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Code pairs with specified parity of the Hamming distances

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Abstract

For code pairs (A,B); $A,B \subset \{0,1,\dots,\alpha-1\}^n$; with mutually constant parity of the Hamming distances a conjecture of the first author concerning the maximal value of $|A||B|$ is established.

Keywords: Constant distance pair of codes; Parity of the Hamming distances; Monochromatic rectangles; Extremal problems

1. Introduction and results

Constant distance codes have been investigated in [7, 9]. The study of pairs of codes with mutually constant distances was initiated in [3] and is continued in Refs. [4, 5]. Weakening of the constant distance property led via [4, 5] to the quite general 4-words inequality of [2]. In another direction, in [1] constant distance code pairs where analysed for *specified* distances and also for *non-binary* alphabets. There, also extremal problems with constant *parity* of the Hamming distance were considered. We quickly report the *results* and conjectures.

$\mathscr{X}_\alpha = \{0,1,\dots,\alpha-1\}$ is a finite set or alphabet. The pair (A,B) with $A,B \subset \mathscr{X}_\alpha^n = \prod_1^n \mathscr{X}_\alpha$ is called an (n,δ)-system (or constant distance code pair with parameters n,δ), if for the Hamming distance function d

$$d(a,b) = \delta \quad \text{for all } a \in A,\ b \in B.$$

Let $\mathscr{S}_\alpha(n,\delta)$ denote the set of those systems and set

$$M_\alpha(n,\delta) = \max\{|A|\,|B| :\ (A,B) \in \mathscr{S}_\alpha(n,\delta)\}, \tag{1.1}$$

$$M_\alpha(n) = \max_{0 \leqslant \delta \leqslant n} M_\alpha(n,\delta). \tag{1.2}$$

The discovery of [3] was

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Theorem AGP.

$$M_2(n) = \begin{cases} 2^n, & \text{if } n \text{ is even} \\ 2^{n-1}, & \text{if } n \text{ is odd}. \end{cases}$$

Next, in [9] $M_\alpha(n,\delta)$ has been related to the functions $F_\alpha(n,\delta)$, where

$$F_2(n,\delta) = \max_{\delta_1+\delta_2=\delta} (4)^{\delta_1} \binom{n-2\delta_1}{\delta_2}, \quad 4 = 2\cdot 2! \tag{1.3}$$

$$F_3(n,\delta) = \max_{2\delta_1+\delta_2=\delta} (18)^{\delta_1} \binom{n-3\delta_1}{\delta_2} 2^{\delta_2}, \quad 18 = 3\cdot 3! \tag{1.4}$$

$$F_\alpha(n,\delta) = \max_{\delta_1+\delta_2=\delta} \left(\left\lfloor\frac{\alpha}{2}\right\rfloor \left\lceil\frac{\alpha}{2}\right\rceil\right)^{\delta_1} \binom{n-\delta_1}{\delta_2} (\alpha-1)^{\delta_2} \quad \text{for } \alpha\geqslant 4. \tag{1.5}$$

Theorem A$_1$. *For $n\in\mathbb{N}$, $0\leqslant\delta\leqslant n$*
(i) $M_2(n,\delta)=F_2(n,\delta)$.
(ii) $M_\alpha(n,\delta)=F_\alpha(n,\delta)$ *for* $\alpha=4,5$.

Conjecture A$_1$.
(iii) $M_3(n,\delta)=F_3(n,\delta)$.
(iv) $M_\alpha(n,\delta)=F_\alpha(n,\delta)$ for $\alpha\geqslant 6$.

Finally we come to the subject of this paper, namely, code pairs with a parity constraint. It is convenient to introduce the function $\Psi : \mathbb{N}\cup\{0\}\to\{0,1\}$, where

$$\Psi(n) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases} \tag{1.6}$$

We consider the parity function Π: $\bigcup_{n=1}^{\infty}(\mathscr{X}_\alpha^n\times\mathscr{X}_\alpha^n)\to\{0,1\}$ defined by

$$\Pi(x^n,y^n) = \Psi\big(d(x^n,y^n)\big). \tag{1.7}$$

The pair (A,B) with $A,B\in\mathscr{X}_\alpha^n$ is said to have p-parity, if

$$\Pi(a,b)=p \quad \text{for all } a\in A, b\in B. \tag{1.8}$$

For $p=0,1$ let $\mathscr{P}_\alpha^p(n)$ denote the set of those p-parity pairs and define

$$Q_\alpha^p(n)=\max\{|A||B| : (A,B)\in\mathscr{P}_\alpha^p(n)\}, \tag{1.9}$$

$$Q_\alpha(n)=\max_{p=0,1} Q_\alpha^p(n). \tag{1.10}$$

This last quantity is known for all n and $\alpha\neq 3$, and $Q_\alpha^p(n)$ is almost known.

Theorem A$_2$ (Ahlswede [1]). *For $n\in\mathbb{N}$ and $\bar\alpha=\lfloor\frac{\alpha}{2}\rfloor\cdot\lceil\frac{\alpha}{2}\rceil$*
(a) $Q_\alpha^p(n)=\bar\alpha^n$, *if* $\Psi(n)=p$ $(\alpha\geqslant 4; p=0,1)$,
(a′) $\bar\alpha^{n-1}\leqslant Q_\alpha^p(n)<\bar\alpha^n$, *if* $\Psi(n)\neq p$ $(\alpha\geqslant 4; p=0,1)$,

(a″) $Q_\alpha(n) = \bar{\alpha}^n$ $(\alpha \geq 4)$,
(b) $Q_2(n) = Q_2^0(n) = Q_2^1(n) = 4^{n-1}$.

For $\alpha = 3$ we have

Conjecture A$_2$.
(c) $Q_3^p(n) = (2^{n-1}+1)(2^{n-1}+1)$, if $\Psi(n) = p = 0$.
(c′) $Q_3^p(n) = (2^{n-1}+1)2^{n-1}$, if $\Psi(n) = p = 1$.
(c″) $Q_3^p(n) = 2^{n-1}\cdot 2^{n-1}$, if $\Psi(n) \neq p$ and $n \neq 3$.

In the exceptional case $\underline{n = 3, p = 0}$ not covered, one readily verifies that

$$(A,B) = (\{111,222,333\},\ \{\text{all permutations of } 123\})$$

is optimal and that therefore $Q_3^0(3) = 18$. A first insight can be gained from the following key tool of [1]. For $B \subset \mathscr{X}_2^n$ and $T \subset \{1,2,\ldots,n\}$ we say that B has parity on T, if the projection $\mathrm{Proj}_T B$ on $\prod_{t\in T}\mathscr{X}_2$ contains only sequences with an odd or only sequences with an even number of ones.

Lemma (Blockwise parity property)

$$\sum_{\substack{T\subset\{1,2,\ldots,n\}\\ B \text{ has parity on } T}} 2^{|T|}|B| \leqslant (2^n+1)2^{n-1} \quad \textit{for every } B \subset \mathscr{X}_2^n.$$

The right-hand bound is assumed, for instance, if B equals the set of all sequences with an even number of ones. The result of this paper is

Theorem. *Conjecture A_2 is true.*

Finally, we draw attention to an open *problem*. For single sets, A has p-parity, if

$$\Pi(a,a') = p \quad \text{for } a,a' \in A \text{ with } a \neq a'.$$

The quantity $q_\alpha^p(n) = \max\{|A| : A \subset \mathscr{X}_\alpha^n \text{ has } p\text{-parity}\}$ has been determined in [1] for $p = 0$ (and all values for α and n). There are only bounds for $q_\alpha^1(n)$. Determine $q_\alpha^1(n)$!

2. Proof of Theorem: the direct part

Our alphabet is $\mathscr{X}_3 = \{0,1,2\}$. Let us define

$$\langle a|x\rangle = \text{number of occurrences of letter } x \text{ in word } a. \tag{2.1}$$

We need the sets

$$\mathscr{E}_2(n) = \{a \in \mathscr{X}_2^n : \langle a|1\rangle \text{ is even}\}, \qquad \mathscr{O}_2(n) = \{a \in \mathscr{X}_2^n : \langle a|1\rangle \text{ is odd}\},$$

and the word

$$\underline{2} = (2,2,\ldots,2) \in \mathscr{X}_3^n. \tag{2.2}$$

We show first how the values for $Q_3^p(n)$ specified in Conjecture A_2 can be achieved. For this choose

(c) $(A,B) = \left(\mathscr{E}_2(n) \cup \{\underline{2}\}, \mathscr{E}_2(n) \cup \{\underline{2}\}\right)$

(c′) $(A,B) = \left(\mathscr{E}_2(n) \cup \{\underline{2}\}, \mathscr{O}_2(n)\right)$ or $(A,B) = \left(\mathscr{E}_2(n), \mathscr{O}_2(n) \cup \{\underline{2}\}\right)$

(c″) $(A,B) = \left(\mathscr{E}_2(n), \mathscr{E}_2(n)\right)$ or $\left(\mathscr{O}_2(n), \mathscr{O}_2(n)\right)$, if $p = 0$, and $(A,B) = \left(\mathscr{O}_2(n), \mathscr{E}_2(n)\right.$, if $p = 1$.

3. Proof of Theorem: the converse part

3.1. Basic concepts and their properties

For $A, B \subset \mathscr{X}_3^n$ define

$$A_{st}^{ij} = \left\{ x^{n-2} \in \prod_{\substack{l \neq i,j \\ 1 \leqslant l \leqslant n}} \mathscr{X}_3 : (x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) \in A \right\}$$

and similarly define B_{st}^{ij}. For simplicity, we consider A_{st}^{12}, B_{st}^{12} and denote them by A_{st} and B_{st}, respectively. Define

$$I = \left\{ \left\{ \begin{pmatrix} s_1 t_1 \\ s_2 t_2 \end{pmatrix} \right\} : A_{s_1,t_1} \cap A_{s_2,t_2} \neq \emptyset \right\}.$$

For

$$a = \begin{pmatrix} s_1 t_1 \\ s_2 t_2 \end{pmatrix},$$

define

$$\mathscr{S}(a) = \{(s,t) : (s,t) \text{ has the same parity with both } (s_1,t_1) \text{ and } (s_2,t_2)\},$$

$$\mathscr{S}(I) = \bigcap_{a \in I} \mathscr{S}(a).$$

Similarly define J, $\mathscr{S}(b)$, and $\mathscr{S}(J)$. Now, if (A,B) has p-parity these sets must have the following properties

(1) $\{(s_1,t_1),(s_2,t_2)\} \in I \;\Rightarrow\; (s_1,t_1),(s_2,t_2) \in \mathscr{S}(J)$,
$\{(s_1,t_1),(s_2,t_2)\} \in J \;\Rightarrow\; (s_1,t_1),(s_2,t_2) \in \mathscr{S}(I)$,

(2) $(s,t) \notin \mathscr{S}(I) \;\Rightarrow\; B_{st} = \emptyset$,
$(s,t) \notin \mathscr{S}(j) \;\Rightarrow\; A_{st} = \emptyset$.

A pair (I,J), which satisfies these properties, is called *matching*, and $(\mathscr{S}(I),\mathscr{S}(J))$ is called *proper*. If (I_1,J_1), (I_2,J_2) are two matching pairs, and $\mathscr{S}(I_1)=\mathscr{S}(I_2)$, $\mathscr{S}(J_1)=\mathscr{S}(J_2)$, then $(I_1\cup I_2,J_1\cup J_2)$ is also a matching pair, and

$$\mathscr{S}(I_1\cup I_2)=\mathscr{S}(I_1)=\mathscr{S}(I_2),\qquad \mathscr{S}(J_1\cup J_2)=\mathscr{S}(J_1)=\mathscr{S}(J_2).$$

This means that given a proper pair $(\mathscr{S}(I),\mathscr{S}(J))$, there exists a maximal matching pair (I,J) reaching this proper pair. Any other matching pair (I',J') such that $\mathscr{S}(I')=\mathscr{S}(I)$, $\mathscr{S}(J')=\mathscr{S}(J)$ satisfies

$$I'\subset I,\qquad J'\subset J.$$

Now we explain why we define these concepts. We are going to use induction to prove the conjecture. We need to divide $\mathscr{S}(I)\times\mathscr{S}(J)$ into smaller rectangles such that for each rectangle

$$\{(s_{11},t_{11}),\dots,(s_{1m},t_{1m})\}\times\{(s_{21},t_{21}),\dots,(s_{2k},t_{2k})\}$$

$\{(s_{1i},t_{1i}),(s_{2j},t_{2j})\}$ should have the same parity. Denote the number of such rectangles, which have parity 0, by α, and the number of such rectangles, which have parity 1, by β. These rectangles cover the whole $\mathscr{S}(I)\times\mathscr{S}(J)$. Therefore we obtain

$$Q_3^p(n)\leqslant \alpha Q_3^p(n-2)+\beta Q_3^{\bar p}(n-2), \tag{3.1}$$

where $\bar p = 1+p \bmod 2$. But these rectangles must have the property that

$$\{(s_{1i},t_{1i}),(s_{1j},t_{1j})\}\notin I \quad\text{and}\quad \{(s_{2i},t_{2i}),(s_{2j},t_{2j})\}\notin J,$$

because for the pairs $(s_1,t_1),(s_2,t_2)$ in I

$$A_{s_1,t_1}\cap A_{s_2,t_2}\neq\emptyset.$$

For the maximal matching pair we denote the corresponding α and β by $\alpha(\mathscr{S}(I),\mathscr{S}(J))$ and $\beta(\mathscr{S}(I),\mathscr{S}(J))$. Then any other matching pair (I',J'), which has the same proper pair $(\mathscr{S}(I),\mathscr{S}(J))$, must satisfy

$$\alpha\leqslant\alpha(\mathscr{S}(I),\mathscr{S}(J)),\qquad \beta\leqslant\beta(\mathscr{S}(I),\mathscr{S}(J)). \tag{3.2}$$

This means that for the induction we need to consider only the maximal matching pairs.

3.2. Determination of all the proper pairs and their corresponding maximal matching pairs

Lemma 1. *We have the following maximal matching pairs:*

(1) $I=\emptyset$, $J=\emptyset$, $\mathscr{S}(I)=\mathscr{S}(J)=\Omega\triangleq\mathscr{X}_3^2$.

(2) $I=\{(00,11),(01,10),(02,20),(00,22),(00,12),(00,21),(11,21),(11,12),(12,21),(22,12),(22,21),(11,22),(01,02),(01,20),(02,10),(10,20)\}$, $J=\emptyset$, $S(I)=\{(00)\}$, $S(J)=\Omega$.

(3) $I = \{(22,12),(00,11),(00,21),(11,21),(01,10),(01,20),(10,20)\}$,
$J = \emptyset$, $\mathscr{S}(I) = \{(00),(01)\}$, $\mathscr{S}(J) = \Omega$.

(4) $J = \emptyset$, $\mathscr{S}(J) = \Omega$, $I = \{(00,11),(00,22),(11,22),(01,10),(02,20),(12,21)\}$,
$\mathscr{S}(I) = \{(00),(11),(22)\}$.

(5) $I = J = \{(00,11),(00,22),(11,22)\}$, $\mathscr{S}(I) = \mathscr{S}(J) = \{(00),(11),(22)\}$.

(6) $J = \{(00,11)\}$, $I = \{(00,11),(00,22),(11,22),(01,10)\}$,
$\mathscr{S}(I) = \{(00),(11),(22),(01),(10)\}$, $\mathscr{S}(J) = \{(00),(11),(22)\}$.

(7) $I = \{(00,01),(10,11),(20,21)\}$, $J = \emptyset$,
$\mathscr{S}(I) = \{(02),(12),(22)\}$, $\mathscr{S}(J) = \Omega$.

(8) $I = \{(20,21)\}$, $J = \{(02),(12)\}$,
$\mathscr{S}(I) = \{(02),(12),(22)\}$, $\mathscr{S}(J) = \{(20),(21),(22)\}$.

(9) $J = \emptyset$, $I = \{(00,11),(01,10)\}$,$\mathscr{S}(J) = \Omega$, $\mathscr{S}(I) = \{(00),(11),(22),(01),(10)\}$.

(10) $I = J = \{(00,11),(01,10),\}$, $\mathscr{S}(I) = \mathscr{S}(J) = \{(00),(11),(22),(01),(10)\}$.

Proof. We search for the maximal matching pairs in the following procedure. First, obviously $(\emptyset,\emptyset)$ should be such a pair and it corresponds to the proper pair (Ω,Ω), where

$$\Omega = \{(00),(01),(10),(11),(02),(20),(12),(21),(22)\}.$$

For other cases we assume $I \neq \emptyset$. Without loss of generality we can assume that either $\{(00),(11)\} \in I$ or $\{(00),(01)\} \in I$, because other cases are equivalent to one of these two cases.

$$\text{Now if } \{(00),(11)\} \in I, \text{ then } \mathscr{S}(I) \subset \{(00),(11),(22),(01),(10)\}, \text{ and}$$
$$\text{if } \{(00),(01)\} \in I, \text{ then } \mathscr{S}(I) \subset \{(02),(12),(22)\}.$$

Subcase $\min\{|\mathscr{S}(I)|,|\mathscr{S}(J)|\} = 1$. There is only one class, that is $\mathscr{S}(I) = \{(00)\}$, $J = \emptyset$, and

$$I = \{(00,11),(01,10),(02,20),(00,22),(00,12),(00,21),(11,21),(11,12),$$
$$(12,21),(22,12),(22,21),(11,22),(01,02),(01,20),(02,10),(10,20)\}.$$

Subcase $\min\{|\mathscr{S}(I)|,|\mathscr{S}(J)|\} = 2$. There are two possibilities: (a) $\mathscr{S}(I) = \{(00),(11)\}$ and (b) $\mathscr{S}(I) = \{(00),(01)\}$.

If $J = \emptyset$, then in case (a) $I = \{(00,11),(00,22),(11,22),(01,10),(02,20),(12,21)\}$. These are all the pairs a with

$$\mathscr{S}(a) \supset \{(00),(11)\} \quad \text{but } \mathscr{S}(I) = \{(00),(11),(22)\}.$$

Therefore $\big(\Omega,\{(00),(11)\}\big)$ is not a proper pair. There is no maximal matching pair in case (a) even if $J \neq \emptyset$. In case (b) $I = \{(22,12),(00,11),(00,21),(11,21),(01,12),(01,20),(10,20)\}$. This is a maximal matching pair. If $|J| = 1$, then $J = \{(00,01)\}$, $\mathscr{S}(J) = \{02,12,22\}$, $I = \{(22,12)\}$, but $\mathscr{S}(I) = \{00,01,02\} \neq \{00,01\}$. This means that there is no maximal matching pair in this case.

Subcase $\min\{|\mathscr{S}(I)|, |\mathscr{S}(J)|\} = 3$. There are the following possibilities:

(a) $\mathscr{S}(J) = \{00, 11, 22\}$, (b) $\mathscr{S}(I) = \{00, 01, 11\}$,

(c) $\mathscr{S}(J) = \{01, 11, 22\}$, (d) $\mathscr{S}(J) = \{02, 12, 22\}$.

It is easy to check that in cases (b) and (c) there is no maximal matching pair, so we consider only (a) and (d).

In case (a) there are only 3 maximal matching pairs, namely

$$I = \{(00,11),(00,22),(11,22),(01,10),(02,20),(12,21)\}, \quad J = \emptyset,$$
$$I = J = \{(00,11),(00,22),(11,22)\}, \quad I = \{(00,11),(00,22),(11,22),(01,10)\},$$
$$J = \{(00,11)\}.$$

In case (d), $J = \emptyset$ and then $I = \{(00,01),(10,11),(20,21)\}$. This is a maximal matching pair. If $|J| = 1$, let $J = \{(02,12)\}$. Then, $\mathscr{S}(J) = \{20,21,22\}$ and $I = \{(20,21)\}$. This is a maximal matching pair. In case $\min\{|I|,|J|\} > 1$, there is no maximal matching pair.

Subcase $\min\{|S(I)|, |\mathscr{S}(J)|\} = 4$. There is no maximal matching pair, because for any two pairs a and b, $|\mathscr{S}(a) \cap \mathscr{S}(b)| = 5$, 3, or 2.

Subcase $\min\{|\mathscr{S}(I)|, |\mathscr{S}(J)|\} = 5$. $\mathscr{S}(I) = \{(00),(11),(22),(01),(10)\}$, $J = \emptyset$, $I = \{(00,11),(01,10)\}$ or $J \neq \emptyset$ and the maximal matching is $I = J = \{(00,11),(01,10)\}$. These are all of the maximal matching pairs.

3.3. *The coefficients α, β for each of the ten maximal matching pairs*

1. We use the parity table, Table 1. From the parity table we take the following 6 squares

$$\{(0t_0),(1t_1),(2t_2)\} \times \{(0t_0),(1t_1),(2t_2)\},$$

Table 1

	00	01	10	11	02	20	12	21	22
00	0	1	1	0	1	1	0	0	0
01	1	0	0	1	1	0	0	1	0
10	1	0	0	1	0	1	1	0	0
11	0	1	1	0	0	0	1	1	0
02	1	1	0	0	0	0	1	0	1
20	1	0	1	0	0	0	0	1	1
12	0	0	1	1	1	0	0	0	1
21	0	1	0	1	0	1	0	0	1
22	0	0	0	0	1	1	1	1	0

Table 2

							00							01			
00	•		•	0	•						1	•					
11	•	•		0		•					1		•				
21		•	•	0			•				1			•			
01	•		•	1				•			0				•		
10	•	•		1					•		0					•	
20		•	•	1						•	0						•
12	•			0			•				0						•
22	•			0		•					0					•	
02				1						•	1			•			

Table 3

						00					11					22		
00	•		•	0					0					0				
11	•	•		0	•				0	•				0	•			
22		•	•	0		•			0		•			0		•		
01	•			1			•		1			•		0		•		
10	•			1				•	1				•	0	•			
02	•			1				•	0		•			1			•	
20	•			1			•		0	•				1				•
12	•			0		•			1				•	1				•
21	•			0	•				1			•		1			•	

and additionally the hyperedge $\{(00,00),(00,11),(00,22)\}$

Table 4

							00				11			22	
00	•		•	0	•					0			0		
11	•	•		0		•				0			0		
21		•	•	0			•			0			0		
01	•			1				•		1	•		0	•	
10	•			1					•	1		•	0		•

and additionally the hyperedges
$\{(00,11),(00,22)\},\{(11,11),(11,22)\},\{(22,11),(22,22)\}$
and $(00,11)$ is a pair in J.

where (t_0,t_1,t_2) is a permutation of $(0,1,2)$. All of them have parity 0 and they cover all 0's in the table. Further, the 9 rectangles $\{(t,s)\}\times\{(i,j):(i,j)$ has parity 1 with $(t,s)\}$ cover all 1's. Thus $\alpha\leqslant 6, \beta\leqslant 9$.

2. $|\mathscr{S}(I)|=1$ and thus $\alpha+\beta\leqslant 9$.

Table 5

				02					12					22		
00	•	1	•				0	•				0	•			
01	•	1		•			0		•			0		•		
10	•	0			•		1			•		0		•		
11	•	0				•	1				•	0	•			
20	•	0				•	0		•			1			•	
21	•	0			•		0	•				1				•
02		0			•		1				•	1				•
12		1		•			0	•				1				•
22		1		•			1				•	0	•			

Table 6

			00			11			01			10			22	
00	•	0			0			1			1			0		
11	•	0	•		0	•		1	•		1	•		0	•	
01	•	1			1			0			0			0		
10	•	1		•	1		•	0		•	0		•	0	•	
02		1		•	0	•		1	•		0		•	1		•
20		1		•	0	•		0		•	1	•		1		•
12		0	•		1		•	0		•	1	•		1		•
21		0	•		1		•	1	•		0		•	1		•
22		0	•		0	•		0		•	0		•	0	•	

and additionally the hyperedges
$\{(00,11),(00,11),(00,22)\}$,
$\{(00,01),(00,10)\}$, $\{(01,00),(01,11)\}$,
$\{(01,01),(01,10),(01,22)\}$.

3. For this we have to use Table 2. Two points in one of the first three columns denote that the pair is in I. For example the first two points in the first column denote that 00 and 11 is a pair in I. The points in the other columns denote the squares. Here we have $\alpha \leqslant 6, \beta \leqslant 6$.

4. For this case we have to use Table 3. Here we have $\alpha \leqslant 7, \beta \leqslant 6$.

5. $|\mathscr{S}(I)| = |\mathscr{S}(J)| = 3$: $\alpha + \beta \leqslant 9$.

6. $\alpha \leqslant 8, \beta \leqslant 4$. This case needs Table 4.

7. $\alpha \leqslant 6, \beta \leqslant 6$. Needed points are given to Table 5.

8. $|\mathscr{S}(I)| = |\mathscr{S}(J)| = 3$, $\alpha + \beta \leqslant 9$.

9. $\alpha \leqslant 7, \beta \leqslant 7$. Needed points are given in Table 6.

10. This is the most complicated case. For this maximal matching pair we got $\alpha = \beta = 8$, which is not good enough for our induction. We will discuss it later.

For the first 9 cases, we have $\alpha + \beta \leqslant 15$ and therefore by Eq. (3.1) and the induction hypothesis

$$Q_3^p(n) \leqslant 15\ (2^{n-3} + 1)^2. \tag{3.3}$$

For $n \geqslant 8$ we have

$$Q_3^p(n) \leqslant 2^{2n-2}. \tag{3.4}$$

The induction works.

3.4. The last case

If we cannot find two positions i,j such that A_{st}^{ij}, B_{st}^{ij} are in one of the first 9 situations, then we prove the conjecture directly. Define

$$A'_{11} = A'_{00} = A_{11} \cup A_{00} \quad \text{and} \quad A'_{01} = A'_{10} = A_{10} \cup A_{01}.$$

Then

$$A' = (22A_{22}) \cup (11A'_{11}) \cup (00A'_{00}) \cup (01A'_{01}) \cup (10A'_{10})$$

and the similarly defined B' are still a parity pair. Without loss of generality we assume for all i,j the existence of a permutation of $(0,1,2)$, say s_0, s_1, s_2, such that $A_{s_0s_0}^{ij} = A_{s_1s_1}^{ij}$, $A_{s_0s_1}^{ij} = A_{s_1s_0}^{ij}$, and $A_{s_0s_0}^{ij}$, $A_{s_0s_1}^{ij}$, $A_{s_2s_2}^{ij}$ are disjoint. The other sets are empty. The same is also true for the B_{st}^{ij}'s. For $(i,j) = (1,2)$, $(3,4), \ldots, (2m-1, 2m), \ldots$, we can assume without loss of generality $s_0 = 0$, $s_1 = 1$, $s_2 = 2$. We claim that for any $(i,j) = (2m-1, 2m)$

$$\text{either } A_{22}^{ij} = \{\underline{2}\} \quad \text{or} \quad A_{22}^{ij} = \emptyset. \tag{3.5}$$

If n is even, an (i,j) exists, say $(1,2)$, such that $(22, \ldots, 0, a, \ldots) \in A_{22}$, where the 0 is in position i. Then we have also $(22, \ldots, 1, \ldots) \in A_{22}$. Look at positions $(2,i)$, where $A_{20}^{2i} \neq \emptyset$ and $A_{21}^{2i} \neq \emptyset$.

If A_{11}^{12} is not empty (otherwise, we can use induction), an element

$$(11, \ldots, \overset{i}{t}, \ldots,)$$

exists in A_{11}^{12} and also an element

$$(00, \ldots, \overset{i}{t}, \ldots,)$$

exists in A_{00}^{12}, because $A_{11} = A_{00}$. If $t = 0$, then A_{10}^{2i} A_{00}^{2i} are not empty. This means that for position $(2,i)$ A_{st}^{2i} is not in the *case* 10, because A_{10}^{2i}, A_{00}^{2i}, and A_{20}^{2i} are not empty. If $t = 1$ we got A_{11}^{2i}, A_{01}^{2i}, and A_{21}^{2i} not empty, and thus we obtain the same conclusion. If $t = 2$, we got A_{20}^{2i}, A_{21}^{2i}, A_{02}^{2i}, and A_{12}^{2i} are not empty. This also contradicts the fact that we should be in the situation 10. For n even we proved that no element which has both 2 and non-2 entries appears in A or B. Then

$$A' = A \setminus A(22, \ldots, 2) \in \{0,1\}^n \quad \text{and} \quad B' = B \setminus B(22, \ldots, 2) \in \{0,1\}^n.$$

Since they have the same parity, if the parity is 0, then

$$|A'| \leqslant 2^{n-1}, \qquad |B'| \leqslant 2^{n-1}, \qquad |A| \leqslant 2^{n-1}+1 \text{ and } \quad |B| \leqslant 2^{n-1}+1$$

and if the parity is 1, then $(22,\ldots,22) \notin A \cap B$ and $|A| \leqslant 2^{n-1}$, $|B| \leqslant 2^{n-1}$.

In fact, we have proved that, even in the case n is odd, in the first $n-1$ positions, 2 and 0, 1 never appear in the same element in A. We prove

$$|\{x^n \in A : x^n = (22,\ldots,2,s)\}| = 1.$$

Otherwise the same argument will lead to the fact that A_{st}^{2n} is not in the situation 10.

After a permutation for position n we get the same result. However,

$$\overbrace{(22,\ldots,2)}^{n}$$

can belong to only one of A and B in case n is odd and $p=1$, and can belong to no one in case n is odd and $p=0$. This proves the conjecture.

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