A diametric theorem for edges

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1 Introduction

Whereas there are vertex — and edge — isoperimetric theorems it went unsaid that diametric theorems are vertex — diametric theorems. We complete the story by introducing edge–diametric theorems into combinatorial extremal theory.

Before we state our new edge–diametric problem and its solution we sketch some key steps in the development of extremal set theory. We keep the notation of earlier papers.

$\mathbb{N}$ denotes the set of positive integers and for $i, j \in \mathbb{N}$, $i < j$, the set $\{i, j+1, \dots, j\}$ is abbreviated as $[i, j]$. We write $[n]$ for $[1, n]$.

For $k, n \in \mathbb{N}$, $k \le n$, we set

$$2^{[n]} = \big\{F : F \subset [1, n]\big\}, \binom{[n]}{k} = \{F \in 2^{[n]} : |F| = k\}.$$

A system of sets $\mathcal{A} \subset 2^{[n]}$ is called t–intersecting, if

$$|A_1 \cap A_2| \ge t \quad \text{for all} \quad A_1, A_2 \in \mathcal{A}.$$

$I(n, t)$ denotes the set of all such systems and we write $I(n)$ for $t = 1$.

We denote by $I(n, k, t)$ the set of all k–uniform t–intersecting systems, that is,

$$I(n, k, t) = \left\{\mathcal{A} \in I(n, t) : \mathcal{A} \subset \binom{[n]}{k}\right\}.$$

The investigation of the function

$$M(n, t) = \max_{\mathcal{A} \in I(n,t)} |\mathcal{A}| \quad \text{and} \quad M(n, k, t) = \max_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}|, 1 \le t \le k \le n,$$

and the structure of maximal systems was initiated by Erdös, Ko, and Rado [6].

We also introduce the sets

$$\mathcal{K}_i(n, t) = \big\{A \in 2^{[n]} : |A \cap [t + 2i]| \ge t + i\big\}, 0 \le i \le \frac{n - t}{2} \tag{1.1}$$

and

$$\mathcal{F}_i(n, k, t) = \left\{F \in \binom{[n]}{k} : |F \cap [t + 2i]| \ge t + i\right\}, 0 \le i \le \frac{n - t}{2}. \tag{1.2}$$

We also use the abbreviation $\mathcal{F}_i = \mathcal{F}_i(n, k, t)$.

There is a well–known result of Katona, which determines the exact value of $M(n, t)$ for all n, t

Theorem Ka [9].

$$M(n, t) = \begin{cases} \big|\mathcal{K}_{\frac{n-t}{2}}(n, t)\big| & \text{if } \ 2 \mid (n - t) \\ \big|\mathcal{K}_{\frac{n-t-1}{2}}(n, t)\big| & \text{if } \ 2 \nmid (n - t). \end{cases}$$

Moreover, in the case $2 \mid (n-t), t \geq 2$, $\mathcal{K}_{\frac{n-t}{2}}(n,t)$ is the unique optimal configuration, while in the case $2 \nmid (n-t), t \geq 2$, $\mathcal{K}_{\frac{n-t-1}{2}}(n,t)$ is the unique solution up to permutations of the ground set $[n]$.

The proof of this Theorem in [9] is essentially based on a result concerning shadows of t–intersecting systems.

Recently we proved a long–standing conjecture concerning the function $M(n,k,t)$.

Theorem AK [2]. *For $1 \leq t \leq k \leq n$ with*

(i) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right)$ *for some $r \in \mathbb{N} \cup \{0\}$, we have*

$$M(n,k,t) = |\mathcal{F}_r|$$

and $\mathcal{F}_r$ is — up to permutations — the unique optimum (by concention $\frac{t-1}{r} = \infty$ for $r = 0$).

(ii) $(k-t+1)\left(2+\frac{t-1}{r+1}\right) = n$ *for $r \in \mathbb{N} \cup \{0\}$ we have*

$$M(n,k,t) = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$$

and an optimal system equals up to permutations — either $\mathcal{F}_r$ or $\mathcal{F}_{r+1}$.

For the proof we introduced the seemingly basic notion of — what we called — generating sets. In [4] we presented a new compression method, which we called "pushing–pulling method" and which led to new proofs for both, Theorem Ka and Theorem AK. The proof of our new result presented below is based on this method.

There is a natural transition from $2^{[n]}$ to $\{0,1\}^n$ — the set of binary words of length n: any set $A \in 2^{[n]}$ can be represented as word $a^n = (a_1, \dots, a_n) \in \{0,1\}^n$, where

$$a_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A \end{cases}$$

and conversely.

The Hamming distance between $a^n = (a_1, \dots, a_n), b^n = (b_1, \dots, b_n) \in \{0,1\}^n$ is defined as follows:

$$d_H(a^n, b^n) = |\{j \in [n] : a_j \neq b_j\}|.$$

We say that $\mathcal{A} \subset \{0,1\}^n$ has a diameter d if

$$\text{diam}(\mathcal{A}) \triangleq \max_{a^n, b^n \in \mathcal{A}} d_H(a^n, b^n) = d.$$

$D(n,d)$ denotes the set of all such systems.

Later, in order to avoid additional notation, we will denote $(0,1)$–images of the sets defined in (1.1) and (1.2) again by $\mathcal{K}_i(n,t)$ and $\mathcal{F}_i(n,k,t)$.

The n–dimensional hypercube Q_n is a graph with vertex set $\mathcal{V}(Q_n) = \{0,1\}^n$ and edge set $\mathcal{E}(Q_n) = \big\{\{a^n, b^n\} : d(a^n,b^n) = 1\big\}$. Clearly, any $\mathcal{A} \subset \{0,1\}^n$ can be embedded into graph Q_n:

$\mathcal{V}(\mathcal{A}) = \mathcal{A}$ (vertex set), $\mathcal{E}(\mathcal{A}) = \big\{(a^n,b^n) : a^n, b^n \in \mathcal{A}, d(a^n,b^n) = 1\big\}$ (edge set).

In the set–theoretical language, we connect by an edge $A_1, A_2 \in 2^{[n]}$, if $|A_1 \triangle A_2| = 1$ (symmetric difference). Two naturally arising functions concerning diametric problems are:

$$V(n,d) = \max_{\mathcal{A} \in D(n,d)} |\mathcal{A}| \quad \text{(vertex–diametric function)}$$

and

$$E(n,d) = \max_{\mathcal{A} \in D(n,d)} |\mathcal{E}(\mathcal{A})| \quad \text{(edge–diametric function).}$$

It seems that the definition of the second function is new.

There is a well–known result of Kleitman, which determines the exact value of $V(n,d)$ for all n, d.

Theorem Kl [10].

$$V(n,d) = \begin{cases} \sum_{i=0}^{\frac{d}{2}} \binom{n}{i}, & \text{if } d \text{ is even} \\ 2\sum_{i=0}^{\frac{d-1}{2}} \binom{n}{i}, & \text{if } d \text{ is odd.} \end{cases}$$

This result and Theorem Ka imply

$$M(n, n-d) = V(n,d).$$

Actually it was shown in [1] that the two theorems can be easily derived from each other by passing through upsets.

In connection with Theorem Kl we mention that recently we solved the problem of determination of $V(n,d)$ (optimal anticodes) for nonbinary alphabets [3].

Now we present our new result. At first recall the definition of the sets $\mathcal{K}_i(n,t)$ and define the set

$$\mathcal{H}(n) = \big\{(a_1, \dots, a_n) \in \{0,1\}^n : a_1 = 1\big\}.$$

Theorem.

$$E(n,d) = \begin{cases} |\mathcal{E}\big(\mathcal{H}(n)\big)|, & \text{if } d = n-1 \\ |\mathcal{E}\big(\mathcal{K}_{\frac{d}{2}}(n, n-d)\big)|, & \text{if } d \le n-2 \text{ and } 2 \mid d \\ |\mathcal{E}\big(\mathcal{K}_{\frac{d-1}{2}}(n, n-d)\big)|, & \text{if } d \le n-2 \text{ and } 2 \nmid d. \end{cases}$$

Remark: In addition to the optimal configuration in the Theorem we have for the case $d = n-2, 2 \mid d$ also the optimal configuration $\mathcal{K}_{\frac{d-2}{2}}(n, n-d)$. Actually we can prove that all other optimal configurations can be obtained by permutations of the ground set $[n]$ and of the alphabets in the components.

2 Reduction to upsets and left–compressed sets

We start with well–known concepts.

Definition 2.1. For any $B \in 2^{[n]}$ we define the upset $\mathcal{U}(B) = \{B' \in 2^{[n]} : B \subset B'\}$. More generally, for $\mathcal{B} \subset 2^{[n]}$ we define the upset

$$\mathcal{U}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{U}(B).$$

For any $\mathcal{C} \subset \{0,1\}^n$ the upset is defined analogously with respect to images.

For any $\mathcal{A} \subset \{0,1\}^n$, any $A = (a_1, \dots, a_n) \in \mathcal{A}$ and $1 \leq j \leq n$ we define the transformation

$$T_j(A) = \begin{cases} (a_1, \dots, a_{j-1}, 1, a_{j+1}, \dots, a_n), & \text{if this is not an element of } \mathcal{A} \\ A, & \text{otherwise} \end{cases}$$

and

$$T_j(\mathcal{A}) = \big\{T_j(A) : A \in \mathcal{A}\big\}.$$

Repeated applications of these transformations yield after finitely many steps an $\mathcal{A}' \subset \{0,1\}^n$, for which

$$T_j(\mathcal{A}') = \mathcal{A}' \quad \text{for all} \quad 1 \leq j \leq n.$$

Clearly, this set is an upset.

For any $\mathcal{A} \subset \{0,1\}^n$ the transformation T_j has the following important properties, which can be easily shown:

(i) It keeps the cardinality unchanged: $|T_j(\mathcal{A})| = |\mathcal{A}|$.

(ii) It does not increase the diameter: $\text{diam}\big(T_j(\mathcal{A})\big) \leq \text{diam}(\mathcal{A})$. (2.1)

(iii) It does not decrease the number of edges: $|\mathcal{E}\big(T_j(\mathcal{A})\big)| \geq |\mathcal{E}(\mathcal{A})|$.

Let $UD(n,d)$ be the set of all upsets in $D(n,d)$. We have

$$E(n,d) = \max_{\mathcal{A} \in D(n,d)} |\mathcal{E}(\mathcal{A})| = \max_{\mathcal{A} \in UD(n,d)} |\mathcal{E}(\mathcal{A})|. \tag{2.2}$$

On the other hand, if $\mathcal{A} \subset \{0,1\}^n$ is an upset and has diameter d, then any $A_1, A_2 \in \mathcal{A}$ have at least $(n-d)$ componentwise common 1's.

Hence

$$E(n,d) = \max_{\mathcal{A}\in D(n,d)} |\mathcal{E}(\mathcal{A})| = \max_{\mathcal{A}\in UD(n,d)} |\mathcal{E}(\mathcal{A})| = \max_{\mathcal{A}\in UI(n,n-d)} |\mathcal{E}(\mathcal{A})| \tag{2.3}$$

where the last formula concerns set systems and $UI(n, n-d)$ denotes the set of all $(n-d)$–intersecting systems which are also upsets.

We note that clearly

$$I(n, n-d) \subset D(n,d) \text{ holds.} \tag{2.4}$$

Another well–known notion is left–compressedness.

Definition 2.2. For any $\mathcal{B} \subset 2^{[n]}$, any $B \in \mathcal{B}$ and $1 \leq i,j \leq n$ we set

$$S_{ij}(B) = \begin{cases} \{i\} \cup \big(B \smallsetminus \{j\}\big), & \text{if } i \notin B, j \in B, \{i\} \cup \big(B \smallsetminus \{j\}\big) \notin \mathcal{B} \\ B, & \text{otherwise} \end{cases}$$

and $S_{ij}(\mathcal{B}) = \big\{S_{ij}(B) : B \in \mathcal{B}\big\}$.

Definition 2.3. $\mathcal{B} \subset 2^{[n]}$ is said to be left–compressed or stable if $S_{ij}(\mathcal{B}) = \mathcal{B}$ for all $1 \leq i \leq j \leq n$.

It can be easily shown that we have the same properties with respect to transformation S_{ij} as for T_j in (2.1).

Therefore, using (2.3) one gets

$$E(n,d) = \max_{\mathcal{A}\in LUI(n,n-d)} |\mathcal{E}(\mathcal{A})|, \tag{2.5}$$

where $LUI(n, n-d)$ is the set of all left–compressed sets from $UI(n, n-d)$.

Definition 2.4. For a set $\mathcal{A} \subset 2^{[n]}$ and $1 \leq i,j \leq n$ we denote by $\mathcal{A}_{i,j}$ the set which is obtained from the set $\mathcal{A}$ by exchanging the coordinates i,j in every $A \in \mathcal{A}$.

Let $\mathcal{A} \in LUI(n,t)$ and $\ell < n$ be the largest integer, such that $\mathcal{A}$ is invariant under exchange operations in $[1,\ell]$, i.e. $\mathcal{A} = \mathcal{A}_{i,j}$ for all $1 \leq i,j \leq \ell$ but $\mathcal{A} \neq \mathcal{A}_{i,\ell+1}$ for some $1 \leq i \leq \ell$.

Moreover let

$$\mathcal{A}' = \{A \in \mathcal{A} : A_{i,\ell+1} \notin \mathcal{A} \text{ for some } 1 \leq i \leq \ell\}. \tag{2.6}$$

We need the following simple, but important

Lemma 1. [4] *Let $\mathcal{A}$ and $\mathcal{A}'$ be sets, which are defined just above and for $A, A_i \in \mathcal{A}'$ introduce the sets $B = A \cap [1,\ell]$, $B_i = A_i \cap [1,\ell]$, $C = A \cap [\ell+1,n]$, and $C_i = A_i \cap [\ell+1,n]$. Then*

(i) $\ell + 1 \notin A$ *for all* $A \in \mathcal{A}'$.

(ii) *Let* $A \in \mathcal{A}'$ *and* $j \in A$, $1 \le j \le \ell$, *then we have* $A_{j,\ell+1} \notin \mathcal{A}$.

(iii) *Let* $A \in \mathcal{A}'$, *then we have* $B' \cup C \in \mathcal{A}'$ *for every* $B' \subset [1,\ell]$ *with* $|B'| = |B|$.

(iv) *Let* $A \in \mathcal{A}'$ *and* $D \in \mathcal{A} \smallsetminus \mathcal{A}'$, *then we have*

$$|A_{i,\ell+1} \cap D| \ge t \quad \text{for all} \quad 1 \le i \le \ell.$$

(v) *Let* $A_1, A_2 \in \mathcal{A}'$ *and* $|B_1| + |B_2| \ne \ell + t$, *then we have*

$$|A_1 \cap A_2| \ge t + 1.$$

(vi) *Let* $A \in \mathcal{A}'$, *then for any* $B' \subset [1,\ell]$ *with* $|B'| < |B|$ *and* $C' \subseteq C$ *we have*

$$(B' \cup C') \notin \mathcal{A}.$$

(vii) *Let* $A \in \mathcal{A}'$, *then for any* $C' \subset C$, $(B \cup C') \in \mathcal{A}$ *implies* $(B \cup C') \in \mathcal{A}'$.

Proof: The statement immediately follows from the left–compressedness of $\mathcal{A}$, the definition of $\mathcal{A}'$, and the maximality of ℓ.

□

The next obvious result shows that the counting of the edges for upsets can be done via cardinalities of the elements.

Lemma 2. *Let $\mathcal{A} \subset 2^{[n]}$ be an upset. Then*

$$|\mathcal{E}(\mathcal{A})| = \sum_{A \in \mathcal{A}} (n - |A|).$$

□

We also need the following result of Harper.

Theorem H. [7] (a special case)

$$\max_{\mathcal{A} \subset \{0,1\}^n, |\mathcal{A}| = 2^{n-1}} |\mathcal{E}(\mathcal{A})| \quad \text{is assumed at the set} \quad \mathcal{H}(n) = \big\{(a_1, \dots, a_n) \in \{0,1\}^n : a_1 = 1\big\},$$

that is,

$$\max_{\mathcal{A} \subset \{0,1\}^n, |\mathcal{A}| = 2^{n-1}} |\mathcal{E}(\mathcal{A})| = |\mathcal{E}\big(\mathcal{H}(n)\big)|.$$

3 An auxiliary result

Lemma 3. *Let $\mathcal{S} \subset 2^{[m]}$ have the properties:*

(i) *$\mathcal{S}$ is complement–closed, that is $A \in \mathcal{S} \Rightarrow \overline{A} \in \mathcal{S}$,*

(ii) *$\mathcal{S}$ is convex, i.e. $A, C \in \mathcal{S}$ and $A \subset B \subset C \Rightarrow B \in \mathcal{S}$.*

Then there exists an $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{S}' \in I(m)$ and

$$\sum_{A \in \mathcal{S}'} (m - |A|) \geq \frac{m-1}{2m} \sum_{A \in \mathcal{S}} (m - |A|) = \frac{m-1}{4} |\mathcal{S}|. \tag{3.1}$$

Moreover, if $\mathcal{S} \neq 2^{[m]}$, then there exists an $\mathcal{S}' \subset \mathcal{S}$, $\mathcal{S}' \in I(m)$ for which strict inequality in (3.1) holds.

Proof: At first we notice that the identity in (3.1) follows from property (i). In the case $\mathcal{S} = 2^{[m]}$, by taking $\mathcal{S}' = \{A \in 2^{[m]} : 1 \in A\}$ we have $\mathcal{S}' \in I(m)$, $|\mathcal{S}'| = \frac{|\mathcal{S}|}{2} = 2^{m-1}$ and easily get (3.1) with equality in this case.

Let now $\mathcal{S} \neq 2^{[m]}$, let $B \in \mathcal{S}$ be any element with minimal cardinality, and let $i \in B$.

We consider the following partition of $\mathcal{S}$:

$\mathcal{S} = \mathcal{S}_1 \dot\cup \mathcal{S}_2 \dot\cup \mathcal{S}_3 \dot\cup \mathcal{S}_4$, where
$\mathcal{S}_1 = \big\{A \in \mathcal{S} : i \in A \text{ and } \big(A \setminus \{i\}\big) \in \mathcal{S}\big\}$ $\mathcal{S}_2 = \big\{A \in \mathcal{S} : i \notin A \text{ and } \big(A \cup \{i\}\big) \in \mathcal{S}\big\}$
$\mathcal{S}_3 = \big\{A \in \mathcal{S} : i \in A \text{ and } \big(A \setminus \{i\}\big) \notin \mathcal{S}\big\}$ $\mathcal{S}_4 = \big\{A \in \mathcal{S} : i \notin A \text{ and } \big(A \cup \{i\}\big) \notin \mathcal{S}\big\}$.

Clearly $|\mathcal{S}_1| = |\mathcal{S}_2|$, $|\mathcal{S}_3| = |\mathcal{S}_4|$ and $\mathcal{S}_3 \neq \varnothing$, since $i \in B \in \mathcal{S}$ and B has minimal cardinality. It is easily seen that

$$\overline{\mathcal{S}_1} = \{\overline{A} : A \in \mathcal{S}_1\} = \mathcal{S}_2 \text{ and } \overline{\mathcal{S}_3} = \{\overline{A} : A \in \mathcal{S}_3\} = \mathcal{S}_4.$$

It is also easily verified that for every $A \in \mathcal{S}_4$ and $A' \in \mathcal{S} \setminus \mathcal{S}_3$, $A \cap A' \neq \varnothing$ holds.

Hence, $(\mathcal{S}_1 \cup \mathcal{S}_4), (\mathcal{S}_1 \cup \mathcal{S}_3) \in I(m)$. Since $\overline{\mathcal{S}_3} = \mathcal{S}_4$, we get

$$\sum_{A \in \mathcal{S}_3 \cup \mathcal{S}_4} (m - |A|) = m \cdot \frac{|\mathcal{S}_3| + |\mathcal{S}_4|}{2}.$$

Hence,

$$\max\left\{\sum_{A \in \mathcal{S}_3} (m - |A|), \sum_{A \in \mathcal{S}_4} (m - |A|)\right\} \geq m \cdot \frac{(|\mathcal{S}_3| + |\mathcal{S}_4|)}{4}. \tag{3.2}$$

On the other hand, by construction of $\mathcal{S}_1, \mathcal{S}_2$, and property $\overline{\mathcal{S}_1} = \mathcal{S}_2$ we have

$$m \cdot \frac{(|\mathcal{S}_1| + |\mathcal{S}_2|)}{2} = \sum_{A \in \mathcal{S}_1} (m - |A|) + \sum_{A \in \mathcal{S}_2} (m - |A|) = 2 \sum_{A \in \mathcal{S}_1} (m - |A|) + \frac{|\mathcal{S}_1| + |\mathcal{S}_2|}{2}.$$

Hence

$$\sum_{A \in \mathcal{S}_1} (m - |A|) = \frac{(m-1)}{4}(|\mathcal{S}_1| + |\mathcal{S}_2|). \tag{3.3}$$

Therefore, from (3.2), (3.3) we get

$$\max \left\{ \sum_{A \in \mathcal{S}_1 \cup \mathcal{S}_3} (m - |A|), \sum_{A \in \mathcal{S}_1 \cup \mathcal{S}_4} (m - |A|) \right\} \geq \frac{m}{4} \cdot (|\mathcal{S}_3| + |\mathcal{S}_4|) +$$

$$+ \frac{m-1}{4}(|\mathcal{S}_1| + |\mathcal{S}_2|) > \frac{m-1}{4}(|\mathcal{S}_1| + |\mathcal{S}_2| + |\mathcal{S}_3| + |\mathcal{S}_4|) = \frac{m-1}{4}|\mathcal{S}|.$$

Corollary. *Let* $\mathcal{S} \subset 2^{[m]}$ *be defined as in Lemma 3 and let (3.1) hold for* $\mathcal{S}' \subset \mathcal{S}$, $\mathcal{S}' \in I(m)$, $|\mathcal{S}'| = \frac{|\mathcal{S}|}{2}$. *Then for any* $c \in \mathbb{R}$

$$\sum_{A \in \mathcal{S}'} (m - |A| + c) \geq \frac{m + 2c - 1}{2(m + 2c)} \sum_{A \in \mathcal{S}} (m - |A| + c). \tag{3.4}$$

Proof: We just notice that (3.4) follows from (3.1) and the identities

$$\frac{m + 2c - 1}{2(m + 2c)} \sum_{A \in \mathcal{S}} (m - |A| + c) = \frac{m + 2c - 1}{2(m + 2c)} \cdot \left(\frac{m}{2} \cdot |\mathcal{S}| + c \cdot |\mathcal{S}| \right)$$

$$= \frac{m + 2c - 1}{4} \cdot |\mathcal{S}| = \frac{m-1}{4}|\mathcal{S}| + \frac{c|\mathcal{S}|}{2} \quad \text{and} \quad \sum_{A \in \mathcal{S}'} (m - |A| + c) = \sum_{A \in \mathcal{S}'} (m - |A|) + \frac{c \cdot |\mathcal{S}|}{2}.$$

□

4 Main step in the proof of the Theorem

Let $\mathcal{A} \in D(n,d)$ be a set with $|\mathcal{E}(\mathcal{A})| = E(n,d)$. According to (2.5) we can assume that $\mathcal{A} \in LUI(n,t)$, where $t = n-d$. The main auxiliary result, which essentially proves the Theorem, is the following

Lemma 4. *Let $\mathcal{A}$ be the set, which is described just above. Then necessarily $\mathcal{A}$ is invariant under exchange operations in*

$$\begin{aligned} &\text{(i)}\ [1,n],\ \textit{if } 2 \mid d \textit{ and } d \le n-3 \\ &\text{(ii)}\ [1,n-2],\ \textit{if } 2 \mid d \textit{ and } d = n-2 \\ &\text{(iii)}\ [1,n-1],\ \textit{if } 2 \nmid d \textit{ and } d \le n-2. \end{aligned} \tag{4.1}$$

Proof: Let ℓ be the largest integer such that $\mathcal{A}_{i,j} = \mathcal{A}$ for all $1 \le i, j \le \ell$. Assume in the opposite to (4.1)

$$\ell < n_1, \tag{4.2}$$

where $n_1 \in \{n-2, n-1, n\}$ depends on the case.

We are going to show that, under assumption (4.2) there exists a $\mathcal{B} \in I(n,t)$ (and hence $\mathcal{B} \in D(n,d)$) with $|\mathcal{E}(\mathcal{B})| > |\mathcal{E}(\mathcal{A})|$, which is a contradiction.

For this we start with a partition $\mathcal{A}' = \bigcup\limits_{i=1}^{\ell} \mathcal{A}(i)$, of the non empty set $\mathcal{A}'$ defined in (2.6), where $\mathcal{A}(i) = \big\{A \in \mathcal{A}' : |A \cap [1,\ell]| = i\big\}$. Of course, some of the $\mathcal{A}(i)$'s can be empty. In fact it follows from Lemma 1 (iv), (v) that $\mathcal{A}(i) = \varnothing$ for all $1 \le i < t$. We will show that all the $\mathcal{A}(i)$'s are empty. Suppose that $\mathcal{A}(i) \neq \varnothing$ for some $i, t \le i \le \ell$. We remind the reader that $\ell + 1 \notin A$ for all $A \in \mathcal{A}'$ (see Lemma 1 (i)). From Lemma 1 (iii) we know that

$$|\mathcal{A}(i)| = \binom{\ell}{i} \cdot |\mathcal{A}^*(i)|, \tag{4.3}$$

where

$$\mathcal{A}^*(i) = \big\{A \cap [\ell+2, n] : A \in \mathcal{A}(i)\big\}. \tag{4.4}$$

Let us note that in the case $n = \ell + 1$ we have $\mathcal{A}^*(i) = \varnothing$ and $|\mathcal{A}^*(i)| = 1$.

Now we consider the set

$$\mathcal{B}(i) = \big\{B : |B \cap [1,\ell]| = i-1, \ell+1 \in B, \big(B \cap [\ell+2, n]\big) \in \mathcal{A}^*(i)\big\}.$$

Clearly

$$|\mathcal{B}(i)| = \binom{\ell}{i-1} \cdot |\mathcal{A}^*(i)| \tag{4.5}$$

and $\mathcal{B}(i) \cap \mathcal{A} = \varnothing$ by Lemma 1 (ii).

With $\mathcal{A}(i)$ and $\mathcal{B}(i)$ we consider also the sets $\mathcal{A}(\ell+t-i)$ and $\mathcal{B}(\ell+t-i)$. Similar to (4.3), (4.5) we have

$$|\mathcal{A}(\ell+t-i)| = \binom{\ell}{\ell+t-i} \cdot |\mathcal{A}^*(\ell+t-i)|$$
$$|\mathcal{B}(\ell+t-i)| = \binom{\ell}{\ell+t-i-1} \cdot |\mathcal{A}^*(\ell+t-i)|. \tag{4.6}$$

We distinguish two cases: a) $i \neq \ell+t-i$, b) $i = \ell+t-i$.

Case a): $i \neq \ell+t-i$

From Lemma 1 (v) it follows that for $B \in \mathcal{B}(i)$, $A \in \mathcal{A}(j)$ with $i+j \neq \ell+t$ $|B \cap A| \geq t$ holds. Hence using this and Lemma 1 (iv) we have

$$\mathcal{H}_1 = \big((\mathcal{A} \smallsetminus \mathcal{A}(\ell+t-i)) \cup \mathcal{B}(i)\big) \in I(n,t) \text{ and } \mathcal{H}_2 = \big((\mathcal{A} \smallsetminus \mathcal{A}(i)) \cup \mathcal{B}(\ell+t-i)\big) \in I(n,t).$$

Let us show that

$$\max\big\{|\mathcal{E}(\mathcal{H}_1)|, |\mathcal{E}(\mathcal{H}_2)|\big\} > |\mathcal{E}(\mathcal{A})| = E(n,d), \tag{4.7}$$

which will be a contradiction.

Using Lemma 1 (vi), (vii) one can easily show that the sets $\mathcal{H}_1$, $\mathcal{H}_2$, $\big(\mathcal{A} \smallsetminus \mathcal{A}(j)\big)$ are all upsets. Therefore, we have (by Lemma 2)

$$|\mathcal{E}(\mathcal{A})| = |\mathcal{E}\big(\mathcal{A} \smallsetminus \mathcal{A}(\ell+t-i)\big)| + \sum_{A \in \mathcal{A}(\ell+t-i)} (n-|A|) = |\mathcal{E}\big(\mathcal{A} \smallsetminus \mathcal{A}(i)\big)| + \sum_{A \in \mathcal{A}(i)} (n-|A|)$$
$$|\mathcal{E}(\mathcal{H}_1)| = |\mathcal{E}\big(\mathcal{A} \smallsetminus \mathcal{A}(\ell+t-i)\big)| + \sum_{A \in \mathcal{B}(i)} (n-|A|) \tag{4.8}$$
$$|\mathcal{E}(\mathcal{H}_2)| = |\mathcal{E}\big(\mathcal{A} \smallsetminus \mathcal{A}(i)\big)| + \sum_{A \in \mathcal{B}(\ell+t-i)} (n-|A|).$$

Hence negation of (4.7) is

$$\sum_{A \in \mathcal{A}(\ell+t-i)} (n-|A|) \geq \sum_{A \in \mathcal{B}(i)} (n-|A|)$$
$$\sum_{A \in \mathcal{A}(i)} (n-|A|) \geq \sum_{A \in \mathcal{B}(\ell+t-i)} (n-|A|). \tag{4.9}$$

Since we have assumed $\mathcal{A}(i) \neq \varnothing$, then clearly $\mathcal{A}(\ell+t-i) \neq \varnothing$ as well, because otherwise the first inequality of (4.9) is false.

Using properties of the sets $\mathcal{A}(i)$, $\mathcal{B}(i)$ (see also (4.3), (4.5), (4.6)) we can write (4.9) in the form

$$\binom{\ell}{\ell+t-i} \cdot \sum_{C \in \mathcal{A}^*(\ell+t-i)} (n-\ell-t+i-|C|) \geq \binom{\ell}{i-1} \cdot \sum_{D \in \mathcal{A}^*(i)} (n-i-|D|)$$
$$(4.10)$$
$$\binom{\ell}{i} \cdot \sum_{D \in \mathcal{A}^*(i)} (n-i-|D|) \geq \binom{\ell}{\ell+t-i-1} \cdot \sum_{C \in \mathcal{A}^*(\ell+t-i)} (n-\ell-t+i-|C|).$$

However (4.10) implies

$$(\ell-i+1)(i+1-t) \geq (\ell+t-i)i,$$

which is false, because $t \geq 2$ and consequently $i > i+1-t$, $\ell+t-i > \ell-i+1$.

Hence $\mathcal{A}(i) = \varnothing$ for all $i \neq \ell+t-i$.

Case b): $i = \ell+t-i$ or $i = \frac{\ell+t}{2}$.
Here necessarily $2 \mid (\ell+t)$ and therefore by assumption (4.2) we have in (4.1)

$$\ell \leq n-2 \text{ in the case (i)}, \ \ell \leq n-4 \text{ in the case (ii)}, \ \ell \leq n-3 \text{ in the case (iii)}. \quad (4.11)$$

Recalling (4.3) and (4.4) we have

$$\left|\mathcal{A}\left(\frac{\ell+t}{2}\right)\right| = \binom{\ell}{\frac{\ell+t}{2}} \cdot \left|\mathcal{A}^*\left(\frac{\ell+t}{2}\right)\right| \quad (4.12)$$

and any $A \in \mathcal{A}\left(\frac{\ell+t}{2}\right)$ can be written in the form $A = B \,\dot\cup\, C$, where $B = \big(A \cap [1,\ell]\big)$ is any element of $\binom{[\ell]}{\frac{\ell+t}{2}}$, $C = \big(A \cap [\ell+2, n\big) \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$.

We remind the reader again that $\ell+1 \notin A$ for all $A \in \mathcal{A}(i) \subset \mathcal{A}'$.

Now we consider any element $A' = B' \,\dot\cup\, C'$, where $B' \in \binom{[\ell]}{\frac{\ell+t}{2}}$, $C \subset C' \subset [\ell+2, n]$ and $C \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$.

Of course, $A' \in \mathcal{A}$, since $\mathcal{A}$ is an upset and $(B' \cup C) \in \mathcal{A}' \subset \mathcal{A}$, $(B' \cup C) \subset (B' \cup C')$. It is also clear by definition that, if $A' \in \mathcal{A}'$, then $A' \in \mathcal{A}\left(\frac{\ell+t}{2}\right)$.

Using Lemma 1 (iv) we can say more:

$A' = B' \cup C' \in \mathcal{A}\left(\frac{\ell+t}{2}\right)$ if and only if there is a $C'' \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ with $C'' \cap C' = \varnothing$, and hence with every $C \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ we have also $\overline{C} = \big([\ell+2, n] \smallsetminus C\big) \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$.

Moreover, it is easily seen that $\mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ is a convex set.

Therefore $\mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ has the properties described in Lemma 3 and we can apply Lemma 3 and the Corollary to get an *intersecting* set $\mathcal{A}_1^*\left(\frac{\ell+t}{2}\right) \subset \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ for which (3.3) holds:

$$\sum_{D \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (m - |D| + c) \geq \frac{m+2c-1}{2(m+2c)} \sum_{D \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)} (m - |D| + c) \tag{4.13}$$

for $m = n - \ell - 1$ and any constant c.

Now we denote by

$$\mathcal{B}_1 = \left\{B : |B \cap [1,\ell]| = \frac{\ell+t}{2} - 1, \ell+1 \in B, \big(B \cap [\ell+2,n]\big) \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)\right\}$$

$$\mathcal{A}_1\left(\frac{\ell+t}{2}\right) = \left\{A \in \mathcal{A}\left(\frac{\ell+t}{2}\right) : \big(A \cap [\ell+2,n]\big) \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)\right\} \tag{4.14}$$

and consider the following competitor of the set $\mathcal{A}$:

$$\mathcal{H}_3 = \left(\left(\mathcal{A} \smallsetminus \mathcal{A}\left(\frac{\ell+t}{2}\right)\right) \cup \mathcal{A}_1\left(\frac{\ell+t}{2}\right) \cup \mathcal{B}_1\right).$$

It is easily seen that $\mathcal{H}_3 \in I(n,t)$.

We are going to show (under assumption (4.11)) that

$$|\mathcal{E}(\mathcal{H}_3)| > |\mathcal{E}(\mathcal{A})|, \tag{4.15}$$

which will be a contradiction.

It is easily verified that both, $\mathcal{H}_3$ and $\left(\mathcal{A} \smallsetminus \mathcal{A}\left(\frac{\ell+t}{2}\right)\right)$, are upsets.

Therefore, by Lemma 2 we can write

$$|\mathcal{E}(\mathcal{A})| = \left|\mathcal{E}\left(\mathcal{A} \smallsetminus \mathcal{A}\left(\frac{\ell+t}{2}\right)\right)\right| + \sum_{A \in \mathcal{A}\left(\frac{\ell+t}{2}\right)} (n - |A|)$$

$$|\mathcal{E}(\mathcal{H}_3)| = \left|\mathcal{E}\left(\mathcal{A} \smallsetminus \mathcal{A}\left(\frac{\ell+t}{2}\right)\right)\right| + \sum_{A \in \mathcal{A}_1\left(\frac{\ell+t}{2}\right) \cup \mathcal{B}_1} (n - |A|).$$

Hence negation of (4.15) is

$$\sum_{A \in \mathcal{A}\left(\frac{\ell+t}{2}\right)} (n - |A|) \geq \sum_{A \in \mathcal{A}_1\left(\frac{\ell+t}{2}\right) \cup \mathcal{B}_1} (n - |A|),$$

which can be written in the form (see (4.12), (4.13))

$$\binom{\ell}{\frac{\ell+t}{2}} \cdot \sum_{D \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)} (n - |D|) \geq \left(\binom{\ell}{\frac{\ell+t}{2}} + \binom{\ell}{\frac{\ell+t}{2} - 1} \right) \cdot \sum_{D \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (n - |D|)$$

$$= \binom{\ell+1}{\frac{\ell+t}{2}} \cdot \sum_{D \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (n - |D|),$$

and this is equivalent to

$$\frac{\ell - t + 2}{2(\ell+1)} \cdot \sum_{D \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)} (n - |D|) \geq \sum_{D \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (n - |D|). \tag{4.16}$$

However (4.13) for $m = n - \ell - 1$, $c = \frac{\ell - t + 2}{2}$ and (4.16) imply

$$\frac{n-t}{n-t+1} \leq \frac{\ell - t + 2}{\ell + 1}, \tag{4.17}$$

which is false, since $t \geq 2$ and (4.11) holds by assumption.

$\square$

5 Final step in the proof of the Theorem

Let $\mathcal{A} \in D(n,d)$ be a set with $|\mathcal{E}(\mathcal{A})| = E(n,d)$. Of course, we can assume that $\mathcal{A}$ is maximal, that is $\mathcal{A} \cup \{A\} \notin D(n,d)$ for all $A \notin \mathcal{A}$. According to (2.5), as in Lemma 4, we can also assume that $\mathcal{A} \in LUI(n, n-d)$.

In the case $d = n-1$, we just notice that any maximal set $\mathcal{B} \in D(n, n-1)$ has cardinality $|\mathcal{B}| = 2^{n-1}$. Now the statement $E(n, n-1) = |\mathcal{E}\big(\mathcal{H}(n)\big)|$ immediately follows from Theorem H.

In the case $2 \mid d$, $d \leq n-3$ we get from Lemma 4 (i):

$|A| \geq n - \frac{d}{2}$ for all $A \in \mathcal{A}$, since $\mathcal{A}$ is invariant in $[1,n]$ and at the same time $\mathcal{A} \in I(n, n-d)$. This implies $\mathcal{A} \subset \mathcal{K}_{\frac{d}{2}}(n, n-d) \in D(n,d)$ and by maximality of $\mathcal{A}$ we get

$$\mathcal{A} = \mathcal{K}_{\frac{d}{2}}(n, n-d).$$

Now we consider the case $2 \mid d$, $d = n-2$. Looking at the proof of Lemma 4 (ii) we see that in (4.17) for $t = n - d = 2$, $\ell = n-2$ we have an equality, which means that Lemma 4 (ii) can be slightly changed to

(ii)* If $2 \mid d$ and $d = n-2$, then there exists an optimal set which is invariant in $[1,n]$.

Therefore, in this case again we have

$$E(n,d) = |\mathcal{E}\big(\mathcal{K}_{\frac{d}{2}}(n, n-d)\big)|.$$

We verify (for $2 \mid d$, $d = n-2$) that

$$|\mathcal{E}\big(\mathcal{K}_{\frac{d-2}{2}}(n, n-d)\big)| = |\mathcal{E}\big(\mathcal{K}_{\frac{d}{2}}(n, n-d)\big)|$$

and hence $\mathcal{K}_{\frac{d-2}{2}}(n, n-d)$ is the second optimal configuration in this case (see the remark after the formulation of the Theorem).

Finally, the case $2 \nmid d$, $d \leq n-2$ follows from Lemma 4 (iii) by similar arguments.

□

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