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On maximal shadows of members in left-compressed sets

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Abstract

Whereas Stanley, Proctor and others studied Sperner properties of the poset $L(k,m)$ we consider shadows of left-compressed sets and derive asymptotic bounds on their sizes. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Fix two positive integers k and m. Draw a $k\times m$ grid of squares on "tilt". Now shade in some of the squares so that there are no unshaded squares below shaded squares – i.e., so that if the shaded squares were blocks in a rectangular frame, none would slide down. Call such a shading a *proper shading*. There is a natural partial ordering of the various proper shadings of a $k\times m$ grid: Order the proper shadings "by containment". Given a proper shading let a_1 be the number of squares shaded in its top right row, a_2 the number shaded in the next row, …, and a_m the number shaded in the bottom left row. Note that $0\leqslant a_1\leqslant a_2\leqslant\cdots\leqslant a_{m-1}\leqslant a_m\leqslant k$. Let the *m-tuple* $a=(a_1,\dots,a_m)$ denote this shading. It is easy to see that there is a 1 : 1 correspondence between the collection of all such *m-tuples* and the set of all proper shadings of a $k\times m$ grid. Now the poset $L(k,m)$ of proper shadings can be described:

$$a\leqslant b$$

if and only if

$$a_1\leqslant b_1,\quad a_2\leqslant b_2,\dots,a_k\leqslant b_k.$$

Equivalently, one can match $a=(a_1,a_2,\dots,a_k)$ with $a_1<a_2+1<a_3+2<\cdots<a_k+k-1$ and in turn match this with $x=(x_1,\dots,x_n)\in\binom{[n]}{k}$, where

$$x_{a_t+t-1}=\begin{cases}1 & \text{for } t=1,2,\dots,k\\ 0 & \text{otherwise}\end{cases}\quad\text{and}\quad n=m+k-1.$$

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This way one gets an isomorphic poset with partial order as defined in Section 2 below. It is the object of our investigation. For an excellent survey on $L(k,m)$ and related posets see [5].

Stanley [9,10] proved the (strong) Sperner property. He noticed that bases for the cohomology rings of certain projective varieties (the Grassmannians for $L(k,m)$) can be labeled in a natural way with elements of $L(k,m)$. His proof uses the hard Lefschetz theorem, a major theorem of algebraic geometry. It was shown in [7] that one can replace here the algebraic geometry with representations of Lie algebras. There are also connections to Dynkin-like diagrams [2] and earlier work in invariant theory by Cayley [1] and Sylvester [11]. $L(k,m)$ was proved by [6] to be rank unimodal. For two other proofs in the context of classical invariant theory see [3,8]. For other proofs using symmetric groups see [12,13,10].

Here we are not concerned with antichains but with another basic combinatorial object, namely shadows, and derive asymptotic bounds on the size. Its consequences for the mathematical areas mentioned remain to be explored.

2. Basic concepts and two of their properties

We consider the k-element subsets of an n-set, that is, $\binom{[n]}{k}$, or in sequence notation $T_k^n = \{x \in \mathscr{X}^n : w(x) = k\}$, where $\mathscr{X} = \{0,1\}$ and w is the weight function counting the ones.

We are interested in the range $k = \kappa n$, $\kappa > 0$. Since we are estimating the sizes of subsets of T_k^n, we can assume $n = m^2$ for some integer m, because we can always extend the elements of T_k^n by adding zeros at the right end.

We recall first the definition of a partial order of T_k^n, which was introduced in [4]. Then we present related concepts.

Definition 1 (*Left-pushing order*). For $x, y \in T_k^n$ we say $x \prec y$ iff for all j, $1 \leqslant j \leqslant n$,

$$\sum_{i=1}^{j} x_i \geqslant \sum_{i=1}^{j} y_i, \quad \text{where } x = (x_1, \ldots, x_n) \quad \text{and} \quad y = (y_1, \ldots, y_n).$$

We call the partial order "$\prec$" the left-pushing order.

Definition 2 (*Left-compressed sets*). Any subset A of T_k^n is called a left-compressed set iff $x \in A$, $y \prec x$ implies $y \in A$.

Definition 3 (*Shadow*). For $x \in T_k^n$ we define the shadow

$$S_k^n(x) = \{y \in T_k^n : y \preccurlyeq x\}.$$

Note that

$$S_k^n(x) \supset S_k^n(z) \quad \text{if } z \prec x. \tag{2.1}$$

Definition 4 (*Distance*)**.** For $x, y \in T_k^n$ define

$$D(x,y) = \max_{1 \leqslant \ell \leqslant k} |j_\ell(x) - j_\ell(y)|, \quad \text{where } j_\ell(x) = \min\left\{ j\colon \sum_{i=1}^{j} x_i = \ell \right\}.$$

We present now the first of two estimates for shadows.

Lemma 1. *For $x, y \in T_k^n$ with $D(x,y) \leqslant d$*

$$|\log_2 |S_k^n(x)| - \log_2 |S_k^n(y)|| \leqslant d \log_2 (k+1).$$

Proof. Let $\underset{\sim}{x}(d) = (0,\ldots,0,x) \in T_k^{n+d}$ and $\tilde{y}(d) = (y,0,\ldots,0) \in T_k^{n+d}$. (Here "$\tilde{\ }$" stands for "right" and "$\underset{\sim}{\ }$" for "left".)

By our definitions $\underset{\sim}{x}(d) \succ \tilde{y}(d)$ and therefore

$$S_k^{n+d}(\underset{\sim}{x}(d)) \supset S_k^{n+d}(\tilde{y}(d)). \tag{2.2}$$

Since

$$|S_k^{n+d}(\tilde{y}(d))| = |S_k^n(y)|, \tag{2.3}$$

we obtain

$$|S_k^{n+d}(\underset{\sim}{x}(d))| \geqslant |S_k^n(y)|. \tag{2.4}$$

Now, clearly,

$$S_k^{n+1}(\underset{\sim}{x}(1)) \leqslant (k+1)|S_k^n(x)|$$

and by iteration

$$S_k^{n+d}(\underset{\sim}{x}(d)) \leqslant (k+1)^d |S_k^n(x)|. \tag{2.5}$$

The claimed inequality follows now from (2.4) and (2.5).

Now comes a new idea. We partition the interval $[1,n]$ into m intervals, each of length m and set

$${}_i\mathscr{X} = \prod_{t=im+1}^{(i+1)m} \mathscr{X}_t. \tag{2.6}$$

For $x = (x_1,\ldots,x_n)$ we write

$${}_ix = (x_{im+1},\ldots,x_{(i+1)m}) \in {}_i\mathscr{X}. \tag{2.7}$$

Obviously, we can write x in the isomorphic way

$$x = ({}_1x,\ldots,{}_mx). \qquad \square \tag{2.8}$$

Definition 5. An element $x \in T_k^n$ is called *piecewise maximal*, if for all i, $1 \leqslant i \leqslant m$,

$${}_ix = (0,\ldots,0,1,\ldots,1).$$

Definition 6 (*Weight vector*).

$$V(x)=(w({}_1x),w({}_2x),\dots,w({}_mx)),\quad x\in T_k^n,$$

is called the *weight vector* of x.

Of course $V(x)\in\{0,1,\dots,m\}^m$.

Definition 7 (*Order of weight vector*). Let $V=(V_1,\dots,V_m)$ and $V'=(V_1',\dots,V_m')$ be two weight vectors. We write

$$V\prec V' \text{ iff } \sum_{i=1}^{j} V_i \geqslant \sum_{i=1}^{j} V_i'$$

for all j, $1\leqslant j\leqslant m$.

As a consequence of Lemma 1 we have

Lemma 2. *If for* $x,y\in T_k^n V(x)=V(y)$, *then*

$$D(x,y)<m \quad \textit{and} \quad |S_k^n(x)|\leqslant|S_k^n(y)|(k+1)^m.$$

Definition 8 (*Product set*). Let V be a weight vector with $\sum_{i=1}^m V_i=k$, then

$$T_k^n(V)\triangleq\{x\in T_k^n\colon w({}_ix)=V_i\}.$$

It has size

$$|T_k^n(V)|=\prod_{i=1}^{m}\binom{m}{V_i}. \tag{2.9}$$

3. A lower bound in terms of the size of a left-compressed set

We give asymptotic results. They are very good for instance for sets $A\subset T_k^n$, where $k=\kappa n$ and $|A|=\mathrm{e}^{\delta n}$ (exponential growth in n).

Theorem 1. *For every left-compressed set* $A\subset T_k^n$ *there is an* $x\in\mathscr{M}(A)$, *the set of maximal elements of* A *in the left-pushing order, such that*

$$|S_k^n(x)|\geqslant|A|\mathrm{e}^{-\mathrm{O}(\sqrt{n}\log n)}.$$

Proof. Define for $A\subset T_k^n$

$$\underset{\sim}{A}(\ell)=\{\underset{\sim}{x}(\ell)\colon x\in A\} \tag{3.1}$$

and consider the shadow

$$S_k^{n+\ell}(\underset{\sim}{A}(\ell))=\bigcup_{x\in A}S_k^{n+\ell}(\underset{\sim}{x}(\ell)). \tag{3.2}$$

Since we can assume that $n=m^2$, we choose $\ell = 2m+1$ so that

$$n+\ell=(m+1)^2. \tag{3.3}$$

Partition the interval $[1, n+\ell]$ into $m+1$ intervals of length $m+1$ each.

Write $\underset{\sim}{x}(\ell)=(0,\ldots,0,x)$ as $\underset{\sim}{x}(\ell)=({}_1\underset{\sim}{x}(\ell), {}_2\underset{\sim}{x}(\ell),\ldots, {}_{m+1}\underset{\sim}{x}(\ell))$ – a concatenation of $m+1$ vectors with $m+1$ components.

Next, define

$$x^* = ({}_1x^*, {}_2x^*, \ldots, {}_{m+1}x^*), \tag{3.4}$$

where

$${}_ix^* = (0,0,\ldots,0,1,1,1,1) \tag{3.5}$$

and

$$\omega({}_ix^*) = w({}_{i+1}\underset{\sim}{x}(\ell)) \quad \text{for } i=1,\ldots,m \text{ and } \omega({}_{m+1}x^*)=0. \tag{3.6}$$

Since, by (2.3), $w({}_1x(\ell))=0$, we have

$$w(x^*)=k.$$

We know, since $\underset{\sim}{x}(\ell)$ is maximal in $\underset{\sim}{A}(\ell)$ if and only if x is maximal in A, that

$$S_k^{n+\ell}(\underset{\sim}{A}(\ell)) = \bigcup_{x\in\mathscr{M}(A)} S_k^{n+\ell}(\underset{\sim}{x}(\ell)). \tag{3.7}$$

Define now

$$A^*(\ell)=\{x^*: x\in A\} \tag{3.8}$$

and conclude that

$$S_k^{n+\ell}(A^*(\ell)) = \bigcup_{x\in\mathscr{M}(A)} S_k^{n+\ell}(x^*). \tag{3.9}$$

Define now

$$\tilde{A}(\ell)=\{\tilde{x}(\ell): x\in A\}. \tag{3.10}$$

Clearly,

$$\tilde{A}(\ell)\subset S_k^{n+\ell}(A^*(\ell)) \tag{3.11}$$

and therefore by (3.10) and (3.11)

$$|A|=|\tilde{A}(\ell)|\leqslant |S_k^{n+\ell}(A^*(\ell))|. \tag{3.12}$$

Since there is a total of at most $(m+2)^{m+1}$ piecewise maximal elements (see Definition 5), there is an $x\in\mathscr{M}(A)$ such that

$$|S_k^{n+\ell}(x^*)| \geqslant |S_k^{n+\ell}(A^*(\ell))|\,(m+2)^{-(m+1)} = |S_k^{n+\ell}(A^*(\ell))|\,\mathrm{e}^{-\mathrm{O}(\sqrt{n}\log n)}. \tag{3.13}$$

Since $x^* \prec \underset{\sim}{x}(\ell)$, we have by (1.1)

$$|S_k^{n+\ell}(\underset{\sim}{x}(\ell))| \geqslant |S_k^{n+\ell}(x^*)|$$

and by Lemma 1

$$|S_k^n(x)| \geqslant |S_k^{n+\ell}(\underset{\sim}{x}(\ell))|(k+1)^{-\ell},$$

we conclude with (2.13) that

$$|S_k^n(x)| \geqslant |S_k^{n+\ell}(A^*(\ell))|\, \mathrm{e}^{-\mathrm{O}(\sqrt{n}\log n)}\, \mathrm{e}^{-\mathrm{O}(\sqrt{n}\log(k+1))}$$

and with (2.12) that

$$|S_k^n(x)| \geqslant |A|\, \mathrm{e}^{-\mathrm{O}(\sqrt{n}\log n)}. \qquad \square$$

4. On the shadow of words with zeros in the beginning

Theorem 2. *For $n = m^2$, let $x = ({}_1x, {}_2x, \dots, {}_mx) \in T_k^n$ have m zeros in the first m positions, that is, ${}_1x = (0, 0, \dots, 0)$, then*

$$|S_k^n(x)| = \max_{V \prec V(x^*)} |T_k^n(V)|\, \mathrm{e}^{\mathrm{O}(\sqrt{n}\log n)},$$

where $x^* = ({}_1x^*, \dots, {}_mx^*)$ is a piecewise maximal element satisfying $w({}_ix^*) = w({}_{i+1}x)$ for $i = 1, \dots, m-1$ and $w({}_mx^*) = 0$.

Proof. Actually, we have

$$S_k^n(x^*) = \bigcup_{V \preccurlyeq V(x^*)} T_k^n(V)$$

and since the number of weight vectors is of the order $\mathrm{e}^{\mathrm{O}(\sqrt{n}\log n)}$, we also have

$$|S_k^n(x^*)| = \max_{V \preccurlyeq V(x^*)} |T_k^n(V)|\, \mathrm{e}^{\mathrm{O}(\sqrt{n}\log n)}. \tag{4.1}$$

Now Lemmas 1 and 2 imply

$$|S_k^n(x^*)| = |S_k^n(x)|\, \mathrm{e}^{\mathrm{O}(\sqrt{n}\log n)}$$

which together with (3.1) leads to the result.

One readily derives via (1.9) the following formula.

Corollary. *For any $x \in T_k^n$, $n = m^2$,*

$$|S_k^n(x)| = \exp\left[\max_{V \preccurlyeq V(x)} m \sum_{i=1}^{m} h\left(\frac{V_i}{m}\right) + \mathrm{O}(m\log m)\right],$$

where h is the binary entropy function.

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