\varepsilon/3)^2}{3 \log (2|S|^n)} \exp \left[ n \left( H_{\min} - \frac{2 \ln 3/e}{n} \log |\mathcal{X}| \right) \right] \tag{3.9}
\]

there exists \( f : \mathcal{X}^n \to \mathcal{M} \) such that \( f(X^n) \) represents \( \varepsilon \)-uniform randomness, for each \( s \in S^n \). Comparison of (3.5) and (3.9) shows that both can be satisfied with \( \delta(\varepsilon, n) \) as in (3.5).

Having available Theorem 3.1, we now prove that for the type of models treated in this paper, CR capacity can be attained with uniform CR. Although we did not give a formal definition of this class of models, we recall from Section II that all our models involve the specification of permissible pairs of RV’s \((K_L, L)\), for each blocklength \( n \). The following definition postulates a property common to all models we are interested in.

Definition 3.1: A model permits independent concatenations if for any pairs of RV’s \((K_1, L_1)\) and \((K_2, L_2)\) permissible for blocklengths \( n_1 \) and \( n_2 \), there exists a pair \((K_L, L)\) permissible for blocklength \( n_1 + n_2 \) such that \( K = (K_1, K_2) \), \( L = (L_1, L_2) \), where \((K_1, L_1)\) and \((K_2, L_2)\) are independent and have the same distribution as \((K_1', L_1')\) and \((K_2', L_2')\).
When the underlying statistics are not uniquely determined but depend on some parameters ("state"), the last condition means that under any permissible statistics for blocklength $n_1 + n_2$, $(K_1, L_1)$ and $(K_2, L_2)$ are independent, with distributions equal to those of $(K'_1, L'_1)$ and $(K'_2, L'_2)$ under one of permissible statistics for blocklength $n_1$ respectively $n_2$.

For models with statistics depending on "states," let $S(n)$ denote the set of possible states for blocklength $n$. We will assume that this set does not grow faster than doubly exponentially, more exactly, that $(1/n) \log \log |S(n)|$ is bounded by a constant. This holds for all models we are aware of, e.g., for the standard AVS and AVC models $|S(n)| = \mathcal{S}^n$ grows only exponentially. Even for the variant of the AVC where the state sequence $s$ may depend on the input sequence $x$, in which case $S(n)$ is the set of all mappings of $\mathcal{X}^n$ into $\mathcal{S}^n$, the growth rate of $|S(n)|$ is "only" doubly exponential.

Theorem 3.2: Let a model permitting independent concatenations be given. If the statistics are not uniquely determined, we assume that $(1/n) \log \log |S(n)|$ is bounded. Then for any fixed $\epsilon > 0$, every $H$ less than CR capacity, and sufficiently large $n$, there exists a permissible pair of RV’s $(K, L)$, both distributed on a set $\mathcal{M}$ satisfying $(1/n) \log |\mathcal{M}| \geq H$, such that

$$\Pr \{ K \neq L \} < \epsilon, \quad \sum_{k \in \mathcal{M}} \Pr \{ K = k \} - \frac{1}{|\mathcal{M}|} < \epsilon$$

(3.10)

for every possible choice of the underlying statistics.

Remark: It will be clear from the proof that the near uniformity of $K$ can be attained also in a stronger sense, namely, in the second inequality in (3.10), instead of a fixed $\epsilon > 0$ one could take a sequence $\epsilon_n$ going to 0 exponentially as $n \to \infty$ (with a sufficiently small exponent). A similar improvement of the first inequality in (3.10) is possible providing that in the definition of CR capacity the fixed $\epsilon > 0$ in (2.3) can be replaced by $\epsilon_n$ going to 0 exponentially, this holds for all the models treated in this paper.

Proof: As $H$ is less than CR capacity, there exists $H' > H$ which is still an achievable CR rate. Applying Definition 2.1 to $H'$ in the role of $H$, with $\delta' = (H' - H)/2$, and $\epsilon > 0$ specified later, it follows that for sufficiently large $m$ there exists a pair $(K', L')$ permissible for blocklength $m$ such that their common range $\mathcal{K}$ satisfies

$$|\mathcal{K}| \leq \exp(\epsilon m)$$

(3.11)

and

$$\Pr \{ K' \neq L' \} < \epsilon'$$

(3.12)

$$\frac{1}{m} H(K') > H' - \delta' = H + \delta'$$

(3.13)

for every choice of the underlying statistics. Clearly, the case of uniquely determined statistics (|S(m)| = 1) need not be considered separately.

As the model permits independent concatenations, for every $r$ there exists a pair $(K'_r, L'_r)$ permissible for blocklength $n = rm$, with $K' = K_1 \cdots K_r$, $L' = L_1 \cdots L_r$, such that for every possible statistics for blocklength $n$ the pairs $(K_i, L_i)$, $i = 1, \cdots, r$ are independent, with distributions equal to that of $(K'_i, L'_i)$ for some possible statistics for blocklength $m$ (possibly different for each $i$). In particular, $K'$ may be regarded as the $r$-length output of an AVS with alphabet $\mathcal{K}$ and state set $S(m)$. For this AVS, $H_{\text{min}} > m(H + \delta')$ by (3.13). Thus by Theorem 3.1, there exists a mapping $f: \mathcal{K}' \to \mathcal{M}$ with

$$\frac{1}{r} \log |\mathcal{M}| > m(H + \delta') - \delta(\epsilon, r)$$

(3.14)

such that the distribution of $f(K'_r)$ is robustly $\epsilon$-close to the uniform distribution on $\mathcal{M}$, where

$$\delta(\epsilon, r) = \frac{2 \ln 1/\epsilon}{r} \log |\mathcal{K}| + \frac{2 \log 1/\epsilon}{r} + \frac{\log \log |S(m)|}{r} + \frac{\log r}{r}.$$  

(3.15)

Using (3.11) and the assumption on the growth rate of $S(m)$, it follows from (3.14) and (3.15) that $(1/m) \log |\mathcal{M}| > H$ if $r$ is sufficiently large, depending on $\epsilon$ but not on $m$.

With such an $r$ we set $K = f(K')$, $L = f(L')$ for blocklength $n = rm$. Then $K$ and $L$ are distributed on $\mathcal{M}$ satisfying $(1/n) \log |\mathcal{M}| > H$, and the second inequality in (3.10) holds for every possible choice of the underlying statistics. Finally, the first inequality in (3.10) follows from (3.12), if we choose $\epsilon' = \epsilon/r$. This completes the proof, because it clearly suffices to restrict attention to blocklengths $n$ which are multiples of a constant $r$.

Theorem 3.3: For all models as in Theorem 3.2, the CR capacity is a lower bound to ID capacity, providing the transmission capacity (for the maximum-error criterion) is positive.

Proof: Immediate from Theorem 3.2 and the result of Ahlswede and Dueck [5].

IV, COMMON RANDOMNESS IN MODELS i), ii), AND iii)

Theorem 4.1: For Model i) described in Section II, the CR capacity equals

$$C_1(R) = \max_{U} [I(U \wedge X) | I(U \wedge X) - I(U \wedge Y) \leq R]$$

(4.1)

if no randomization is permitted, and

$$\hat{C}_1(R) = \begin{cases} C_1(R), & \text{if } R \leq H(X \wedge Y) \\ R + I(X \wedge Y), & \text{if } R > H(X \wedge Y) \end{cases}$$

(4.2)

if Terminal $\mathcal{X}$ is allowed to randomize. Here the maximum is for all RV’s $U$ that satisfy the Markov condition $U \rightarrow X \rightarrow Y$, and the range constraint $|\mathcal{U}| \leq |\mathcal{X}|$, and $R$ is the capacity of the noiseless channel in the model. Moreover, the CR capacity of the variant of Model ii), where the noiseless channel is replaced by a DMC, is still given by (4.1) respectively (4.2), with $R$ replaced by the capacity of that DMC.

Remark: If $\mathcal{X}$ is permitted to randomize, a trivial way to create CR is that $\mathcal{X}$ generates $nR$ random bits and transmits them to $\mathcal{Y}$, disregarding the DMMS. Theorem 4.1 shows that this is suboptimal, and for $R \geq H(X|Y)$ the CR capacity exceeds $R$ (attained by the trivial scheme) by exactly $I(X \wedge Y)$. This means that although mutual information does not
represent a “common information” (as shown in [12]), it does
represent a kind of hidden common randomness that can be
recovered if sufficient transmission capacity is available. It is
interesting to compare this interpretation of mutual information
with that obtained in Part I in the context involving secrecy.

Proof: A short proof is available using standard results
of multiuser information theory, cf. the proof of Theorem 4.2
below. Here we prefer an independent proof, which later will
be extended to the case of two-way communication.

We state, as also for later reference, an identity also used in
Part I (Lemma 4.1; cf. also [16, p. 409]): For arbitrary RV’s
S, T and sequences of RV’s X^n, Y^n
\[ I(S \wedge X^n|T) = I(S \wedge Y^n|T) \]
\[ = \sum_{i=1}^{n} [I(S \wedge X_i|X_1, \ldots, X_{i-1}, Y_{i+1} \ldots Y_n)
- I(S \wedge Y_i|X_1, \ldots, X_{i-1}, Y_{i+1} \ldots Y_n)]
= n[I(S \wedge X_j|V) - I(S \wedge Y_j|V)] \]
where J is an RV independent of all the previous ones,
uniformly distributed on \{1, \ldots, n\}, and
\[ V = X_1 \ldots X_{j-1} Y_{j+1} \ldots Y_n T J. \]

a) Converse Part: Consider first the “no-randomization”
case. Suppose \((K, L)\) satisfy (2.1), (2.4). Write
\[ H(K|Y^n) = I(K \wedge f(X^n)|V^n) + H(K|Y^n, f(X^n)). \]
Here the first term is \(\leq nR\), by (2.1), and the second term
is \(\leq H(K|L) \leq \varepsilon + 1\) by (2.2) and Fano’s inequality, using
(2.3) and (2.4). Thus we have
\[ H(K|Y^n) \leq H(K|Y^n) \leq nR + \varepsilon + 1. \]

Apply (4.3) to the present \(X^n, Y^n\) with \(S = K, T = \emptyset\).
Then \(V\) in (4.4) is independent of \((X_j, Y_j)\), hence the last
line in (4.3) can also be written as
\[ I(U \wedge X_j) - I(U \wedge Y_j), \quad \text{with } U = KV. \]

Thus
\[ H(K) - I(K \wedge Y^n) = I(K \wedge X^n) - I(K \wedge Y^n)
= n[I(K \wedge X_j) - I(U \wedge Y_j)] \]
where \(U = X_1 \ldots X_{j-1} Y_{j+1} \ldots Y_n\) satisfies the Markov
condition \(U \wedge X_j \wedge Y_j\). Notice also that
\[ H(K) = I(K \wedge X^n) = \sum_{i=1}^{n} I(K \wedge X_i|X_1, \ldots, X_{i-1})
= n[I(K \wedge X_j|X_1, \ldots, X_{j-1}) \leq nI(U \wedge X_J). \]

As \(X_j, Y_j\) may be identified with the generic variables
\(X, Y\) of our DMMS, (4.6)-(4.8) show that \((1/n)H(K)\) is
upper-bounded by the maximum of \(I(U \wedge X)\) subject to
\(I(U \wedge X) - I(U \wedge Y) \leq R + \varepsilon + (1/n)\), for RV’s \(U\) satisfying
the Markov condition \(U \wedge X \wedge Y\). It is routine to show that
there exists \(U\) attaining the maximum that satisfies the range
constraint \([U] \leq \lvert X \rvert\) (direct application of the Support Lemma
of [16, p. 310] gives \([U] \leq \lvert X \rvert + 1\), but for a \(U\) yielding
an extremal value, this bound can be improved by 1, cf. [19]).

This completes the proof for the “no-randomization” case,
Notice that in (4.1) necessarily \(I(U \wedge Y) \leq I(X \wedge Y)\) hence
\(C_I(R) \leq R + I(X \wedge Y)\), equality holds if \(R = H(X|Y)\).

When \(X\) may randomize, we will conveniently regard his
randomization RV \(M_X\) as an i.i.d. sequence \(M^n\) (of course,
independent of \(X^n, Y^n\)). This reduces the present case to the
previous one, replacing \(X\) by \(X M\), where \(M\) is independent
of \(X, Y\). Thus we need to maximize \(I(U \wedge X M)\) subject to
\[ I(U \wedge X M) - I(U \wedge Y) \leq R, \quad U \wedge X M \wedge Y. \]

It follows similarly to (4.9) that (4.10) implies \(I(U \wedge X M) \leq R + I(X \wedge Y)\), thus for the case \(R \geq H(X|Y)\) we are done.
For \(R < H(X|Y)\), notice that since
\[ I(U \wedge X M) = I(U \wedge X) + I(U \wedge M|X) \]
and the Markov condition in (4.10) implies \(U \wedge X \wedge Y\), it follows from (4.9) that
\[ I(U \wedge X M) \leq C_I(R - I(U \wedge M|X)) + I(U \wedge M|X). \]

It is easy to check that the function defined by (4.1) is concave,
hence, by (4.9), its slope is \(\geq 1\) if \(R \leq H(X|Y)\). Thus the
right-hand side of (4.11) is \(\leq C_I(R)\). This completes the proof
for the randomized case.

Finally, if the channel from \(X\) to \(Y\) is not noiseless but a
DMC, the only modification needed in the above proof is to
replace \(f(X^n)\) in (4.5) by the output of that DMC.

Denote the input of this DMC by \(T^n\) and the output by \(Z^n\).
Whether or not terminal \(X\) randomizes, the Markov condition
\(Y^n \wedge X^n \wedge T^n \wedge Z^n\) must hold. Thus the first term in (4.5)
with \(f(X^n)\) replaced by \(Z^n\) can be bounded as
\[ I(K \wedge Z^n|Y^n) \leq I(X^n K \wedge Z^n|Y^n) \leq I(T^n \wedge Z^n|Y^n) \leq nC \]
establishing our claim.

b) Direct Part: It suffices to consider the case \(R \leq H(X|Y)\),
with no randomization, By contiguity, it suffices to show that
\(C_I(R)\) is an achievable CR rate for every \(R < R\).
We are going to show this by exhibiting for arbitrary \(U\) satisfying
\[ U \wedge X \wedge Y, \quad I(U \wedge X) - I(U \wedge Y) < R \]
and for any \(\varepsilon > 0, \delta > 0\), and sufficiently large \(n\), a

permissible pair \(K, L\) as defined by (2.1) and (2.2), such that
\(K, L\) satisfy (2.3)-(2.5) with \(H = I(U \wedge X)\).

Assuming without any loss of generality that the distribution
of \(U\) is a possible type for blocklength \(n\), select at random

\[ \text{exp} \{n[I(U \wedge X) + \delta]\} \text{sequences } u_i \in U^n \text{ of type } P_U, \]
denoted \(u_{ij}\), \(1 \leq i \leq N_1, 1 \leq j \leq N_2\), with
\[ N_1 = \text{exp} \{n[I(U \wedge X) - I(U \wedge Y) + 3\delta]\}
N_2 = \text{exp} \{n[I(U \wedge Y) - 2\delta]\}. \]

Then for every \(X\)-typical \(x \in X^n\), the probability that neither
\(u_{ij}\) is jointly \(UX\)-typical with \(x\) is doubly exponentially small.
Hence with probability close to 1, every typical \(x\) is jointly
typical with some \(u_{ij}\).
Let $K(x)$ be equal to an $u_{ij}$ jointly typical with $x$ (either one if there are several), and let $f(x) = i$ if $K(x) = u_{ij}$; both functions are set constant when $x$ is not further. Let $L(y, f(x)) = u_{ij}$ if $f(x) = i$ and $u_{ij}, y$ are jointly $U$-typical. If there is no such $u_{ij}$ or there are several, $L$ is set equal to a constant. Then, by (4.12) and (4.13), the rate constraint (2.1) on $f$ is satisfied if $f$ is sufficiently small, and $K = K(X^n), L = L(Y^n, f(X^n))$ obviously satisfy (2.4). Also (2.5) is satisfied since

$$\Pr \{ K = u_{ij} \} \leq P^n_{st}(\{ x : (u_{ij}, x) \text{ jointly typical} \}) = \exp (-nI(U \wedge X) + o(n))$$

implies that $H(K) \geq nI(U \wedge X) + o(n)$.

It remains to check (2.3), i.e., $\Pr \{ K = L \} \leq \varepsilon$. Notice that for any jointly $U$-$X$-typical pair $(u, x)$, the set of $y$’s jointly typical with $(u, x)$ has conditional probability arbitrarily close to 1 on the condition $U^n = u, X^n = x$, and hence by Markovity, also on the condition $X^n = x$. It follows that the set $A$ of those pairs $(x, y)$ for which $(K(x), x, y)$ are jointly $U$-$X$-typical has $P^n_{st}(A)$ arbitrarily close to 1. Let us denote by $A$ the set of those pairs $(x, y) \in A$ for which in addition to $u_{ij} = K(x)$, some other $u_{ij}$ (with the same first index $i$) is also jointly typical with $y$. To complete the proof, it suffices to show that $P^n_{st}(A)$ will be arbitrarily small, with large probability, with respect to the random choice of $\{u_{ij}\}$.

Now, for fixed $(x, y)$, the probability that $A$ determined by the random $\{u_{ij}\}$ contains $(x, y)$, is upper-bounded by

$$\exp \left[ -nI(U \wedge Y) + o(n) \right] = \exp \left[ -n\delta + o(n) \right]. \tag{4.15}$$

Here we used (4.13) and that the $u_{ij}$ are independent, chosen with uniform distribution from the sequences of type $P^n$. Hence the expectation of $P^n_{st}(A)$, as an RV depending on $\{u_{ij}\}$, is also upper-bounded by $\exp \left[ -n\delta + o(n) \right]$. This completes the proof.

Consider now the following generalization of Model i) to generating CR at $r+1$ (rather than 2) terminals. Given a DMM with $r+1$ components, with generic variables $X, Y_1, \ldots, Y_r$, Terminal $X$ can observe $X^n$ and send messages $f_i(X^n)$ to terminals $Y_i, i$, subject to rate constraints

$$\frac{1}{n} \log ||f_i|| \leq R_i, i = 1, \ldots, r. \tag{4.16}$$

Terminal $Y_i$ can observe $Y_i^n$, and the message $f_i(X^n)$ sent by him to Terminal $Y$. Achievable CR rates and CR capacity are defined by the natural extension of Definition 2.1, namely, the role of permissible pairs $(K, L)$ is now played by permissible $(r+1)$-tuples $(K, L_1, \ldots, L_r)$ defined in analogy to (2.2), and the role of condition (2.3) is played by $r$ similar conditions

$$\Pr \{ K \neq L_i \} < \varepsilon, i = 1, \ldots, r.$$

Theorem 4.2: For the above model, with no randomization permitted, the CR capacity equals the maximum of $I(U \wedge X)$ subject to the constraints

$$I(U \wedge X) - I(U \wedge Y_i) \leq R_i, i = 1, \ldots, k, \quad U \in X \oplus Y_1, \ldots, Y_k. \tag{4.17}$$

where $U$ may be supposed to satisfy the range constraint $|U| \leq |X| - r - 1$. If Terminal $X$ is permitted to randomize, the CR capacity is still the same if $R_i < H(X|Y_i)$ for some $i$, and it equals

$$\min_{1 \leq i \leq k} \left( R_i + I(X|Y_i) \right), \quad i = 1, \ldots, k.$$

Proof: If $H$ is an achievable CR rate and $\delta > 0$, take (for large $n$) $K = K(X^n)$ that can be $e$-reproduced at each terminal by $L_i = L_i(Y_i^n, f_i(X^n)), i = 1, \ldots, r$, and such that

$$\frac{1}{n} \log H(K) - H < \delta. \tag{4.18}$$

Although the definition of achievable CR rates postulates $\frac{1}{n} H(K) > H - \delta$ only, it clearly does not restrict generality to require $\frac{1}{n} H(K) < H + \delta$, as well. In order that $K$ could be $e$-reproduced at $Y_i$, it is necessary that

$$\frac{1}{n} \log H(K|Y_i^n) \leq R_i + \delta \tag{4.19}$$

(formally, $H(K|Y_i^n)$ can be written as a sum of two terms as in (4.5), and bounded as there).

On the other hand, if $H$ is a number $H$ for all $\delta > 0$ and sufficiently large $n$ there exists a function $K = K(X^n)$ that satisfies (4.18) and (4.19), then $H$ is an achievable entropy rate. Indeed, from $Y_i^n$ and a suitable code $f_i(K^n)$ of rate $n^{-r} \log ||f_i|| \leq R_i$ of the $r$-fold repetition of $K$, Terminal $Y_i$ can reproduce $K'$ with arbitrarily small probability of error, by Slepian–Wolf. Although the permissible rate is only $R_i$, this can be remedied by taking blocklength $N = n^r + n'$ slightly larger than $n$, satisfying $n^r R_i \leq n(R_i + \delta), i = 1, \ldots, r$, and disregarding the last $N-n'$ source outputs. Thus for blocklength $N$, the terminals can produce $e$-common randomness of rate

$$\frac{1}{N} H(K) = \frac{1}{n} H(K)$$

arbitrarily close to $H$.

Thus we have obtained a “product space characterization” of achievable CR rates, namely, that $H$ is achievable iff for every $\delta > 0$ and sufficiently large $n$ there exists a function $K = K(X^n)$ satisfying (4.18) and (4.19). This can be easily single-letterized, using results available in the literature. To this, notice that on account of (4.18), in our product space characterization we may replace (4.19) by

$$\frac{1}{n} H(X^n|K) - (H(X) - H) < \delta \tag{4.20}$$

Now, by [16, p. 352], an $(r+1)$-tuple $\hat{R}_0, \hat{R}_1, \ldots, \hat{R}_r$ has the property that for every $\delta > 0$ and sufficiently large $n$ there
exists a function $f(X^n)$ satisfying

$$\frac{1}{n} H(f(X^n)) - \tilde{R}_0 < \delta \quad \frac{1}{n} H(Y^n|f(X^n)) \leq \tilde{R}_i + \delta$$

iff there exists a RV $U$ with $U \rightarrow X \rightarrow Y_1, \ldots, Y_r$ such that

$$H(X|U) = \tilde{R}_0 \quad H(Y_i|U) \leq \tilde{R}_i.$$

Substituting here $\tilde{R}_0 = H(X) - H, \tilde{R}_i = R_i - H + H(Y_i),$ we get

$$I(U \land X) = H \quad I(U \land Y) \leq R_i$$

and this completes the proof for the no-randomization case (up to the routine range constraint).

If randomization is permitted, we replace $X$ by $X|M$ and proceed as in the proof of Theorem 4.1.

**Theorem 4.3:** For Model ii) described in Section II, the CR capacity equals

$$C_2(R) = \max_X I(X \land Y) + \min (R, H(X|Y)) \tag{4.21}$$

if randomization at $X$ is not permitted, and

$$\tilde{C}_2(R) = C(W) + R \tag{4.22}$$

if Terminal $\mathcal{Y}$ is allowed to randomize. In (4.21), the maximum is taken for random inputs $X$ to the DMC $\{W\}$ given in the model, $Y$ denoting the corresponding output, $R$ is the capacity of the backward noiseless channel in the model, and $C(W)$ is the capacity of $\{W\}$. Moreover, the CR capacity of the variant of Model ii), where the backward channel is replaced by a DMC, is still given by (4.21) respectively (4.22), with $R$ replaced by the capacity of that DMC.

**Remarks:**

1) Comparing (4.21) and (4.22) shows that if $R$ is no larger than $H(Y|X)$ for a capacity achieving $X$ then $\tilde{C}_2(R) = C_2(R).$ Thus similarly to Model i), randomization helps only when $R$ is “large.” For many DMC’s, the maximum of $H(Y)$ is attained for a capacity achieving $X$. In those cases $\tilde{C}_2(R) = H(Y)$ for all $R$ “large” in the above sense.

2) One possible strategy of Terminal $\mathcal{X}$ in Model ii) is to use an i.i.d. sequence $X^n$ as channel input, which leads to the situation of Model i), with the roles of $\mathcal{X}$ and $\mathcal{Y}$ reversed. Comparing Theorems 4.1 and 4.3 shows that this reduction to Model i) suffices to achieve CR capacity for Model ii) when the max in (4.21) is attained for some $X$ and $H(X|Y) \leq R_i$ but not otherwise.

**Proof:**

a) **Direct Part:** If Terminal $\mathcal{Y}$ can randomize (recall that Terminal $\mathcal{X}$ always can in this model), a CR rate as in (4.22) can be attained in a trivial way: For large blocklength $n$, Terminal $\mathcal{X}$ generates and transmits to $\mathcal{Y}$ an RV uniformly distributed on a set of size $\exp \left[\frac{n}{2} (H(Y^n) - \delta)\right]$. $\mathcal{Y}$ can decode it with small probability of error. Terminal $\mathcal{Y}$, in general, generates an RV uniformly distributed on a set of size $\exp nR$, and transmits it to $\mathcal{X}$.

If $\mathcal{Y}$ cannot randomize, a CR rate as in (4.21) can be attained as follows. For large $n$, take $(X, Y)$ almost attaining the maximum in (4.21) such that $P_X$ is a possible type for blocklength $n$. Terminal $\mathcal{X}$ generates an RV $M$ uniformly distributed on a set of size $\exp \left[\frac{n}{2} (H(X^n) - \delta)\right]$ and transmits it to $\mathcal{Y}$ using a code of fixed composition $P_X$. $\mathcal{Y}$ sends nothing back until he has received all $n$ outputs. Then $\mathcal{Y}$ decodes $M$. Terminal $\mathcal{Y}$ can decode with small probability of error, and he gets access to additional randomness from the channel output. Namely, he can enumerate the words in each of his decoding sets from 1 to $\exp \left[\frac{n}{2} (H(Y^n) + o(n))\right]$, then the RV $Z$ equal to the number assigned to the observed output sequence will be almost independent of $M$ and have entropy $\frac{n}{2} (H(Y^n) + o(n))$.

If $R > H(Y|X)$, this $Z$ can be transmitted back to $\mathcal{X}$, and if $R < H(Y|X)$ then a suitable function of $Z$ of entropy $nR + o(n)$ can be transmitted back.

b) **Converse Part:** Let $(K, L)$ be a permissible pair for blocklength $n$, thus $K = K(M, g_1, \ldots, g_n), L = L^n(X^n),$ with $g = (g_1, \ldots, g_n)$ satisfying (2.6). Supposing that $(K, L)$ satisfies the condition in Definition 2.1, we decompose $H(L|M)$ in analogy to (4.5), replacing $f(X^n)$ there by $g = (g_1, \ldots, g_n)$. Bounding as there we obtain

$$H(L|M) = I(L \land g[M]) + H(L|M, g) \leq nR + \text{enc} + 1,$$  

(4.23)

Further, (2.7) and the memoryless character of the DMC $\{W\}$ imply

$$I(L \land M) \leq I(M \land Y^n) = \sum_{i=1}^n I(M \land Y_i | Y_1^{i-1})$$

$$\leq \sum_{i=1}^n I(X_i \land Y_i | Y_1^{i-1})$$

$$\leq \sum_{i=1}^n I(X_i \land Y_i) \leq nI(X \land Y) \tag{4.24}$$

where $J$ is an auxiliary RV uniformly distributed on $\{1, \ldots, n\}$, independent of $(M, Y^n)$. On the other hand, $H(L) \leq H(Y^n) \leq \sum_{i=1}^n H(Y_i) \leq H(Y)$. Combining this with (4.23) and (4.24) we get that

$$\frac{1}{n} H(L) \leq \min \left[ I(X \land Y) + R + \text{enc} + \frac{1}{n} H(Y) \right]$$

$$\leq I(X \land Y) + \min [R, H(Y|X)] + \text{enc} + \frac{1}{n}.$$

As $Y$ is the channel output for input $X$, this completes the converse proof also for the no-randomization case. When the backward channel is not noiseless but a DMC, then denoting its input and output by $T^n$ and $Z^n$, the only difference will be that $g$ in (4.23) has to be replaced by $Z^n$. Then the first term will be bounded as

$$I(L \land Z^n|M) \leq I(T^n \land Z^n|M) \leq nC$$

where $C$ is the capacity of the backward channel.

**Theorem 4.4:** For Model iii) described in Section II, the CR capacity without randomization is equal to

$$C_3(R_1, R_2) = \max_{U, V} I(U \land X) + I(V \land Y) - I(U \land X|U) - I(V \land Y|U) \tag{4.25}$$
where the maximization is for RV’s $U$ and $V$ satisfying the Markov conditions

$$U \perp X \perp Y \quad X \perp Y \perp U \perp V.$$  \hspace{1cm} (4.26)

However, the range sizes of $U$ and $V$ can be bounded by $|X|+2$ and $|Y|$, respectively.

**Remark:** It is reassuring to check that (4.25) reduces to the expected simple results when either $R_1 \geq H(X|Y)$ or $R_2 \geq H(Y|X)$. In the first case, $U = X$ is a permissible choice, then the Markov condition for $V$ becomes void, and it follows that

$$C_3(R_1, R_2) = H(X) + \min(R_1, H(Y|X)).$$

if $R_1 \geq H(Y|X)$.  \hspace{1cm} (4.27)

In the second case $V = Y$ is a permissible choice, which leads to

$$I(U \land X) + I(V \land Y|U) = I(U \land X) + H(Y|U)
= I(U \land X) - I(U \land Y) + H(Y).$$

Hence

$$C_3(R_1, R_2) = H(Y) + \min(R_1, H(X|Y)), $$

if $R_2 \geq H(Y|X)$.  \hspace{1cm} (4.28)

**Proof:**

a) **Converse Part:** Let $(K, L)$ be a permissible pair for Model iii) without randomization, i.e., $K = K(X^n, g, L = L(Y^n, f), f = f(X^n), g = g(Y^n, f)$, where $f$ and $g$ satisfy the rate constraints (2.1) and (2.8). Suppose that $(K, L)$ satisfy the conditions (2.3) and (2.4) of Definition 2.1.

Our key tool is the identity (4.3), which will be applied twice. First, we get

$$n R_1 \geq H(f) = I(f \land X^n) \geq I(f \land X^n) - I(f \land Y^n) = n[I(U \land X) - I(U \land Y)]$$ \hspace{1cm} (4.29)

with

$$X = X_J, Y = Y_J, U = fX_1 \cdots X_{J-1}Y_{J+1} \cdots Y_n, \ J$$ \hspace{1cm} (4.30)

(where we proceed as in the derivation of (4.7), the role of $K$ there now played by $f$).

Notice now that just as $H(K|Y^n)$ was bounded in (4.5), we have the bound

$$H(L|X^n) \leq n R_2 + ecn + 1. \hspace{1cm} (4.31)$$

Applying the identity (4.3) again, we get

$$-H(I|X^n) = H(L|X^n, f) = I(L \land X^n,f) - H(L) = I(L \land X^n) - I(L \land Y^n) = n[I(U \land X) - I(U \land Y)]$$ \hspace{1cm} (4.32)

where $X, Y, U$ are (luckily) the same as in (4.30).

By (4.31) (with sufficiently small $e$) and (4.32) we have for any fixed $\delta > 0$

$$R_2 \geq I(L \land Y|U) - I(L \land X|U) - \delta. \hspace{1cm} (4.33)$$

Finally, we can write

$$I(L \land X^n) = \sum_{i=1}^{n} I(L \land X_i|X_1 \cdots X_{i-1})$$

$$= \sum_{i=1}^{n} I(LX_1 \cdots X_{i-1} \land X_i)$$

$$\leq n[I(U \land X) - I(U \land Y)]. \hspace{1cm} (4.34)$$

Combining (4.32) and (4.34) gives

$$H(L) = I(L \land X^n) + H(L|X^n)$$

$$\leq n[I(U \land X) + I(L \land Y|U) - I(L \land X|U)]$$

$$= n[I(U \land X) + I(L \land Y|U)]. \hspace{1cm} (4.35)$$

Replacing $L$ with $V$, we have thus proved that achievable CR rates are bounded above by an expression as in (4.25) (the Markov conditions (4.26) are easily verified), perhaps with $R_2$ replaced by $R_2 + \delta$; the latter is inconsequential, by continuity.

b) **Direct Part:** As in the proof of Theorem 4.1, it suffices to prove that $I(X \land U) + I(V \land Y|U)$ is an achievable CR rate whenever $U$ and $V$ satisfy (in addition to (4.26)) the inequalities in (4.25) with strict inequality. The form of (4.25) suggests that in the first round, CR of rate $I(U \land X)$ ought to be generated, and in the second round, additional CR of rate $I(V \land Y)$.

We use the same construction as in the proof of Theorem 4.1. First we generate $(u_{ij}, 1 \leq i \leq N_1, 1 \leq j \leq N_2)$ and associate with them functions $K_1(x)$ and $f(x)$ and $L_1(y, i)$ as there (we write $K_1$ and $L_1$ rather than $K$ and $L$, for now these functions will represent only the first part of the CR).

Then, by the proof of Theorem 4.1, for every pair $(x, y)$ not in the exceptional set $A' \cup B$ of arbitrarily small $P_{X,Y}$-probability, $K_1(x) = L_1(y, f(x))$.

Next, for each $u_{ij}$ as above, we generate at random $\exp[n(I(V \land Y|U) + \delta)]$ sequences $v \in \mathbb{V}^n$ of joint type with $u_{ij}$ equal to $v_{k_1}$, denoted as $v_{k_1}$, $1 \leq k_1 \leq M_1$, $1 \leq i \leq M_2$, where

$$M_1 = \exp[n(I(V \land Y|U) - I(V \land X|U) + 3\delta)]$$

$$M_2 = \exp[n(I(V \land X|U) - 2\delta)]. \hspace{1cm} (4.36)$$

Then for every $y \in \mathbb{Y}^n$ jointly $UY$-typical with $u_{ij}$, the probability that neither $v_{k_1}$ is jointly $UY$-typical with $(u_{ij}, y)$ is doubly exponentially small. Hence with probability close to 1, for every jointly typical pair $(u_{ij}, y)$ there is a $v_{k_1}$ such that $(u_{ij}, y, v_{k_1})$ is jointly typical; we denote by $L(y, u_{ij})$ such a $v_{k_1}$ (either one if there are several). Then for each $y$ and $1 \leq i \leq N_1$ we take for $u_{ij} = L_1(y, i)$ the unique $u_{ij}$ with the given first index $i$ which is jointly typical with $y$, or a constant if no or several such $u_{ij}$ exist, and define $L_2(y, i)$ as the $v_{k_1}$ selected for this $u_{ij}$ and $y$

$$L_2(y, i) = L(y, i) \cdot L_1(u_{ij}, y, i) = v_{k_1}. \hspace{1cm} (4.37)$$

Moreover, we define $g(y, i)$ to equal the first index $k_1$ of $v_{k_1}$ in (4.37). Finally, for $x \in \mathbb{X}^n$ and $1 \leq k_1 \leq M_1$ we define $K_2(x, k)$ as the unique $v_{k_1}$ jointly typical with $(u_{ij}, x)$
where \( u_k = K_1(x) \), or set \( K_2(x, k) = \text{const} \) if no or several such \( v \) exist.

Then, by (4.36), \( g \) satisfies the rate constraint (2.8) if \( \delta \) is sufficiently small. It is also clear that

\[
K = (K_1(X^n), K_2(X^n, g(Y^n, f(X^n)))) \\
L = (L_1(Y^n, f(X^n)), L_2(Y^n, f(X^n)))
\]

represent a permissible pair for Model iii), satisfying (2.4), and one shows as in the proof of Theorem 4.1 that

\[
\frac{1}{n} H(L) \geq I(U \wedge X) + I(V \wedge Y|U) - \delta.
\]

It remains only to show that the condition (2.3), i.e., \( \Pr \{K = L\} > 1 - \varepsilon \) is also satisfied, at least with large probability with respect to the random selections, \( \Pr \{K_1 = L_1\} > 1 - \varepsilon \) has already been demonstrated in the proof of Theorem 3.1. The remaining part \( \Pr \{K_2 = L_2\} > 1 - \varepsilon \) can be proved similarly, with a little more work.

V. COMMON RANDOMNESS, IDENTIFICATION, AND TRANSMISSION FOR ARBITRARILY VARYING CHANNELS

Recall the definition of an AVC in Section II by a class \( \mathcal{W} = \{W(\cdot, s), s \in \mathcal{S}\} \) of channels \( W(\cdot, s): \mathcal{X} \rightarrow \mathcal{Y} \). There also CR capacities, ID capacities, and transmission capacities have been defined for various models involving an AVC. We present now our results.

A. Model A: AVC Without Feedback and Any Other Side Information

First we recall some well-known results for transmission capacities, cf. [16].

A random code \( (C_1, \ldots, C_M, Q) \) is defined by deterministic codes \( C_1, \ldots, C_M \) of the same blocklength \( n \) and a PD \( Q \) on \( \{1, \ldots, M\} \), with the understanding that \( C_i \) will be used with probability \( Q(i) \). The error criterion is that the maximum or the average (for \( k \)) of \( \sum_{i=1}^{M} Q(i) c_k(i, s) \) be small for every \( s \in \mathcal{S} \), where \( c_k(i, s) \) denotes the probability of not decoding correctly the message \( k \) when the code \( C_i \) is used and the state sequence is \( s \). Both criteria lead to the same random code capacity \( C_R \). Notice that random codes can be used for transmission only if sender and receiver have access to CR, the outcome of a random experiment with distribution \( Q \).

It was shown in [8] that

\[
C_R = \max_{P} \min_{W \in \mathcal{W}} I(P; W) = \min_{W \in \mathcal{W}} C(W). \tag{5.1}
\]

Here \( I(P; W) \) denotes the mutual information of input and output RV's with joint distribution \( P(x)W(y|x) \), \( C(W) = \max_{P} I(P; W) \) is the Shannon capacity of the channel \( W \), and \( \mathcal{W} \) denotes the convex hull of \( W \).

By an elimination technique—based on an idea called “de-randomization” in computer science—it was shown in [1] that \( C_R \) can be attained by random codes \( (C_1, \ldots, C_M, Q) \) with \( M \) not larger than the capacity, the square of the blocklength \( n \) and with uniform \( Q \). As a consequence, the capacity for deterministic codes and the average probability of error criterion, denoted by \( \overline{C} \), satisfies

\[
\overline{C} = C_R, \quad \text{if } \overline{C} > 0. \tag{5.2}
\]

Random codes should be distinguished from codes with randomized encoding, which do not need CR, the decoding being deterministic. It was also shown in [1] that with randomized encoding, both the maximum and average error criteria lead to the same capacity, and

capacity under randomized encoding = \( \overline{C} \). \tag{5.3}

We note for later reference that (5.2) and (5.3) remain valid also for AVC's with noisy feedback, if \( \overline{C} \) is replaced by \( C_f \), the average error capacity for deterministic codes with feedback.

A necessary and sufficient condition for \( \overline{C} > 0 \), given in [1], is that for some \( n \) there exist PD's \( Q_1, Q_2 \) on \( \mathcal{X}^n \) and disjoint subsets \( D_1, D_2 \) of \( \mathcal{Y}^n \) such that

\[
\min_{s \in \mathcal{S}} \sum_{x \in \mathcal{X}^n} Q_1(x) W^n(D_1|x, s) > \frac{1}{2^n}, \quad i = 1, 2. \tag{5.4}
\]

A single-letter necessary and sufficient condition for \( \overline{C} > 0 \) was given in [10], \( \overline{C} > 0 \) iff \( \mathcal{W} \) is not symmetrizable, where symmetrizzability of \( \mathcal{W} \) means the existence of a channel \( \overline{U}: \mathcal{X} \rightarrow \mathcal{S} \) such that

\[
\sum_{s \in \mathcal{S}} U(s|x') W(y|x, s) = \sum_{s \in \mathcal{S}} U(s|x) W(y|x'), s \tag{5.5}
\]

every \( x, x' \) in \( \mathcal{X} \) and \( y \) in \( \mathcal{Y} \).

With these results and Theorems 3.2 and 3.3, the following theorem is readily obtained.

Theorem 5.1: For an AVC without feedback, both ID capacity and CR capacity with sender permitted to randomize are equal to average error transmission capacity for deterministic codes

\[
C_{1D} = C_{CR} = \overline{C}. \tag{5.6}
\]

Their common value equals \( C_R \) given by (5.1) if \( \mathcal{W} \) is not symmetrizable, and 0 otherwise.

Proof:

i) \( C_{CR} = \overline{C} \): the nontrivial part \( C_{CR} \geq \overline{C} \) follows from Theorem 3.2 and (5.3). Indeed, a permissible pair \( (K, L) \) for every choice of \( s \in \mathcal{S}^n \) gives rise to a code with randomized encoder of rate \( \frac{1}{n} \log |\mathcal{M}| \) and average probability of error < \( 2^e \).

ii) \( C_{1D} \geq C_{CR} = \overline{C} \): In the nontrivial case \( \overline{C} > 0 \), this is a consequence of Theorem 3.3 and of the fact that \( \overline{C} \) equals the maximum error capacity for randomized encoding.

iii) \( C_{1D} \leq \overline{C} \): Notice that \( C_{1D} > 0 \) implies \( \overline{C} > 0 \), because \( Q_1 \) and \( Q_2 \) as in (2.12) with \( D'_1 = D_1 \setminus D_2 \), \( D'_2 = D_2 \setminus D_1 \) satisfy (5.4) if \( e < 1/4 \) in (2.12). Thus on account of (5.2), it suffices to show that \( C_{1D} \leq C_R \). It follows from (2.12) that an \( (N, n, e) \) ID code for the AVC is, for each \( W \in \mathcal{W} \), also an \( (N, n, e) \) code for the DMC \( \{W\} \). Since the ID capacity of a DMC equals its transmission capacity, this and (5.1) imply the claimed inequality.
B. Model B: AVC with Noiseless (Passive) Feedback

Let $C_{CRF}$ and $C_{CRF}'$ denote the CR capacity and $C_{DF}$ and $C_{DF}'$ the identification capacity of the AVC with noiseless (passive) feedback, according as Terminal $X$ is permitted to randomize or not. As now $X$ knows everything that $Y$ does, $C_{CRF}$ equals the limit as $n \to \infty$ of the maximum, for all protocols as described in the passage containing (2.10), of

$$\frac{1}{n} \min_{s \in S^n} H(Y^n).$$

(5.7)

$C_{CRF}$ is obtained similarly, with the maximum taken for the deterministic protocols (formally, with $M = \text{const}$ in (2.10)).

**Theorem 5.2:** For an AVC with noiseless feedback,

$$C_{CRF} = \max_P \min_{W \in \mathcal{W}} H(PW)$$

(5.8)

$$C_{CRF}' = \max_P \min_{W \in \mathcal{W}} H(W|P), \quad \text{if } C_{CRF} > 0$$

(5.9)

$$C_{CRF} > 0, \text{ iff } \min_{W \in \mathcal{W}} H(W|\{x\}) > 0 \text{ for some } x \in X.$$  

(5.10)

Here $H(PW)$ and $H(W|P)$ denote the entropy $H(Y)$ and conditional entropy $H(Y|X)$ for RV's $X, Y$ with joint distribution $P(x)W(y|x)$. This proves that the right-hand side of (5.8) is an achievable CR rate. For the converse, we prove by induction that for any given protocol

$$\min_{s \in S^n} H(Y^k) \leq k \max_P \min_{W \in \mathcal{W}} H(PW)$$

(5.11)

for $k = 1, \ldots, n$. Indeed, (5.11) clearly holds for $k = 1$. Now, if $\min_{s \in S^n} H(Y^k)$ is attained for $s = s_1 \cdots s_k$, let $P$ denote the distribution of $X_k+1$ when (the given protocol is used) and $s_i = s_i, i = 1, \ldots, k$. Then

$$\min_{s \in S^n} H(Y^{k+1}) \leq \min_{s : s^* \in S^k} (H(Y^k) + H(Y_{k+1}))$$

$$\leq \min_{s \in S^n} H(Y^k) + \min_{W \in \mathcal{W}} H(PW).$$

(5.12)

Hence, (5.11) holds for $(k + 1)$ if it does for $k$.

**ii** For a deterministic protocol, when $X_i$ is a function of $Y^{i-1}$, we have

$$H(Y^n) = \sum_{i=1}^{n} H(Y_i|Y^{i-1}) = \sum_{i=1}^{n} H(Y_i|Y^{i-1}, X_i)$$

$$= \sum_{i=1}^{n} H(Y_i|X_i).$$

(5.13)

Using (5.13), an induction as above shows that the right-hand side of (5.9) is an upper bound to (5.7) for any deterministic protocol.

Now, let $P^*$ be the PD achieving the maximum in (5.9). Supposing $C_{CRF} > 0$, it follows from Theorem 3.2 that to any $\varepsilon > 0$ there exists $k = k(\varepsilon)$, a protocol of blocklength $k$, and a mapping $f$ of $Y^k$ into $X$, such that the distribution of $f(Y^k)$ differs by less than $\varepsilon$ from $P^*$, in variation distance, no matter what is the state sequence $s \in S^k$. We extend this protocol to blocklength $n$, by letting $X_i = f(Y^k)$ for $i = k + 1, \ldots, n$. Then, by (5.13), the limit of (5.7) as $n \to \infty$ will be arbitrarily close to the right-hand side of (5.9), if $\varepsilon > 0$ is sufficiently small.

**ii** Obviously, the condition in (5.10) is sufficient for $C_{CRF} > 0$. To prove its necessity, suppose indirectly that to each $x \in X$ there is an $s = s(x)$ such that $W(x, s)$ is the point mass at some $y = y(x)$. Given any deterministic protocol, consider $x \in X^n$, $s \in S^n$, and $y \in Y^n$ defined recursively such that $s_i = s(x_i), y = y(x)$, and $x_{i+1}$ is the input symbol that the given protocol specifies when the past output sequence is $y_1 \cdots y_i$. For this particular state sequence $s$, the given protocol leads to a unique output sequence $y$, proving that quantity (5.7) is equal to 0 for every deterministic protocol, hence $C_{CRF} = 0$.

Our result on the CR capacity leads to a noticeable conclusion for the classical transmission problem.

**Theorem 5.3:** The average error capacity $C_{IF}$ of an AVC with noiseless feedback, for deterministic coding, is always equal to $C_R$ given by (5.1). Further,

$$C_{DF} = C_{DF}, C_{DF} = C_{CRF}, \quad \text{if } C_R > 0$$

(5.14)

$$C_{DF} = C_{DF}, \quad \text{if } C_R = 0.$$  

(5.15)

**Proof:**

**i** The random code $(C_1, \ldots, C_M, Q)$ in the paragraph containing (5.2) can be used for transmission if $X$ and $Y$ have access to $2 \log n$ bits of robust UCR, i.e., to RV's $K, L$ satisfying (3.10) for every $s \in S^n$ with $|M| = n^2$. Since $C_R > 0$ implies $C_{CRF} > 0$, cf. (5.1) and (5.8), such UCR may be generated using a protocol of blocklength $n' = c \log n$, by Theorem 3.2. This proves that $C_R$ is an achieveable transmission rate, at least if randomization is permitted (randomization may be needed in the CR-generating protocol of negligible blocklength $n' = c \log n$, whose outcome will identify the $C_i$ actually used). The proof is completed by reference to the feedback versions of (5.2) and (5.3).

**ii** If $C_{IF} = C_R > 0$, the inequalities $C_{DF} \geq C_{CRF}, C_{DF} \geq C_{DF}$ are proved analogously to the proof of Theorem 5.1, part ii). The reversed inequalities follow by the method of [5], where the 1D capacity of a DMC with feedback has been determined. If $C_R = 0$ then $C(W) = 0$ for some $W \in \mathcal{W}$. Then the feedback 1D capacity of the DMC $\{W\}$ is 0 by [5], and (5.15) follows.

C. Model C: Strongly Arbitrarily Varying Channel (SAVC)

It is assumed here that the jammer can make his choice of $s \in S^n$ depend on the sent $x \in X^n$. Formally, the parameter determining the statistics is now an arbitrary mapping from $X^n$ to $S^n$. 
Since the number of such mappings $s$ is doubly exponential in $n$, the hypothesis of Theorem 3.2 is still satisfied. The criterion (2.11) for an $(N, n, \varepsilon)$ ID code becomes
\[
\sum_{x \in \mathcal{X}^n} Q_j(x) \max_{s} W^n(D_j| x, s) \leq \varepsilon
\]
for any $N, n, \varepsilon$ transmission code with randomized encoding.

Any $(N, n, \varepsilon)$ transmission code with randomized encoding gives rise to a deterministic $(N, n, \varepsilon)$ code, with codewords $x_j = \arg \min_{s} \max_{a} W^n(D_j| x, s)$.

Hence the maximum error capacity of an SAVC for deterministic and randomized encoding is the same. It is also (well known and) easy to see that this capacity coincides with the average error capacity for deterministic codes, and it equals the maximum error capacity for deterministic codes of the AVC defined by the same $W$. We shall denote this capacity by $\overline{C}$.

As shown in [14], $\overline{C} > 0$ if, and only if, there exists $x, x'$ in $\mathcal{X}$ with $T(x) \cap T(x') = \emptyset$ where $T(x)$ denotes the convex hull of the set of PD's $W(x, s), s \in S$.

The row-convex hull $\overline{W}$ of $W$ is the set of all channels $W: \mathcal{X} \to \mathcal{Y}$ such that $W(x) = T(x), x \in \mathcal{X}$. Write
\[
D = \min_{W \in \overline{W}} C(W).
\]

Theorem 5.4: For an SAVC, the CR capacity $C_{CR}$ and ID capacity $C_{ID}$ (with $\varepsilon$ permitted to randomize) satisfy
\[
\overline{C} \leq C_{CR} \leq C_{ID} \leq D
\]  
(5.18)
\[
C_{ID} \geq 0, \quad \text{if } \overline{C} > 0.
\]  
(5.19)

Remark: Under not too restrictive hypotheses, $\overline{C} = D$, cf. [21] for $W$ satisfying $T(x) \cap T(x') = \emptyset$ whenever $x \neq x'$, and [13] under a weaker hypothesis: there are, however, examples of $0 < \overline{C} < D$. For $W$ with $\overline{C} = D$, Theorem 5.4 gives a conclusive result, but we do not know whether $C_{ID} = C_{CR}$ and/or $C_{CR} = \overline{C}$ hold for every SAVC. $C_{CR}$ always equals the average error capacity for randomized encoding, but it appears unknown whether the latter can ever be larger than $\overline{C}$.

Proof: The first inequality of (5.18) is obvious, and if $\overline{C} > 0$, the second inequality follows from Theorem 3.3. It remains to prove that $C_{ID} \leq D$ and that $\overline{C} = 0$ implies $C_{CR} = C_{ID} = 0$.

Consider an auxiliary model where at each instant $i$ the state $s_i$ may depend on $x_i$ but not on the other $x'_j$'s. Formally, this is an AVC model, with state set $\mathcal{S}^*$ consisting of all mappings $s^*: \mathcal{X} \to \mathcal{S}$, defined by the set of channels
\[
W^* = \{W^*(\cdot, \cdot), s^* \in S^* \} \quad W^*(\cdot, s^*) = W^*(\cdot, s^*(\cdot)).
\]  
(5.20)

Clearly, the CR and ID capacities of this AVC are upper bounds to $C_{CR}$ and $C_{ID}$. Thus on account of Theorem S.1, it is sufficient to show that i) the random code capacity of the AVC defined by (5.20) equals $D$ and ii) $W^*$ is symmetrizable if $\overline{C} = 0$.

i) is obvious from (5.1) and (5.17) since $\overline{W} = \overline{\mathcal{W}}$.

To prove ii), use either the Strong Separation Lemma of [11] or, alternatively, recall that $C = 0$ iff $T(x) \cap T(x')$ is never empty, i.e., for suitable PD's $U_j(x, x')$ on $\mathcal{S}$,
\[
\sum_{x \in \mathcal{S}} U(x|x', y)W(y|x, s) = \sum_{x \in \mathcal{S}} U(x|x', y)W(y|x', s)
\]  
(5.21)
for every $x, x'$, and $y$. Equation (5.21) means that $W^*$ satisfies (5.5), with $U^*: \mathcal{X} \to \mathcal{S}^*$ defined by
\[
U^*(s^*|x) = \prod_{x \in \mathcal{X}} U(s^*(x)|x, x).
\]  
(5.22)

Remark: Work relevant for problems concerning feedback with noise can be found in [20].

APPENDIX

Proof of Lemma 3.1: Choosing $f$ at random as in the proof of Lemma 1.1, with $Z_i(v)$ as there, we have
\[
P(f^{-1}(i) \cap E(P, d)) = \sum_{v \in E(P, d)} p(v)Z_i(v).
\]  
(A.1)

Chernoff bounding gives that for any $A \subset \mathcal{V}$
\[
Pr \left\{ \sum_{v \in A} p(v)Z_i(v) > \frac{1 + \varepsilon}{k} P(A) \right\}
\]
\[
= Pr \left\{ \exp \left[ \frac{\beta}{k} \sum_{v \in A} p(v)Z_i(v) \right] > \exp \left( \frac{\beta}{k} \frac{1 + \varepsilon}{k} P(A) \right) \right\}
\]
\[
\leq E \left\{ \exp \left[ \frac{\beta}{k} \sum_{v \in A} p(v)Z_i(v) \right] \right\} \exp \left( - \frac{\beta}{k} \frac{1 + \varepsilon}{k} P(A) \right)
\]
\[
= \exp \left( - \frac{1 + \varepsilon}{k} P(A) \right) \prod_{v \in A} \left[ 1 + \frac{1}{k} (\exp(\beta p(v)) - 1) \right]
\]  
(A.2)

where $\beta > 0$ is arbitrary, and similarly
\[
Pr \left\{ \sum_{v \in A} p(v)Z_i(v) < \frac{1 - \varepsilon}{k} P(A) \right\}
\]
\[
\leq \exp \left( \frac{\beta}{k} \frac{1 - \varepsilon}{k} P(A) \right) \prod_{v \in A} \left[ 1 + \frac{1}{k} (\exp(-\beta p(v)) - 1) \right].
\]  
(A.3)

Apply (A.2) to $A = E(P, d)$ with $\beta = \varepsilon d$. Then for $v \in A = E(P, d)$ we have $\beta p(v) < \varepsilon$, by (3.1), and, therefore,
\[
\exp (\beta p(v)) - 1 = \sum_{j=1}^{\infty} \frac{(\beta p(v) \ln 2)^j}{j!}
\]
\[
< \beta p(v) \left[ 1 + \frac{1}{2} \sum_{j=1}^{\infty} (\varepsilon \ln 2)^j \ln 2 \right] \ln 2
\]
\[
= \beta p(v) (1 + \varepsilon^*) \ln 2
\]
where
\[
\varepsilon^* = \frac{\ln 2}{2(1 - \varepsilon \ln 2)}.
\]
Using the inequality $1 + t \ln 2 \leq \exp t$, it follows that the last product in (A.2) is upper-bounded by
\[
\exp \left[ \sum_{v \in E(P, d)} \frac{1}{k} \beta p(v) (1 + \varepsilon^*) \right] = \exp \left[ \frac{\varepsilon^*}{k} (1 + \varepsilon^*) P(E(P, d)) \right].
\]
Thus (A.2) gives, using the assumption (3.2) and recalling that $\beta = ed$,
\[
\Pr \left\{ \sum_{v \in E(P, d)} p(v) Z_v(v) > \frac{1 + \varepsilon}{k} P(E(P, d)) \right\} < \exp \left[ -\frac{\varepsilon}{k} (1 + \varepsilon^*) P(E(P, d)) \right] < \exp \left( -\frac{\varepsilon^2 d}{3k} \right). \tag{A.4}
\]
Here, in the last step, we used that
\[
(e - \varepsilon^*)(1 - e) = 1 - \frac{\ln 2}{2(1 - \ln 2)} (1 - e) > \frac{\varepsilon}{3}
\]
if $\varepsilon < 3 - 2 \ln 2$, and that condition does hold by the assumption $\varepsilon \leq \frac{1}{3}$. It follows from (A.3) in a similar but even simpler way (as $\exp(-\beta p(v))$ can be bounded by $\beta p(v)(-1 + \frac{1}{2} \varepsilon \ln 2) \ln 2$) that the left-hand side of (A.3) is also bounded by $\exp(-\varepsilon^2/3kd)$. Recalling (A.1), we have thereby shown that the probability that (3.3) does not hold for a randomly chosen $f$ is $< 2N \exp(-\varepsilon^2/3kd)$. Hence this probability is less than $1$ if $k \leq \varepsilon^2/3 \ln 2$. This completes the proof of Lemma 3.1, because (3.4) is an immediate consequence of (3.3).

A. Completion of the Proof of Theorem 3.1

We have to show that the $P$-probability of the set (3.1) with $P = P(\{s\})$ defined by (2.9) is $\leq \varepsilon$ if $d$ is as in (3.7), with $\xi$ given by (3.8). This probability can be written as

\[
\Pr \{ P_{\{s\}} > \exp \left[ -n(H_{\text{min}} - \xi) \right] \} \tag{A.5}
\]

where $\Pr$ denotes probability under $P(\{s\})$. Now, for every $t > 0$

\[
\Pr \{ P_{\{s\}} > \exp \left[ -n(H_{\text{min}} - \xi) \right] \}
= \Pr \{ P_{\{s\}} > \exp \left[ -n(H_{\text{min}} - \xi) \right] \}
< \exp \left[ -n(H_{\text{min}} - \xi) \right] E(P_{\{s\}})
= \exp \left[ -n(H_{\text{min}} - \xi) \right] \prod_{i=1}^{n} \sum_{x \in X} P^{1+t}(x|s_i) \tag{A.6}
\]

where $E$ denotes expectation under $P(\{s\})$. To bound the last product in (A.6), notice that for any $\mathcal{P}_x = \{p(x)\}$ on $\mathcal{X}$

\[
\sum_{x \in X} P^{1+t}(x) = \sum_{x \in X} p(x) \sum_{j=0}^{\infty} \frac{(t \ln p(x))^j}{j!}
\leq 1 + t \sum_{x \in X} p(x) \ln p(x)
+ \frac{t^2}{2} \sum_{x \in X} (p(x) \ln p(x))^2
\leq 1 - t H(P) \ln 2
+ \frac{t^2}{2} \sum_{x \in X} (p(x) \ln p(x))^2. \tag{A.7}
\]

Calculus shows that the last sum in (A.7) is maximum when $P$ is the uniform distribution on $\mathcal{X}$, providing $|\mathcal{X}| \geq 3$. Hence,

\[
\sum_{x \in \mathcal{X}} P^{1+t}(x) \leq 1 - t H(P) \ln 2 + \frac{t^2}{2} (\ln |\mathcal{X}|)^2 \leq \exp \left[ -t H(P) \ln 2 + \frac{t^2}{2} (\ln |\mathcal{X}|)^2 \ln 2 \right] \tag{A.8}
\]

with the understanding (as also in the rest of the proof) that $|\mathcal{X}|$ should be replaced by $3$ if $|\mathcal{X}| = 2$.

As $H(P(s)) \geq H_{\text{min}}$ by definition, (A.6) and (A.8) give that the probability (A.5) is upper-bounded by

\[
\exp \left[ -n \xi + n(t^2/2) (\ln |\mathcal{X}|)^2 \ln 2 \right]
\]
for each $t > 0$. Setting $t = \xi/(\log |\mathcal{X}|)^2 \ln 2$, we get

\[
\Pr \{ P_{\{s\}} \} > \exp \left[ -n(H_{\text{min}} - \xi) \right]
< \exp \left[ -\frac{n \xi^2}{2(\log |\mathcal{X}|)^2 \ln 2} \right]. \tag{A.9}
\]

For $\xi$ given by (3.8), the right-hand side of (A.9) is equal to $\epsilon/3$, establishing our claim.

Completion of the Proof of Theorem 4.4

Here we show that to any RV’s $U, V$ satisfying the Markov conditions (4.26) there exist $\tilde{U}, \tilde{V}$ satisfying the same conditions, with range sizes $|U| \leq |\mathcal{X}| + 2, |V| \leq |\mathcal{Y}|$ such that

\[
I(\tilde{U} \wedge \tilde{X}) - I(\tilde{U} \wedge \tilde{Y}) = I(U \wedge X) - I(U \wedge Y) \tag{A.10}
\]

\[
I(\tilde{V} \wedge \tilde{X}|\tilde{U}) - I(\tilde{V} \wedge \tilde{Y}|\tilde{U}) \leq I(V \wedge Y|U) - I(V \wedge X|U) \tag{A.11}
\]

\[
I(\tilde{U} \wedge \tilde{X}) + I(\tilde{V} \wedge \tilde{Y}|\tilde{U}) \geq I(U \wedge X) + I(V \wedge Y|U). \tag{A.12}
\]

1) Given $U, V$ satisfying (4.26), introduce an equivalence relation on $U$ by letting $u_1 \sim u_2$ iff $P_{X|u_1=u_2} = P_{X|u_1=u_2}$. Our first claim is that $U, V$ can be replaced by $\tilde{U}, \tilde{V}$ without changing the relevant mutual informations, such that no distinct elements of the range of $U'$ are equivalent in the above sense.

Let $f(u)$ denote the equivalence class of $u$. Then clearly

\[
P_{XY|f(u)} = P_{XY|f(u)} \tag{A.13}
\]

hence

\[
I(U \wedge X) = I(f(U) \wedge X), I(U \wedge Y) = I(f(U) \wedge Y).
\]
This, in turn, implies that
\[ I(V \land X | U) = I(UV \land X) - I(U \land X) \]
\[ = I(UV f(U) \land X) - I(f(U) \land X) \]
\[ = I(UV \land X | f(U)) \]
and similarly
\[ I(V \land Y | U) = I(UV \land Y | f(U)). \]
Thus \( U^* = f(U), V^* = UV \) satisfy our claim, since \( U^* \land X \land Y \) is obvious from \( U \land X \land Y \) and (A.13), and \( X \land Y U \land V^* \) follows as
\[ P_X | Y = y, f(U) = f(u), U = u, V = v = P_X | Y = y, U = u, V = v \]
\[ = P_X | Y = y, U = u \]
where the second equality holds by \( X \land Y U \land V \) and the third by (A.13).

2) If \( 1 \), it suffices to consider \( U, V \) satisfying (4.26) such that the PD's \( P_{U} = P_{X | Y = y, U \in U} \), \( U \in U \) are all distinct. This will enable us to use the Support Lemma (see [19] or [16, p. 310]) to reduce the range size of \( U \). To this end, define the stochastic matrix valued function \( F(P) \) for \( P \in \{ P_{U} : U \in U \} \) by letting \( \text{Pr}(V = v | Y = y, U = u) \) be the \( (y, v) \) entry of \( F(P) \). Then extend \( F(P) \) continuously but otherwise arbitrarily to the set \( P(X) \) of all PD's on \( X \). Now apply the Support Lemma to the following functions on \( P(X) \):
\[ f_1(P) = H(X) - H(Y) - H(P) + H(P_W) \]
where \( W = F_1X \)
\[ f_2(P) = H(X) - H(P) + I(P_W, F(P)) \]
\[ f_3(P) = I(P_W, F(P)) - I(P, W | F(P)) \]
\[ f_4(P) = P_{x_1 \ldots x_i} \]
where \( X = \{ x_1, \ldots, x_i \} \).

It follows that there exist PD's
\[ P_i \in P(X), \ i = 1, \ldots, \ |X| + 2 \]
and a PD \( \{ c_1, \ldots, c_{|X|+2} \} \) on \( \{ 1, \ldots, |X| + 2 \} \) such that
\[ \sum_{u \in U} \text{Pr}(U = u) f_2(P_u) = \sum_{i=1}^{|X|+2} a_i f_2(P_i), \ j = 1, \ldots, |X| + 2. \] (A.14)

The last \( |X| - 1 \) identities in (A.14) mean that an RV \( U \) with range \( U = \{ 1, \ldots, |X| + 2 \} \) and distribution \( \{ c_1, \ldots, c_{|X|+2} \} \) exists such that
\[ P_X | Y = y = P_i, \ i = 1, \ldots, |X| + 2. \]

Letting this \( U \) satisfy \( U \land X \land Y \), the first identity in (A.14) gives (A.10). Further, letting \( V \) be such that \( X \land Y \land V \) is \( F(P) \), the second and third identities in (A.14) mean that (A.11) and (A.12) hold with equality.

3) Finally, it remains to show that \( V \) in 2) can be replaced by some \( V' \) with range size \( \leq |Y| \) and \( X \land Y \land V' \) such that \( V' \) still satisfies (A.11) and (A.12). Now, by the range constraint result of Theorem 4.1, applied to RV's with joint distribution \( P_{X | Y = y, U_i = u} \) in the role of \( X, Y \), for each fixed \( u \in U \) there exists \( V_0 \) distributed on a set of size \( \leq |Y| \) and conditionally independent of \( X \) on the conditions \( Y = y, U = u \), for every \( y \in Y \), such that
\[ I(V_0 \land Y | U = u) \geq I(\hat{V} \land Y | U = u) \]
\[ I(V_0 \land Y | U = u) - I(V_0 \land X | U = u) \]
\[ \leq I(V \land Y | U = u) - I(V \land X | U = u). \]

But then we can define an RV \( \hat{V} \) with \( X \land Y \land \hat{V} \) such that
\[ P_{\hat{V} | Y = y, U = u} = P_{V_0 | Y = y, U = u} \]
for every \( y \in Y, u \in U \). This \( \hat{V} \) of range size \( \leq |Y| \) will satisfy the last inequalities for every \( u \in U \), and hence also (A.11) and (A.12), as required.

REFERENCES