An isoperimetric theorem for sequences generated by feedback and feedback-codes for unequal error protection

Rudolf Ahlswede Ning Cai Christian Deppe

Abstract

We derive an isoperimetric theorem for sequences genenerated by feedback and consider block codes for the binary broadcast channel with two receivers and noiseless feedback. We get an outer bound by applying the isoperimetric theorem on the achievable rates for special cases of these codes with unequal error protection. We get a lower bound with a generalized Varshamov-Gilbert construction.

1 Introduction

In this paper we derive a new isoperimetric theorem. We consider the sequences generated as output sequences by a family of feedback-encoding functions and show that these sets fulfill an isoperimetric theorem. We get this theorem by generalizing the Eckhoff-Wegner inequality [7]. With this theorem we get an outer bound for the following broadcast channel. We consider a communication system with one sender (or encoder) E and two receivers (or decoders) D_1 and D_2 with noiseless feedback and unequal error protection. The sender E wants to send a message $i \in \mathcal{M}_1$ to D_1 (decoder 1) and a message $j \in \mathcal{M}_2$ to D_2 (decoder 2) simultaneously and he encodes each pair (i, j) of messages into a binary sequence of length n.

This sequence is sent to the two receivers via two independent channels. Because of the noise in the channels the output sequences received by the two receivers may have errors at most of $t_1 = n\tau_1$ and $t_2 = n\tau_2$ positions respectively. Moreover we assume that there is noiseless feedback. That means, the sender is allowed to choose the m-th position of input according to the first $(m-1)$ positions of both output sequences. An encoding function is of the form: $f_{i,j}^n\left(y_1^{n-1}, y_2^{n-1}\right) = \left(f_{i,j}^{(1)}, \dots, f_{i,j}^{(k)}\left(y_1^{k-1}, y_2^{k-1}\right), \dots, f_{i,j}^{(n)}\left(y_1^{n-1}, y_2^{n-1}\right)\right)$, where $f_{i,j}^{(k)} : \mathcal{Y}_1^{k-1} \times \mathcal{Y}_2^{k-1} \to \mathcal{X}$ is a function for the k-th coordinate, which depends on the $(k-1)$ positions which have been received by decoder 1 and 2, where $\mathcal{X} = \mathcal{Y} = \{0, 1\}$. We call a family of encoding functions $\{f_{i,j}^n : i \in \mathcal{M}_1, j \in \mathcal{M}_2\}$ a binary $(n, M_1, M_2, (t_1, t_2))$ feedback code (or briefly an fb-code) for the broadcast channel where $|\mathcal{M}_k| = M_k$ for $k = 1, 2$.

The first paper about the broadcast channel [6] was published by T. M. Cover in 1972. Boyarinov [5] introduced codes with unequal error protection. The best known upper and lower bounds for the code rate without feedback can be found in [2]. In [9] the authors obtained lower bounds by using linear codes. It is shown in [3] that these constructions gave no better results than in [2]. We consider here the model with feedback. It can also be viewed as an extension of the model which was introduced by E. R. Berlekamp [4]. He considered the model with one sender and one receiver with feedback. We consider $(n, M_1, M_2, (0,t))$ feedback codes. In this model, the feedback increases the rate already by time-sharing.

2 An isoperimetric theorem

We will present an isoperimetric theorem, therefore we need an isoperimetric inequality. For a subset $A \subset \{0,1\}^n$, let $\Gamma^t(A) = \{x^n :$ there exists an $a^n \in A$ such that $d_H(x^n, a^n) \leq t\}$ for $1 \leq t \leq n$. Then the isoperimetric problem for binary Hamming space asks which sets achieve $\min_{A:|A|=u} |\Gamma^t(A)|$. This problem was solved by Harper [8]. Later Katona [10] gave another proof. It is shown that the initial segments of the following order are always optimal for the minimization. Let x^n and y^n be two binary sequences with the same Hamming weight. Then we say that x^n precedes y^n in the squashed order if $y_i = 1$ at the largest component i where $x_i \neq y_i$. A binary sequence x^n precedes a binary sequence y^n in Harper-order, briefly H-order, if the Hamming weight of x^n is less than the Hamming weight of y^n or they have the same Hamming weight and $1^n - x^n$ precedes $1^n - y^n$ in the squashed order. In the binary Hamming space of n dimensions the uth initial segment in H-order is denoted by $S_{n,u}$. Clearly

$$\Gamma^t(S_{n,u}) \subset \Gamma^t(S_{n,v}) \tag{2.1}$$

for all n, t if $u < v$. We present the outer bound in terms of the function G which can be found in [10]. For given n any non-negative integer u can be uniquely represented as

$$u = \binom{n}{n} + \ldots + \binom{n}{k+1} + \binom{\alpha_k}{k} + \ldots + \binom{\alpha_t}{t} \tag{2.2}$$

with $n > \alpha_k > \ldots > \alpha_t \geq t \geq 1$. We define the function G as

$$G(n,u) = \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k} + \binom{\alpha_k}{k-1} + \ldots + \binom{\alpha_t}{t-1}. \tag{2.3}$$

Moreover we rewrite G as $G^{\circ 1}$ and define $G^{\circ t}(n, \cdot) = G(n, G^{\circ t-1}(n, \cdot))$ recursively. The Isoperimetric Theorem ([8], [10]) says that

$$\min_{A:|A|=u} |\Gamma^t(A)| = |\Gamma^t(S_{n,u})| = G^{\circ t}(n,u). \tag{2.4}$$

Eckhoff and Wegner [7] proved that if $0 \leq u_1 \leq u_0$ and $u = u_0 + u_1$,

$$G(n,u) \leq \max[u_0,\ G(n-1,u_1)] + G(n-1,u_0). \tag{2.5}$$

To obtain our outer bound we need its generalization.

Lemma 1 *Let u_0 and u_1 be non-negative integers and $u = u_0 + u_1$, then*

$$G^{\circ t}(n,u) \leq \max[G^{\circ t}(n-1,u_0), G^{\circ t-1}(n-1,u_1)] + \max[G^{\circ t-1}(n-1,u_0), G^{\circ t}(n-1,u_1)] \tag{2.6}$$

for all n and $0 \leq t \leq n$.

Proof: Let us assume to the contrary that for some n, t and $u = u_0 + u_1$

$$\begin{aligned} G^{\circ t}(n,u) &> \max[G^{\circ t}(n-1,u_0),\ G^{\circ t-1}(n-1,u_1)] \\ &+ \max[G^{\circ t-1}(n-1,u_0),\ G^{\circ t}(n-1,u_1)]. \end{aligned} \tag{2.7}$$

Let $A_0 = \{(a^{n-1},0) : a^{n-1} \in S_{n-1,u_0}\}$, $A_1 = \{(a^{n-1},1) : a^{n-1} \in S_{n-1,u_1}\}$, and $A = A_0 \cup A_1$. For $k = 0,1$ the elements in the subset $A_k \subset \{0,1\}^n$ are obtained by adding a letter k after a sequence of length $n-1$ in the u_kth initial segment in the H-order in $n-1$ dimensional binary Hamming space. Then $A_0 \cap A_1 = \emptyset$, $|A_0| = u_0$, $|A_1| = u_1$ and $|A| = u = u_0 + u_1$. Note that $\Gamma^t(A)$ consists of two disjoint parts according to whether the last components of their members are 0 or 1. The former is

$$\{(b^{n-1},0) : b^{n-1} \in \Gamma^t(S_{n-1,u_0})\} \cup \{(b^{n-1},0) : b^{n-1} \in \Gamma^{t-1}(S_{n-1,u_1})\}$$

and the latter is

$$\{(b^{n-1},1) : b^{n-1} \in \Gamma^{t-1}(S_{n-1,u_0})\} \cup \{(b^{n-1},1) : b^{n-1} \in \Gamma^t(S_{n-1,u_1})\}.$$

Thus by (2.1) and (2.4) we have

$$|\Gamma^t(A)| = \max[G^{\circ t}(n-1,u_0), G^{\circ t-1}(n-1,u_1)] + \max[G^{\circ t-1}(n-1,u_0), G^{\circ t}(n-1,u_1)],$$

which with (2.7) yields $\Gamma^t(A) < G^{\circ t}(n,u)$. A contradiction to (2.4).

Let us now turn to our problem of binary error correcting codes with feedback. We will obtain the theorem by counting the number of possible output sequences. Consider a family of encoding functions of a binary feedback code $f_m^n : \{0,1\}^{n-1} \longrightarrow \{0,1\}^n$, $m \in \mathcal{M}$,

$$f_m^n(y^{n-1}) = (f_m^{(1)}, f_m^{(2)}(y_1), f_m^{(3)}(y_1,y_2), \ldots, f_m^{(n)}(y_1,y_2,\ldots,y_{n-1})). \tag{2.8}$$

When they are input to a channel (of one receiver) with noiseless feedback, the output $y^n = (y_1, \ldots, y_n)$ with

$$y_1 = f_m^{(1)} + e_1 \text{ and } y_t = f_m^{(t)}(y_1, y_2, \ldots y_{t-1}) + e_t \text{ for t = 2,3, \ldots, n} \tag{2.9}$$

is uniquely determined by the encoding function f_m^n and the binary error pattern $e^n = (e_1, e_2, \ldots, e_n)$ occurring in the transmission and so can be regarded as their function $\Phi(f_m^n, e^n)$. For a family of encoding functions $\{f_m^n : \ m \in \mathcal{M}\}$ and a set $\mathcal{E}$ of error patterns we write

$$\Phi(f_{\mathcal{M}}, \mathcal{E}) = \{y^n : \text{ there exist } m \in \mathcal{M} \text{ and } e^n \in \mathcal{E} \text{ such that } y^n = \Phi(f_m^n, e^n)\}. \tag{2.10}$$

We believe that the following theorem is independently interesting in Combinatorics because it can be considered as an isoperimetric theorem for the sequences generated by feedback. By choosing as the set of error patterns $\mathcal{E} = \mathcal{E}(n,t)$, the set of binary sequences of length n whose Hamming weight is not exceeding t, we have

Theorem 1 *For any family $\{f_m^n : \ m \in \mathcal{M}\}$ of encoding functions,*

$$|\Phi(f_{\mathcal{M}}^n, \mathcal{E}(n,t))| \geq G^{\circ t}(n, |\mathcal{M}|). \tag{2.11}$$

Proof (Induction on n): It is trivial when $n = 1$. Suppose that we have shown that the theorem is true for $n-1$ and we are given a family $\{f_m^n : \ m \in \mathcal{M}\}$ of encoding feedback functions. We first partition $\mathcal{M}$ into two parts $\mathcal{M}(0)$ and $\mathcal{M}(1)$, according to the first components of f_m^n's, $f_m^{(1)} = 0$ or 1. According to the first components of the output sequences, 4 families of encoding functions of length $n-1$ are further generated by the original encoding functions as follows.

$$f_m^{*n-1}(y_2, \ldots, y_n) = (f_m^{(2)}(0), f_m^{(3)}(0, y_2), \ldots, f_m^{(n)}(0, y_2, \ldots, y_n)) \text{ for } m \in \mathcal{M}(0), \tag{2.12}$$

$$f_m^{*n-1}(y_2, \ldots, y_n) = (f_m^{(2)}(1), f_m^{(3)}(1, y_2), \ldots, f_m^{(n)}(1, y_2, \ldots, y_n)) \text{ for } m \in \mathcal{M}(0), \tag{2.13}$$

$$f_m^{*n-1}(y_2, \ldots, y_n) = (f_m^{(2)}(0), f_m^{(3)}(0, y_2), \ldots, f_m^{(n)}(0, y_2, \ldots, y_n)) \text{ for } m \in \mathcal{M}(1), \tag{2.14}$$

and

$$f_m^{*n-1}(y_2, \ldots, y_n) = (f_m^{(2)}(1), f_m^{(3)}(1, y_2), \ldots, f_m^{(n)}(1, y_2, \ldots, y_n)) \text{ for } m \in \mathcal{M}(1). \tag{2.15}$$

We set the index sets $\mathcal{M}^*(k,l)$, for $k,l = 0,1$ and label the members of the above 4 families of encoding functions by the elements in the index sets $\mathcal{M}^*(0,0)$, $\mathcal{M}^*(0,1)$, $\mathcal{M}^*(1,0)$ and $\mathcal{M}^*(1,1)$. Then

$$|\mathcal{M}^*(0,0)| = |\mathcal{M}^*(0,1)| = |\mathcal{M}(0)| \text{ and } |\mathcal{M}^*(1,0)| = |\mathcal{M}^*(1,1)| = |\mathcal{M}(1)| \tag{2.16}$$

and

$$|\mathcal{M}| = |\mathcal{M}(0)| + |\mathcal{M}(1)|. \tag{2.17}$$

We note that any output sequence $y^n = (y_1, \dots, y_n) \in \Phi(f^n_{\mathcal{M}}, \mathcal{E}(n,t))$ with $y_1 = 0$ can be obtained by adding a "0" in front of a sequence $(y_2, \dots, y_n)$ in $\Phi(f^{*n-1}_{\mathcal{M}^*(0,0)}, \mathcal{E}(n-1,t))$ or in $\Phi(f^{*n-1}_{\mathcal{M}^*(1,0)}, \mathcal{E}(n-1,t-1))$ whereas any sequence $y^n = (y_1, \dots, y_n) \in \Phi(f^n_{\mathcal{M}}, \mathcal{E}(n,t))$ with $y_1 = 1$ can be obtained by adding a "1" in front of a sequence $(y_2, \dots, y_n)$ in $\Phi(f^{*n-1}_{\mathcal{M}^*(0,1)}, \mathcal{E}(n-1,t-1))$ or in $\Phi(f^{*n-1}_{\mathcal{M}^*(1,1)}, \mathcal{E}(n-1,t))$. Thus by the previous lemma, (2.12)-(2.17), and the induction hypothesis, we have that

$$\begin{aligned}
|\Phi(f^n_{\mathcal{M}}, \mathcal{E}(n,t))| &= |\Phi(f^{*n-1}_{\mathcal{M}^*(0,0)}, \mathcal{E}(n-1,t)) \cup \Phi(f^{*n-1}_{\mathcal{M}^*(1,0)}, \mathcal{E}(n-1,t-1))| \\
&+ |\Phi(f^{*n-1}_{\mathcal{M}^*(0,1)}, \mathcal{E}(n-1,t-1)) \cup \Phi(f^{*n-1}_{\mathcal{M}^*(1,1)}, \mathcal{E}(n-1,t)| \\
&\geq \max[|\Phi(f^{*n-1}_{\mathcal{M}^*(0,0)}, \mathcal{E}(n-1,t))|,\ |\Phi(f^{*n-1}_{\mathcal{M}^*(1,0)}, \mathcal{E}(n-1,t-1))|] \\
&+ \max[|\Phi(f^{*n-1}_{\mathcal{M}^*(0,1)}, \mathcal{E}(n-1,t-1))|,\ |\Phi(f^{*n-1}_{\mathcal{M}^*(1,1)}, \mathcal{E}(n-1,t))|] \\
&\geq \max[G^{\circ t}(n-1, |\mathcal{M}^*(0,0)|),\ G^{\circ t-1}(n-1, |\mathcal{M}^*(1,0)|)] \\
&+ \max[G^{\circ t-1}(n-1, |\mathcal{M}^*(0,1)|),\ G^{\circ t}(n-1, |\mathcal{M}^*(1,1)|)] \\
&\geq G^{\circ t}(n, |\mathcal{M}|),
\end{aligned} \tag{2.18}$$

where the first inequality follows from the induction hypothesis and the last inequality follows from (2.6), (2.16), and (2.17).

3 Hamming-type bounds

Berlekamp proved the following Hamming bound for binary t error correcting codes with feedback [4].

Proposition 1 *(The Hamming bound) Let $N, t \in \mathbb{N}$, then for every binary t error correcting codes with feedback we have $N \leq \frac{2^n}{\sum_{j=0}^{t} \binom{n}{j}}$, where N is the cardinality of the message-set.*

By counting output sequences, one can obtain a very simple proof to it. Let $\{f^n_i : i = 1, \dots, M\}$ be the set of encoding functions of such a code. Then for $(i, e^n) \neq (i', e'^n)$, in terminology from above, $\Phi(f^n_i, e^n) \neq \Phi(f^n_{i'}, e'^n)$ because (1) the decoder is able to distinguish the messages i and i' if $i \neq i'$ and (2) the tth position of $\Phi(f_i^n, e^n)$ and $\Phi(f_{i'}^n e'^n)$ must be different if $i = i'$, $e^n \neq e'^n$, and the tth position is the first position where e^n and e'^n are different. So $M|\mathcal{E}(n,t)| = M \sum_{k=0}^{t} \binom{n}{k} \leq 2^n$.

Corollary 1 *Let $n \to \infty$, $\tau = \frac{t(n)}{n}$, then for t error correcting codes with feedback holds $R \leq 1 - h(\tau)$.*

We also get an outer bound for an $(n, M_1, M_2, (0,t))$ feedback code.

Theorem 2 *If there exists an $(n, M_1, M_2, (0,t))$ binary fb-codes for broadcast channels, then*

$$M_2 \leq \frac{2^n}{G^{\circ t}(n, M_1)}. \tag{3.1}$$

Proof: Let $\{f_{i,j}^n : i = 1, \ldots, M_1 \text{ and } j = 1, \ldots, M_2\}$ be the set of encoding functions of a given $(n, M_1, M_2, (0,t))$ fb-code. Denote by $\mathcal{M}'_j = \{(i,j) : i = 1, \ldots, M_1\}$ and partition the set of encoding functions into

$$\{f_{i,j}^n : (i,j) \in \mathcal{M}'_j\},\ j = 1, \ldots, M_2.$$

Then by Lemma 2 we have that

$$\Phi(f_{\mathcal{M}'_j}^n, \mathcal{E}(n,t)) \geq G^{\circ t}(n, M_1) \tag{3.2}$$

for all j, since $|\mathcal{M}'_j| = M_1$. On the other hand, for $j \neq j', \Phi(f_{\mathcal{M}'_j, \mathcal{E}}(n,t))$ and $\Phi(f_{\mathcal{M}'_{j'}, \mathcal{E}}(n,t))$ must be disjoint because the decoder is able to distinguish the messages j and j'. Thus it follows from (3.2) that

$$M_2\ G^{\circ t}(M_1) \leq 2^n, \tag{3.3}$$

because the whole output space contains 2^n binary sequences. Therefore (3.1) holds.

Corollary 2 *Let $n \to \infty$, $\tau = \frac{t(n)}{n}$ and p be chosen such that $R_1 = h(p)$, then $R_2 \leq 1 - h(p + \tau)$.*

The asymptotic version of Berlekamp's Hamming-Bound (Corollary 1) is tight, if $R \leq R_0 = 0.29650$. For bigger rates Berlekamp proved that a tangent to the Hamming bound through the point R_0 is tight. We conjecture that we will have a similar behaviour in the present case, this means, that our outer bound is tight for small rates.

4 A code construction

We present a binary $(n, M_1, M_2, (0,t))$ feedback code, which gives some better rates than the best known code-construction without feedback. We send the data just in two rounds. In the first round the sender sends the first $k = \lceil \log M_2 \rceil$ positions according to the message in $\mathcal{M}_2$ for D_2, then by receiving the feedback of the output of the first round and the message of D_1, he sends the remaining positions in the second round. The idea is to transmit $i \in \mathcal{M}_1$ to D_1 and $j \in \mathcal{M}_2$ to D_2 in the following way. Let $\mathcal{M}_i = \{0, \dots, M_i - 1\}$, the sender transmits in the first k positions the binary representation of the message j. After this transmission the sender knows the error pattern of D_2 ($E_k = \{\binom{[k]}{l} : 0 \leq l \leq t\}$). In the remaining $n-k$ positions the sender encodes the message $i \in \mathcal{M}_1$ for D_1 and the error pattern $e^k \in E_k$ for D_2. We assume that $|\mathcal{M}_1| = M_1 = \sum_{l=0}^{r} \binom{n-k}{l}$ and $M_2 = 2^k$. Let f be the number of errors which occurred in the first k positions ($w(e^k) = f$). For $x^n \in \{0,1\}^n$ we denote by $B_{n,r}(x^n) = \{y^n \in \{0,1\}^n : d_H(x^n, y^n) \leq r\}$ the ball of radius r. The sender knows that at most $t - f$ errors can occur in the remaining $n-k$ positions. For each $e^k \in E_k$ we construct a Hamming ball with the radius r with the center $x(e^k) \in \{0,1\}^{n-k}$. If the error-pattern e^k occurred the sender sends a sequence in the corresponding Hamming ball, depending on the message for D_1. If for all error patterns $e_1^k, e_2^k \in E_k$ ($e_1^k \neq e_2^k$) holds $d_H(B_{n-k,r}(x(e_1^k)), B_{n-k,r}(x(e_2^k))) = \min\{d_H(x^{n-k}, y^{n-k}) : x^{n-k} \in B_{n-k,r}(x(e_1^k)), y^{n-k} \in B_{n-k,r}(x(e_2^k))\} \geq 2t - w(e_1^k) + w(e_2^k)$, D_1 can decode his message and D_2 can decode the error pattern. We will give a formal construction of the code and calculate its rate. The idea of this construction is given in [2]. For the construction we will use the fact that for two sets $V, W \subset \{0,1\}^n$ holds

$$\frac{\min_{x^n \in \{0,1\}^n} |(x^n + V) \cap W|}{|V|} \leq \frac{|W|}{2^n}. \qquad (4.1)$$

This is true, because $2^n \min_{x^n \in \{0,1\}^n} |(x^n + V) \cap W| \leq \sum_{x^n \in \{0,1\}^n} |(x^n + V) \cap W| = |V||W|$. For $W \subset \{0,1\}^n$ we denote by $\mathcal{M}_{n,r}(W) = \bigcup_{x \in W} B_{n,r}(x)$ the r-neighborhood of W.

Let $1 < g \leq g_0 = |E_k|$, $V \subset \{0,1\}^{n-k}$ and $u_1, \dots, u_{g_0} \in \{0,1\}^{n-k}$. We set $m(V, n, k, t, j) = M_{n-k,2i}(u_j + V)$, where $i = \min\{h \in \mathbb{N} : j \leq \sum_{l=0}^{h} \binom{k}{t-l}\}$ and we define

$$\mathcal{M}(V, n, k, t, u_1, \dots, u_g) = \bigcup_{j=1}^{g} m(V, n, k, t, j).$$

Now we describe the construction. Let $V \subset \{0,1\}^{n-k}$, we construct inductively the following translations of V: $u_1 + V, \dots, u_{g_0} + V$. Let $u_1 = 0^{n-k}$. Assume $u_1, \dots, u_{g-1}$ have already been chosen, then we take a point u_g which satisfies the inequality

$$\frac{|(u_g + V) \cap \mathcal{M}(V, n, k, t, u_1, \dots, u_{g-1})|}{|V|} \leq \frac{|\mathcal{M}(V, n, k, t, u_1, \dots, u_{g-1})|}{2^{n-k}}. \qquad (4.2)$$

Such a point always exists, because of (4.1). If after the procedure we have

$$\frac{|\mathcal{M}(V,n,k,t,u_1,\ldots,u_{g_0})|}{2^{n-k}} \leq \frac{1}{2}, \qquad (4.3)$$

then by (4.2) and (4.3) we have

$$\frac{|(u_g+V)\cap\mathcal{M}(V,n,k,t,u_1,\ldots,u_{g-1})|}{|V|} \leq \frac{1}{2}$$

for all $g \leq g_0$. We set $V_i = (u_i+V)\setminus\mathcal{M}(V,n,k,t,u_1,\ldots,u_{i-1})$. Thus $\min_{i=1,\ldots,g_0}|V_i| \geq \frac{|V|}{2}$ holds. Thus, if we assume that all neighborhoods are disjoint and choose V as a Hamming ball, we get $|\mathcal{M}(V,n,k,t,u_1,\ldots,u_{g_0})| = \sum_{i=0}^{t}[\binom{k}{t-i}\sum_{j=0}^{r+2i}\binom{n-k}{j}]$. Now we can use V_i to decode the message of D_1. Depending on which V_i we use, D_2 can decode the error pattern. Let $R_i = \frac{\log M_i(n)}{n}$ for $i=1,2$. In our construction we have $2^{nR_1-1} = |V|$ and $R_2 = \frac{k}{n}$. It follows

Proposition 2 *For $M_2,n,t \in \mathbb{N}$, there exists an $(n,M_1,M_2,(0,t))$ feedback code with $M_1 = \frac{1}{2}\sum_{j=0}^{r}\binom{n-k}{j}$, if $\sum_{i=0}^{t}[\binom{k}{t-i}\sum_{j=0}^{r+2i}\binom{n-k}{j}] \leq 2^{n-k-1}$, where $k = \lceil\log M_2\rceil$.*

Definition 1 *For $0 \leq \beta,\rho < 1$ we set*

$$I(\beta,\rho) = \min_{0\leq\gamma\leq\min\{\frac{\beta}{2},\tau\}}\{(1-\beta)(1-h(\frac{\rho+2\tau-2\gamma}{1-\beta}))-\beta h(\frac{\gamma}{\beta})\}.$$

We will bound the region of permissible values (R_1,R_2,τ), $\tau n = t$ for all large n. With our code construction we get the following bound.

Theorem 3 *Let $0\leq\beta\leq 1$, $0\leq\tau\leq\frac{1}{2}$ and $\rho\leq\frac{1-\beta}{2}-2\tau$, then there exists an $N_0\in\mathbb{N}$ such that for all $n\geq N_0$ there exists an $(n,M_1,M_2,(0,t))$ feedback code with the rates:*
$R_1 = (1-\beta)h(\frac{\rho}{1-\beta})$ and $R_2=\beta$, if $I(\beta,\rho)\geq 0$.

Proof: There exists an $(n,M_1,M_2,(0,t))$ feedback code, if $0 \leq 2^{n-k-1}-(\sum_{i=0}^{t}[\binom{k}{t-i}\sum_{j=0}^{r+2i}\binom{n-k}{j}])$ (Proposition 2). Let $n\to\infty$ and $\frac{t}{n}\to\tau$. We get the statement of the theorem

Acknowledgment: The authors are grateful to L. A. Bassalygo for drawing their attention to a basic extension of a Varshamov-Gilbert type code in [2], which leads to the present code construction (improving our original result).

References

[1] R. Ahlswede, "A constructive proof of the coding theorem for discrete memoryless channels with feedback", Transactions of the Sixth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes (Tech. Univ., Prague), 39-50, 1971.

[2] L. A. Bassalygo, V. A. Zinov'ev, V. V. Zyablov, M. S. Pinsker, G. Sh. Poltyrev, "Bounds for codes with unequal protection of two sets of messages", Problems Inform. Transmission 15, no. 3, 190–197, (1980); translated from Problemy Peredachi Informatsii 15, no. 3,40–49, 1979 (Russian).

[3] L. A. Bassalygo, M. S. Pinsker, "A remark on a paper by T. Kasami, S. Lin, V. K. Wei and S. Yamamura: "Coding for the binary symmetric broadcast channel with two receivers" [IEEE Trans. Inform. Theory 31 (1985), no. 5, 616–625; MR 87c:94030]", (Russian) Problemy Peredachi Informatsii 24, no. 3,102–106, 1988; translation in Problems Inform. Transmission 24, no. 3, 253–257, 1989.

[4] E.R. Berlekamp, "Block coding for the binary symmetric channel with noiseless, delayless feedback" in *H.B.Mann, "Error Correcting Codes", Wiley, 61-85, 1968.*

[5] I. M. Boyarinov, "Codes with unequal symbol protection", Fifth Conf. on Coding Theory and Information Transmission, vol. 2, Moscow-Gorki, 22–24, 1972.

[6] T. M. Cover, "Broadcast channels", IEEE Trans. Inf. Theory 18, no. 1, 2-14, 1972.

[7] J. Eckhoff, G. Wegner, "Über einen Satz von Kruskal", Period. Math. Hungar. 6, no. 2, 137–142, 1975.

[8] L. H. Harper, "Optimal numberings and isoperimetric problems on graphs", J. Combinatorial Theory 1, 385–393, 1966.

[9] Tadao Kasami, Shu Lin,Victor K. Wei, Saburo Yamamura, "Coding for the binary symmetric broadcast channel with two receivers", IEEE Trans. Inf. Theory 31, no. 5, 616–625, 1985.

[10] G. Katona, "The Hamming-sphere has minimum boundary", Studia Scientarum Mathematicarum Hungarica 10, 131-140, 1975.