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On the Counting Function of Primitive Sets of Integers

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Erdős has shown that for a primitive set $A \subset \mathbb{N}$ $\sum_{a \in A} 1/(a \log a) < \text{const.}$ This implies that $A(x) < x/(\log \log x \log \log \log x)$ for infinitely many x. We prove that this is best possible apart from a factor $(\log \log \log x)^{\varepsilon}$. © 1999 Academic Press

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1. INTRODUCTION AND RESULTS

We explain first our terminology for our study of primitive sets.

The set of the positive integers and positive square-free integers are denoted by $\mathbb{N}$ and $\mathbb{N}^*$, respectively, and we write $\mathbb{N}(n) = \mathbb{N} \cap [1, n]$, $\mathbb{N}^*(n) = \mathbb{N}^* \cap [1, n]$. The smallest and greatest prime factors of the positive integer n are denoted by $p(n)$ and $P(n)$, respectively. $\omega(n)$ denotes the number of distinct prime factors of n, while $\Omega(n)$ denotes the number of prime factors of n counted with multiplicity,

$$\omega(n) = \sum_{p \mid n} 1, \qquad \Omega(n) = \sum_{p^{\alpha} \parallel n} \alpha.$$

$\mu(n)$ denotes the Möbius function.

The counting function of a set $\mathscr{A} \subset \mathbb{N}$, denoted by $A(x)$, is defined by

$$A(x) = |\mathscr{A} \cap [1, x]|.$$

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The upper density $\bar{d}(\mathscr{A})$ and the lower density $\underline{d}(\mathscr{A})$ of the infinite set $\mathscr{A} \subset \mathbb{N}$ are defined by

$$\bar{d}(\mathscr{A}) = \limsup_{x \to \infty} \frac{A(x)}{x}$$

and

$$\underline{d}(\mathscr{A}) = \liminf_{x \to \infty} \frac{A(x)}{x},$$

respectively, and if $\bar{d}(\mathscr{A}) = \underline{d}(\mathscr{A})$ then the density $d(\mathscr{A})$ of $\mathscr{A}$ is defined as

$$d(\mathscr{A}) = \bar{d}(\mathscr{A}) = \underline{d}(\mathscr{A}).$$

The upper logarithmic density $\bar{\delta}(\mathscr{A})$ of the infinite set $\mathscr{A} \subset \mathbb{N}$ is defined by

$$\bar{\delta}(\mathscr{A}) = \limsup_{x \to \infty} \frac{1}{\log x} \sum_{\substack{a \in \mathscr{A} \\ a \leqslant x}} \frac{1}{a},$$

and the definitions of the lower logarithmic density $\underline{\delta}(\mathscr{A})$ and logarithmic density $\delta(\mathscr{A})$ are similar.

For $\mathscr{A} \subset \mathbb{N}$, $s > 1$ write

$$f_{\mathscr{A}}(s) = \sum_{a \in \mathscr{A}} a^{-s}.$$

Then the lower and upper Dirichlet densities of $\mathscr{A}$ are defined by

$$\underline{D}(\mathscr{A}) = \liminf_{s \to 1^+} (s-1) f_{\mathscr{A}}(s)$$

and

$$\bar{D}(\mathscr{A}) = \limsup_{s \to 1^+} (s-1) f_{\mathscr{A}}(s),$$

respectively. If $\bar{D}(\mathscr{A}) = \underline{D}(\mathscr{A})$, then the Dirichlet density $D(\mathscr{A})$ of $\mathscr{A}$ is defined as

$$D(\mathscr{A}) = \bar{D}(\mathscr{A}) = \underline{D}(\mathscr{A}).$$

It is known that for every $\mathscr{A} \subset \mathbb{N}$ we have

$$\bar{\delta}(\mathscr{A}) = \bar{D}(\mathscr{A}), \qquad \underline{\delta}(\mathscr{A}) = \underline{D}(\mathscr{A})$$

and

$$0 \leqslant \underline{d}(\mathscr{A}) \leqslant \underline{\delta}(\mathscr{A}) \leqslant \bar{\delta}(\mathscr{A}) \leqslant \bar{d}(\mathscr{A}) \leqslant 1.$$

A set $\mathscr{A} \subset \mathbb{N}$ is said to be *primitive* if there are no a, a' with $a \in \mathscr{A}$, $a' \in \mathscr{A}$, $a \neq a'$ and $a \mid a'$. Let $F(n)$ denote the cardinality of the greatest primitive set selected from $\{1, 2, ..., n\}$. Then it is easy to see [8] that

$$F(n) = n - [n/2] (= (\tfrac{1}{2} + o(1))\, n). \tag{1.1}$$

By the results of Besicovitch [2] and Erdős [5], for all $\varepsilon > 0$

$$\text{there is an infinite primitive set } \mathscr{A} \subset \mathbb{N} \text{ with } \bar{d}(\mathscr{A}) > \tfrac{1}{2} - \varepsilon. \tag{1.2}$$

Behrend [3] proved that if $\mathscr{A} \subset \{1, 2, ..., N\}$ and $\mathscr{A}$ is primitive then we have

$$\sum_{a \in \mathscr{A}} \frac{1}{a} < c_1 \frac{\log N}{(\log \log N)^{1/2}} \tag{1.3}$$

(so that an infinite primitive set must have O logarithmic density) and Erdős [4] proved that if $\mathscr{A} \subset \mathbb{N}$ is a (finite or infinite) primitive set then

$$\sum_{a \in \mathscr{A}} \frac{1}{a \log a} < c_2. \tag{1.4}$$

This easily implies (proving by contradiction and using partial summation) that

Corollary. *If $\mathscr{A} \subset \mathbb{N}$ is primitive then we have*

$$A(x) < \frac{x}{\log \log x \log \log \log x} \tag{1.5}$$

for an unbounded sequence of values x.

One might like to know how far the upper bound in (1.5) is from the best possible. This is closely related to one of the favourite problems of Erdős. In [7] this problem is formulated in the following way (and he mentioned it in numerous problem papers as well): "The following problem seems difficult: Let $b_1 < \cdots$ be an infinite sequence of integers. What is the necessary and sufficient condition that there should exist a primitive sequence $a_1 < \cdots$ satisfying $a_n < cb_n$ for every n? From (1.4) ... we obtain that we must have

$$\sum_{i=1}^{\infty} \frac{1}{b_i \log b_i} < \infty \cdots. \tag{1.6}$$

We know that (1.6) is not sufficient—it is not clear whether a simple necessary and sufficient condition exists."

This is followed by a lengthy discussion of the problem how large one can make $\sum_{a \leqslant x} 1/a$ uniformly in x for a primitive set $a_1 < \cdots$ (see also [6]).

It seems to be a more natural (although more difficult) problem to replace here the sum $\sum_{a \leqslant x} 1/a$ by the counting function $A(x)$, i.e., to study the problem how large one can make $A(x)$ uniformly in x for a primitive set $\mathscr{A}$. We will provide a quite satisfactory answer by proving that (1.5) is best possible apart from a factor $(\log\log\log x)^{\varepsilon}$:

THEOREM. *For all $\varepsilon > 0$ there is an infinite primitive set $\mathscr{A} \subset \mathbb{N}$ such that for $x > x_0(\varepsilon)$ we have*

$$A(x) > \frac{x}{\log\log x (\log\log\log x)^{1+\varepsilon}}.$$

Our recent interest in primitive sets arose while we investigated the two related new concepts "prefix-free sets" and "suffix-free sets" (see [13]). The present result and the results of [13] were obtained in parallel with mutual influences of ideas.

2. PROOF OF THE THEOREM

It is well known that $\sum_{p \leqslant x} 1/p = \log\log x + c_3 + o(1)$ and therefore we may split the set $\mathscr{P}$ of the primes into two parts so that

$$\mathscr{P} = \mathscr{Q} \cup \mathscr{R}, \qquad \mathscr{Q} \cap \mathscr{R} = \emptyset,$$

$$\sum_{\substack{p \in \mathscr{Q} \\ p \leqslant x}} \frac{1}{p} = \frac{1}{2} \log\log x + c_4 + o(1), \quad \sum_{\substack{p \in \mathscr{R} \\ p \leqslant x}} \frac{1}{p} = \frac{1}{2} \log\log x + c_4 + o(1) \tag{2.1}$$

with some absolute constant c_4.

Set

$$\mathscr{Q}' = \left\{ q \colon q \in \mathscr{Q},\, q > \frac{5}{\varepsilon} \right\} = \{q_1, q_2, ...\}$$

(with $q_1 < q_2 < \dots$). Define j_1 by

$$\frac{1}{q_1} + \cdots + \frac{1}{q_{j_1}} < \frac{\varepsilon}{5} \leqslant \frac{1}{q_1} + \cdots + \frac{1}{q_{j_1}} + \frac{1}{q_{j_1+1}},$$

let $\mathscr{Q}_1 = \{q+1, ..., q_j\}$, and if $q_{j_1}, \mathscr{Q}_1, ..., q_{j_{k-1}}, \mathscr{Q}_{k-1}$ have been defined already, then define j_k by

$$\sum_{i=1}^{j_k} \frac{1}{q_i} < \frac{\varepsilon}{5} \sum_{i=1}^{k} \frac{1}{i} \leqslant \sum_{i=1}^{j_k+1} \frac{1}{q_i} \tag{2.2}$$

and let $\mathscr{Q}_k = \{q_{j_{k-1}+1}, ..., q_{j_k}\}$ so that clearly

$$\sum_{q \in \mathscr{Q}_k} \frac{1}{q} = (1+o(1)) \frac{\varepsilon}{5k} \qquad (\text{as} \quad k \to \infty), \tag{2.3}$$

and it follows from (2.1) and (2.3) that for large k we have

$$Q_k \subset [1, e^{k^{\varepsilon/2}}]. \tag{2.4}$$

For $k \in \mathbb{N}$ set $\mathscr{R}_k = \{r: r \in \mathscr{R}, r > 100 \cdot 2^k\} = \{r_1, r_2, ...\}$ (with $r_1 < r_2 < \cdots$).
Define $j_1 = j_1(k)$ by

$$\sum_{\ell=1}^{j_1} \frac{1}{r_\ell} < \frac{1}{100 \cdot 2^k} \leqslant \sum_{\ell=1}^{j_1+1} \frac{1}{r_\ell}$$

and let $\mathscr{R}_k^{(1)} = \{r_1, r_2, ..., r_{j_1}\}$. If $j_1, \mathscr{R}_k^{(1)}, ..., j_{i-1} \mathscr{R}_k^{(i-1)}$ (with $1 \leqslant i \leqslant 3 \cdot 2^k$) have been defined already, then define j_i by

$$\sum_{\ell=1}^{j_i} \frac{1}{r_\ell} < \frac{i}{100 \cdot 2^k} \leqslant \sum_{\ell=1}^{j_i+1} \frac{1}{r_\ell}$$

and let $\mathscr{R}_k^{(i)} = \{r_{j_{i-1}+1}, ..., r_{j_i}\}$ so that, as is easy to see,

$$\frac{1}{200 \cdot 2^k} < \sum_{r \in \mathscr{R}_k^{(i)}} \frac{1}{r} = \sum_{\ell = j_{i-1}+1}^{j_i} \frac{1}{r_\ell} < \frac{1}{50 \cdot 2^k} \tag{2.5}$$

and whence

$$\sum_{i=1}^{3 \cdot 2^k} \sum_{r \in \mathscr{R}_k^{(i)}} \frac{1}{r} < 3 \cdot 2^k \cdot \frac{1}{50 \cdot 2^k} < \frac{1}{10}. \tag{2.6}$$

Thus, by (2.1), for large k we have

$$\bigcup_{i=1}^{3 \cdot 2^k} \mathscr{R}_k^{(i)} \subset (100 \cdot 2^k, 2^{2k}]. \tag{2.7}$$

For $k \in \mathbb{N}$, $i = 0, 1, 2, ..., 3 \cdot 2^k$, write $x_k^{(i)} = e^{e^{2^{2k} + i \cdot 2^k}}$, $Q_k = \prod_{p \in \bigcup_{j=1}^{k} \mathcal{Q}_j} p$, and $R_k = \prod_{p \in \bigcup_{i=1}^{3 \cdot 2^k} \mathcal{R}_k^{(i)}} p$, and for $k \in \mathbb{N}$, $i = 1, 2, ..., 3 \cdot 2^k$ let $\mathcal{A}_k^{(i)}$ denote the set of the integers of the form

$$a = qrt \qquad \text{with} \quad q \in \mathcal{Q}_k, \quad r \in \mathcal{R}_k^{(i)}, \quad (t, Q_k P_k) = 1,$$
$$\Omega(t) = [\log \log x_k^{(i-1)}],$$

write $\mathcal{B}_k^{(i)} = \mathcal{A}_k^{(i)} \cap (x_k^{(i-1)}, x_k^{(i)}]$, and let $\mathcal{A} = \bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{3 \cdot 2^k} \mathcal{B}_k^{(i)}$. We will show that this set $\mathcal{A}$ has the desired properties.

We have to show two facts:

$$\mathcal{A} \text{ is primitive} \tag{2.8}$$

and

$$A(x) > \frac{x}{\log \log x (\log \log \log x)^{1+\varepsilon}} \qquad \text{for} \quad x > x_0. \tag{2.9}$$

To prove (2.8), we have to show that if $a, a' \in \mathcal{A}$, $a < a'$, then $a \nmid a'$. We have to distinguish three cases.

Case 1. Assume that $a \in \mathcal{A}_k^{(i)}$, $a' \in \mathcal{A}_{k'}^{(i')}$ with $k \neq k'$; then by $a < a'$ we have $k < k'$. By the construction of $\mathcal{A}$ there is a prime q such that $q \in \mathcal{Q}_k$, $q \mid a$ and $q \nmid a'$, and thus $a \nmid a'$.

Case 2. Assume that $a \in \mathcal{A}_k^{(i)}$, $a' \in \mathcal{A}_k^{(i')}$ with $i \neq i'$; then by $a < a'$ we have $i < i'$. By the construction of $\mathcal{A}$ there is a prime r such that $r \in \mathcal{R}_k^{(i)}$, $r \mid a$ and $r \nmid a'$, and thus $a \nmid a'$.

Case 3. Assume that $a \in \mathcal{A}_k^{(i)}$, $a' \in \mathcal{A}_k^{(i)}$. Then $\Omega(a) = \Omega(a')$; since $a \neq a'$ this implies $a \nmid a'$.

To prove (2.9), consider a large x and define k and i ($1 \leqslant i \leqslant 3 \cdot 2^k$) by

$$x_k^{(i-1)} < x \leqslant x_k^{(i)}.$$

(By $x_k^{(3 \cdot 2^k)} = x_{k+1}^{(0)}$ there is a unique pair (k, i) with this property.) Then we have

$$\begin{aligned} A(x) &\geqslant B_k^{(i)}(x) + B_k^{(i-1)}(x) \\ &= (A_k^{(i)}(x) - A_k^{(i)}(x_k^{(i-1)})) + (A_k^{(i-1)}(x_k^{(i-1)}) - A_k^{(i-1)}(x_k^{(i-2)})) \\ &\qquad \text{for} \quad i \geqslant 2 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} A(x) &\geqslant B_k^{(1)}(x) + B_{k-1}^{(3\cdot 2^{k-1})}(x) \\ &= (A_k^{(1)}(x) - A_k^{(1)}(x_k^{(0)})) + (A_{k-1}^{(3\cdot 2^{k-1})}(x_k^{(3\cdot 2^{k-1})}) \\ &\quad - A_{k-1}^{(3\cdot 2^{k-1})}(x_k^{(3\cdot 2^{k-1}-1)})) \qquad \text{for} \quad i=1. \end{aligned} \tag{2.11}$$

Since each term in these lower bounds is of the form

$$A_k^{(i)}(z) - A_k^{(i)}(x_k^{(i-1)}) \tag{2.12}$$

for some i, k and for

$$x_k^{(i-1)} < z \leqslant x_k^{(i)}, \tag{2.13}$$

thus it remains to estimate (2.12) with z satisfying (2.13). This estimate will be based on the following lemma:

LEMMA 1. *Assume that* $x > x_0$,

$$\frac{1}{2} < \frac{k}{\log\log x} < \frac{3}{2} \tag{2.14}$$

and

$$1 \leqslant y < \tfrac{1}{4} \log\log x. \tag{2.15}$$

Write $P_y = \prod_{p \leqslant y} p$ *and*

$$S_y(x, k) = |\{n\colon n \leqslant x,\ \Omega(n) = k,\ (n, P_y) = 1\}|. \tag{2.16}$$

Then we have

$$\begin{aligned} |S_y(x,k)| = \left(\prod_{p \leqslant y}\left(1 - \frac{1}{p}\right) + O\left(\frac{(\log y)^2\,|k - \log\log x| + (\log y)^4}{\log\log x}\right)\right) \\ \times \frac{x}{\log x}\,\frac{(\log\log x)^{k-1}}{(k-1)!}. \end{aligned}$$

This lemma will be proved in the next section. First in this section we will complete the proof of the Theorem by using Lemma 1.

Let y denote the greatest prime in $\mathcal{Q}_k$. Then, writing $\ell = [\log\log x_k^{(i-1)}]$, by the definition of $\mathcal{A}_k^{(i)}$ we have

$$\begin{aligned} &A_k^{(i)}(z) - A_k^{(i)}(x_k^{(i-1)}) \\ &\quad = \sum_{q \in \mathcal{Q}_k} \sum_{r \in \mathcal{R}_k^{(i)}} |\{t\colon x_k^{(i-1)} < qrt \leqslant z,\ (t, Q_k R_k) = 1,\ \Omega(t) = \ell\}|. \end{aligned} \tag{2.17}$$

Using notation (2.16), clearly we have

$$
\begin{aligned}
&|\{t\colon x_k^{(i-1)} < qrt \leqslant z, (t, Q_k R_k) = 1, \Omega(t) = \ell\}| \\
&\quad \geqslant |\{t\colon x_k^{(i-1)}/qr < t \leqslant z/qr, (t, Q_k) = 1, \Omega(t) = \ell\}| \\
&\qquad - \sum_{p \mid R_k} |\{t\colon x_k^{(i-1)}/qr < t \leqslant z/qr, p \mid t, (t, Q_k) = 1, \Omega(t) = \ell\}| \\
&\quad = |\{t\colon x_k^{(i-1)}/qr < t \leqslant z/qr, (t, Q_k) = 1, \Omega(t) = \ell\}| \\
&\qquad - \sum_{p \mid R_k} |\{u\colon x_k^{(i-1)}/qrp < u \leqslant z/qrp, (u, Q_k) = 1, \Omega(u) = \ell - 1\}| \\
&\quad = (S_y(z/qr, \ell) - S_y(x_k^{(i-1)}/qr, \ell)) \\
&\qquad - \sum_{p \mid P_k} (S_y(z/qrp, \ell - 1) - S_y(x_k^{(i-1)}/qrp, \ell - 1))
\end{aligned} \tag{2.18}
$$

(we substituted $t = up$). We will estimate each term by Lemma 1. It follows from (2.13) and the definition of $x_k^{(i)}$ that

$$\log\log\log z = k \log 4 + O(1) \tag{2.19}$$

and

$$
\begin{aligned}
\ell = [\log\log x_k^{(i-1)}] &\leqslant \log\log z \leqslant \log\log x_k^{(i)} \\
&\leqslant \log\log x_k^{(i-1)} + (\log\log x_k^{(i-1)})^{1/2} \\
&< \ell + 2(\log\log z)^{1/2}.
\end{aligned} \tag{2.20}
$$

By (2.4), (2.6), and (2.19) we have

$$y \leqslant \exp(k^{\varepsilon/2}) = \exp((\log\log\log z)^{\varepsilon/2}) \tag{2.21}$$

and

$$
\begin{aligned}
qr < qrp &= \exp(O(k^{\varepsilon/2}) + O(k) + O(k)) \\
&= \exp(O(k)) = (\log\log z)^{O(1)}.
\end{aligned} \tag{2.22}
$$

By (2.19), (2.20), (2.21), and (2.22), Lemma 1 can be applied first with z/qr and ℓ; with $x_k^{(i-1)}/qr$ and ℓ; with z/qrp and $\ell - 1$; finally, with $x_k^{(i-1)}/qrp$ and $\ell - 1$ in place of x and k, respectively. We obtain from (2.6), (2.18), (2.20), and (2.21) that

$$\begin{aligned}
&|\{t\colon x_k^{(i-1)}<qrt\leqslant z,\,(t,Q_kR_k)=1,\,\Omega(t)=\ell\}|\\
&=\left(\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\frac{z-x_k^{(i-1)}}{qr\log z}+O\left(\frac{z(\log\log\log z)^{\varepsilon}}{qr\log z(\log\log z)^{1/2}}\right)\right)\\
&\quad\times\frac{(\log\log z)^{\ell-1}}{(\ell-1)!}-\left(\sum_{p'\mid R_k}\left(\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\frac{z-x_k^{(i-1)}}{p'qr\log z}\right.\right.\\
&\quad\left.\left.+O\left(\frac{z(\log\log\log z)^{\varepsilon}}{p'qr\log z(\log\log z)^{1/2}}\right)\right)\right)\frac{(\log\log z)^{\ell-1}}{(\ell-1)!}\\
&=\left(\left(1-\sum_{p'\mid R_k}\frac{1}{p'}\right)\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\frac{z-x_k^{(i-1)}}{qr\log z}\right.\\
&\quad\left.+O\left(\left(1+\sum_{p'\mid R_k}\frac{1}{p'}\right)\frac{z(\log\log\log z)^{\varepsilon}}{qr\log z(\log\log z)^{1/2}}\right)\right)\frac{(\log\log z)^{\ell-1}}{(\ell-1)!}\\
&<\left(\frac{9}{10}\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\frac{z-x_k^{(i-1)}}{qr\log z}+O\left(\frac{z(\log\log\log z)^{\varepsilon}}{qr\log z(\log\log z)^{1/2}}\right)\right)\\
&\quad\times\frac{(\log\log z)^{\ell-1}}{(\ell-1)!}.
\end{aligned}\tag{2.23}$$

By Merten's formula and (2.21) we have

$$\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)>\frac{c_4}{\log y}>\frac{c_5}{(\log\log\log z)^{\varepsilon/2}}.\tag{2.24}$$

Moreover, by using Stirling's formula, it follows from (2.20) that

$$\frac{(\log\log z)^{\ell-1}}{(\ell-1)!}>c_6\log z(\log\log z)^{-1/2}.\tag{2.25}$$

By (2.23), (2.24), and (2.25) we have

$$\begin{aligned}
&|\{t\colon x_k^{(i-1)}<qrt\leqslant z,\,(t,Q_kR_k)=1,\,\Omega(t)=\ell\}|\\
&\quad>c_7\frac{z-x_k^{(i-1)}}{qr(\log\log\log z)^{\varepsilon/2}(\log\log z)^{1/2}}+O\left(\frac{z(\log\log\log z)^{\varepsilon}}{qr\log\log z}\right)
\end{aligned}\tag{2.26}$$

so that, from (2.3), (2.5), (2.13), and (2.26),

$$
\begin{aligned}
&A_k^{(i)}(z)-A_k^{(i)}(x_k^{(i-1)})\\
&\quad>\Bigg(\sum_{q\in\mathcal{Q}_k}\sum_{r\in\mathcal{R}_k^{(i)}}\frac{1}{qr}\Bigg)\Bigg(c_7\,\frac{z-x_k^{(i-1)}}{(\log\log\log z)^{\varepsilon/2}\,(\log\log z)^{1/2}}\\
&\qquad+O\bigg(\frac{z(\log\log\log z)^{\varepsilon}}{\log\log z}\bigg)\Bigg)\\
&\quad=\Bigg(\sum_{q\in\mathcal{Q}_k}\frac{1}{q}\Bigg)\Bigg(\sum_{r\in\mathcal{R}_k^{(i)}}\frac{1}{r}\Bigg)\Bigg(c_7\,\frac{z-x_k^{(i-1)}}{(\log\log\log z)^{\varepsilon/2}\,(\log\log z)^{1/2}}\\
&\qquad+O\bigg(\frac{z(\log\log\log z)^{\varepsilon}}{\log\log z}\bigg)\Bigg)\\
&\quad>\frac{\varepsilon}{6k}\cdot\frac{1}{200\cdot 2^k}\Bigg(c_7\,\frac{z-x_k^{(i-1)}}{(\log\log\log z)^{\varepsilon/2}\,(\log\log z)^{1/2}}\\
&\qquad+O\bigg(\frac{z(\log\log\log z)^{\varepsilon}}{\log\log z}\bigg)\Bigg)\\
&\quad>\frac{\varepsilon}{\log\log\log z}\cdot\frac{1}{(\log\log z)^{1/2}}\Bigg(c_8\,\frac{z-x_k^{(i-1)}}{(\log\log\log z)^{\varepsilon/2}\,(\log\log z)^{1/2}}\\
&\qquad+O\bigg(\frac{z(\log\log\log z)^{\varepsilon}}{\log\log z}\bigg)\Bigg)\\
&\quad>\frac{z-x_k^{(i-1)}}{\log\log z(\log\log\log z)^{1+2\varepsilon/3}}\\
&\qquad+O\bigg(\frac{z}{(\log\log\log z)^{1-\varepsilon}\,(\log\log z)^{3/2}}\bigg).
\end{aligned}
\tag{2.27}
$$

Using (2.27) to estimate each of the terms in (2.10) and (2.11), and also using the fact that

$$x_k^{(i-1)}=(x_k^{(i)})^{o(1)},$$

we obtain in both cases that

$$
\begin{aligned}
A(x)&>\frac{x}{\log\log x(\log\log\log x)^{1+2\varepsilon/3}}+O\bigg(\frac{x}{(\log\log\log x)^{1-\varepsilon}\,(\log\log x)^{3/2}}\bigg)\\
&>\frac{x}{\log\log x(\log\log\log x)^{1+\varepsilon}},
\end{aligned}
$$

which completes the proof of the Theorem.

3. PROOF OF LEMMA 1

Write

$$\sigma_k(x) = |\{n: n \leqslant x, \Omega(n) = k\}|$$

and

$$f(s, z) = \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z.$$

The proof of Lemma 1 will be based on

LEMMA 2. *For $\varepsilon > 0$, $x \to \infty$, $k < (2 - \varepsilon) \log \log x$ we have*

$$\sigma_k(x) = \frac{x}{\log x} \frac{f\left(1, \frac{k-1}{\log \log x}\right)}{\Gamma\left(1 + \frac{k-1}{\log \log x}\right)} \frac{(\log \log x)^{k-1}}{(k-1)!} + O_\varepsilon\left(\frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \frac{k}{(\log \log x)^2}\right)$$

(*where the O_ε notation means that the implied constant may depend on* ε).

Proof. This is Selberg's theorem [10].

LEMMA 3. *For $\varepsilon > 0$, $x \to \infty$,*

$$k < (2 - \varepsilon) \log \log x \tag{3.1}$$

we have

$$\sigma_k(x) = \left(1 + O_\varepsilon\left(\frac{|k - \log \log x| + 1}{\log \log x}\right)\right) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

Proof. This follows from Lemma 2 since for $O \leqslant z < 2 - \varepsilon$ we have

$$f(1, z) = f(1, 1) \cdot O_\varepsilon(\exp(|z - 1|)) = O_\varepsilon(\exp(|z - 1|))$$

and

$$\Gamma(1 + z) = \Gamma(z)\, O(\exp(|z - 1|)) = O(\exp(|z - 1|))$$

so that

$$\frac{f(1, z)}{\Gamma(1+z)} = O_\varepsilon(\exp(|z-1|)) = 1 + O_\varepsilon(|z-1|).$$

Proof of Lemma 1. Write

$$\sigma_k(x, d) = |\{n\colon n \leqslant x, \Omega(n) = k, d \mid n\}|$$

so that $\sigma_k(x, 1) = \sigma_k(x)$. Then, by

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1, \end{cases}$$

we have

$$\begin{aligned} S_y(x, k) &= |\{n\colon n \leqslant x, \Omega(n) = k, (n, P_y) = 1\}| \\ &= \sum_{\substack{n \leqslant x \\ \Omega(n) = k}} \sum_{d \mid (n, P_y)} \mu(d) = \sum_{d \mid P_y} \mu(d) \sum_{\substack{n \leqslant x, d \mid n \\ \Omega(n) = k}} 1 = \sum_{d \mid P_y} \mu(d) \sum_{\substack{dt \leqslant x \\ \Omega(dt) = k}} 1 \\ &= \sum_{d \mid P_y} \mu(d) \sum_{\substack{t \leqslant x/d \\ \Omega(t) = k - \Omega(d)}} 1 = \sum_{d \mid P_y} \mu(d)\, \sigma_{k - \Omega(d)}(x/d). \end{aligned} \tag{3.2}$$

If $d \mid P_y$ then we have

$$\Omega(d) = \omega(d) \leqslant \pi(y) \qquad (\leqslant y) \tag{3.3}$$

and, clearly,

$$d \leqslant y^{\omega(d)} \tag{3.4}$$

and

$$d \leqslant P_y \leqslant \exp(c_9 y). \tag{3.5}$$

It follows from (2.14), (2.15), (3.3), and (3.5) that (3.1) holds with $\frac{1}{5}$, $k - \Omega(d)$ and x/d in place of ε, k and x, respectively, so that Lemma 3 can be applied to estimate $\sigma_{k-\Omega(d)}(x/d)$. We obtain for all $d \mid P_y$ that

$$\begin{aligned} \sigma_{k-\Omega(d)}(x/d) = {} & \left(1 + O\left(\frac{|k - \log\log(x/d)| + 1}{\log\log(x/d)}\right)\right) \\ & \times \frac{x/d}{\log(x/d)} \frac{(\log\log(x/d))^{k - \Omega(d) - 1}}{(k - \Omega(d) - 1)!}. \end{aligned} \tag{3.6}$$

By (2.14), (2.15), and (3.5) we have

$$\frac{1}{\log(x/d)} = \frac{1}{\log x}\left(1 + O\left(\frac{\log d}{\log x}\right)\right), \tag{3.7}$$

$$\log \log(x/d) = \log \log x + O\left(\frac{\log d}{\log x}\right), \tag{3.8}$$

$$\begin{aligned}(\log \log(x/d))^{k-\Omega(d)-1} &= \left(\log \log x + O\left(\frac{\log d}{\log x}\right)\right)^{k-\Omega(d)-1} \\ &= (\log \log x)^{k-\Omega(d)-1}\left(1 + O\left(\frac{\log d}{\log x}\right)\right)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\frac{(k-1)(k-2)\cdots(k-\Omega(d))}{(\log \log x)^{\Omega(d)}} \\ &= \prod_{i=1}^{\Omega(d)}\left(1 + \frac{(k-\log \log x)-i}{\log \log x}\right) \\ &= \exp\left(\Omega(d)\,\frac{|k-\log \log x| + \Omega(d)}{\log \log x}\right) \\ &= 1 + O\left(\Omega(d)\,\frac{|k-\log \log x| + \Omega(d)}{\log \log x}\right).\end{aligned} \tag{3.10}$$

It follows from (3.6), (3.7), (3.8), (3.9), and (3.10) that

$$\begin{aligned}\sigma_{k-\Omega(d)}(x/d) = &\left(1 + O\left(\frac{\log d}{\log x} + \Omega(d)\,\frac{|k-\log \log x| + \Omega(d)}{\log \log x}\right)\right) \\ &\times \frac{x}{d \log x}\,\frac{(\log \log x)^{k-1}}{(k-1)!}\end{aligned}$$

for all $d \mid P_y$. Thus we obtain from (3.2) that

$$\begin{aligned}S_y(x, k) &= \left(\sum_{d \mid P_y} \frac{\mu(d)}{d} + R\right)\frac{x}{\log x}\,\frac{(\log \log x)^{k-1}}{(k-1)!} \\ &= \left(\prod_{p \leqslant y}\left(1 - \frac{1}{p}\right) + R\right)\frac{x}{\log x}\,\frac{(\log \log x)^{k-1}}{(k-1)!},\end{aligned} \tag{3.11}$$

where

$$R = O\left(\left(\sum_{d \mid P_y} \frac{\log d}{d}\right)\frac{1}{\log x} + \left(\sum_{d \mid P_y} \frac{\omega(d)}{d}\right)\frac{|k - \log\log x|}{\log\log x} + \sum_{d \mid P_y} \frac{(\omega(d))^2}{d} \frac{1}{\log\log x}\right). \tag{3.12}$$

By $\omega(d) \leqslant 2^{\omega(d)}$ and (3.4) here we have

$$\sum_{d \mid P_y} \frac{\omega(d)}{d} \leqslant \sum_{d \mid P_y} \frac{2^{\omega(d)}}{d} = \prod_{p \leqslant y}\left(1 + \frac{2}{p}\right) < c_{10}(\log y)^2,$$

$$\sum_{d \mid P_y} \frac{(\omega(d))^2}{d} \leqslant \prod_{p \leqslant y}\left(1 + \frac{4}{p}\right) < c_{11}(\log y)^2$$

and

$$\sum_{d \mid P_y} \frac{\log d}{d} \leqslant \sum_{d \mid P_y} \frac{\omega(d)\log y}{d} < c_{12}(\log y)^3$$

so that, from (3.12),

$$R = O\left((\log y)^3 \frac{1}{\log x} + (\log y)^2 \frac{|k - \log\log x|}{\log\log x} + (\log y)^4 \frac{1}{\log\log x}\right) = O\left(\frac{(\log y)^2 \, |k - \log\log x| + (\log y)^4}{\log\log x}\right). \tag{3.13}$$

The conclusion of Lemma 1 follows from (3.11) and (3.13).

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