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Forbidden (0,1)-Vectors in Hyperplanes of $\mathbb{R}^n$: The Restricted Case

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Abstract. In this paper we continue our investigation on "Extremal problems under dimension constraint" introduced in [2].

Let $E(n,k)$ be the set of (0,1)-vectors in $\mathbb{R}^n$ with k one's. Given $1 \le m,\ w \le n$ let $X \subset E(n,m)$ satisfy $\operatorname{span}(X) \cap E(n,w) = \varnothing$. How big can $|X|$ be?

This is the main problem studied in this paper. We solve this problem for all parameters $1 \le m,\ w \le n$ and $n > n_0(m,w)$.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers. For the set $\{i, i+1, \dots, j\}$ $(i, j \in \mathbb{N})$ we use the notation $[i, j]$. For $k, n \in \mathbb{N}$, $k \le n$ we set

$$2^{[n]} = \{A : A \subset [1,n]\},\ \binom{[n]}{k} = \left\{A \in 2^{[n]} : |A| = k\right\}.$$

For any subset $X \in 2^{[n]}$ define its characteristic vector $\chi(X) = (x_1, \dots, x_n)$, where $x_i = 1$ if $i \in X$ and $x_i = 0$, if $i \notin X$. We also define $\chi(\mathcal{A}) = \{\chi(X) : X \in \mathcal{A}\}$ for any family $\mathcal{A} \subset 2^{[n]}$ and as a shorthand mostly just write A for $\chi(\mathcal{A})$ or B for $\chi(\mathcal{B})$ etc.

The set of (0,1)-vectors in $\mathbb{R}^n$ is denoted by $E(n) = \{0,1\}^n$. Correspondingly for "k-uniform" vectors we use the notation

$$E(n,k) = \{x^n \in E(n) : x^n \text{ has } k \text{ ones}\}.$$

We consider the following problem. Given $m, w \in \mathbb{N}$ determine

$$F(n, w, m) = \max\{|X| : X \subset E(n, m), \operatorname{span}(X) \cap E(n, w) = \varnothing\}.$$

An equivalent formulation of the function $F(n, w, m)$ is as follows:
Let V be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ so that $V \cap E(n, w) = \varnothing$. Then

$$F(n, w, m) = \max_V |V \cap E(n, m)|.$$

To see the equivalence of these formulations note that any subspace $U \subset \mathbb{R}^n$ of dimension $k < n-1$ can be embedded in a subspace V of dimension $n-1$ so that

$$U \cap E(n) = V \cap E(n).$$

We state now our main results.

THEOREM 1.

(i) For $m \nmid w$, $m < w$ and $n > n_0(w, m)$ we have

$$F(n, w, m) = \max_{\substack{1 \le \ell < n \\ 1 \le i \le m-1}} \binom{\ell}{i}\binom{n-\ell}{m-i} = \binom{t}{1}\binom{n-t}{m-1}, (n = tm + r, 0 < r < m).$$

(ii) For $w < m$ we have

$$F(n, w, m) = \max_{\substack{1 \le \ell < n \\ 1 \le i \le m-1}} \binom{\ell}{i}\binom{n-\ell}{m-i} = \binom{t}{1}\binom{n-t}{m-1}, \ (n = tm + r, 0 \le r < m).$$

THEOREM 2. *For $w = sm$, $s \in \mathbb{N}$ and $n > n_0(w, m)$ we have*

$$F(n, sm, m) = (s-1)\binom{n-s+1}{m-1}.$$

2. Auxiliary Results and Tools

Let $\mathcal{A} \subset 2^{[n]}$. $\mathcal{A}$ is called an antichain if $A_1 \not\subset A_2$ holds for all $A_1, A_2 \in \mathcal{A}$. Correspondingly $\mathcal{A}$ is called a chain if $A_1 \subset A_2$ or $A_1 \supset A_2$ holds for all $A_1, A_2 \in \mathcal{A}$. We need the following result from [3].

LEMMA 1. *Let the ground set $[1, n]$ be partitioned into two parts $[1, n] = [1, \ell] \cup [\ell+1, n]$. Let $\mathcal{A} \subset 2^{[n]}$ be a family with the property*

(P) For any two members A and B of $\mathcal{A}$ one has the following properties: if $A \cap [1, \ell]$ and $B \cap [1, \ell]$ form a chain then $A \cap [\ell+1, n]$ and $B \cap [\ell+1, n]$ form an antichain.

Define

$$\alpha_{ij} = \#\{A \in \mathcal{A} : |A \cap [1, \ell]| = i, |A \cap [\ell+1, n] = j\}.$$

Then we have the following LYM type inequality (for LYM see e.g. [11])

$$\sum_{i,j} \frac{\alpha_{ij}}{\binom{\ell}{i}\binom{n-\ell}{j}} \leq 1.$$

LEMMA 2.

(a) *Given* $0 < i < k < n$*,* $n = tk + k_1$*, with* $t, k_1 \in \mathbb{N}$*,* $t \geq 2$*,* $0 < k_1 < k$*, then*

$$\max_{i \leq \ell < n} \binom{\ell}{i}\binom{n-\ell}{k-i} = \binom{\ell_i}{i}\binom{n-\ell_i}{k-i}, \text{ where } \ell_i = it + \left\lfloor \frac{i(k_1+1)}{k} \right\rfloor. \quad (1)$$

(b) *Given* $0 < \ell, k < n$*, then*

$$\max_{1 \leq i < k} \binom{\ell}{i}\binom{n-\ell}{k-i} = \binom{\ell}{i_\ell}\binom{n-\ell}{k-i_\ell}, \text{ where } i_\ell = \left\lfloor \frac{(\ell+1)(k+1)}{n+2} \right\rfloor. \quad (2)$$

(c) *Given* $0 < k < n$*, then*

$$\max_{\substack{1 \leq \ell < n \\ 1 \leq i < k}} \binom{\ell}{i}\binom{n-\ell}{k-i} = \binom{t}{1}\binom{n-t}{k-1}. \quad (3)$$

Proof.

(a) Suppose the maximum in (1) (with a fixed $0 < i < k$) is attained for some ℓ, $i \leq \ell < n$. Then we have

$$\binom{\ell}{i}\binom{n-\ell}{k-i} \geq \binom{\ell-1}{i}\binom{n-\ell+1}{k-i} \quad \text{and} \quad \binom{\ell}{i}\binom{n-\ell}{k-i} \geq \binom{\ell+1}{i}\binom{n-\ell-1}{k-i},$$

which implies that

$$i(n+1) \geq \ell k \quad \text{and} \quad (\ell+1)k \geq i(n+1).$$

Hence

$$\frac{i(n+1)}{k} - 1 \leq \ell \leq \frac{i(n+1)}{k}. \qquad \blacksquare$$

Note that for $n = kt + k - 1$ we have two choices for ℓ_i:

$$\ell_i = i(t+1) \text{ or } \ell_i = i(t+1) - 1.$$

(b) Suppose now ℓ is fixed and the maximum in (2) is attained for some $1 \leq i \leq k$. Then we use the inequalities

$$\binom{\ell}{i}\binom{n-\ell}{k-i} \geq \binom{\ell}{i-1}\binom{n-\ell}{k-i+1} \quad \text{and} \quad \binom{\ell}{i}\binom{n-\ell}{k-i} \geq \binom{\ell}{i+1}\binom{n-\ell}{k-i-1},$$

which give

$$(\ell+1)(k+1) \geq i(n+2) \geq (\ell+1)(k+1) - (n+2)$$

and (2) follows. $\blacksquare$

(c) We have $n = tk + k_1$, $0 \le k_1 \le k-1$. In view of (a) it suffices to prove that

$$\binom{t}{1}\binom{(k-1)t+k_1}{k-1} > \binom{it+\alpha}{i}\binom{(k-i)t+k_1-\alpha}{k-i}, \tag{4}$$

where $\alpha = \lfloor i(k_1+1)k \rfloor$, $i \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$.

We proceed by induction on k_1 and k.

Induction beginning $k_1 = 0$

CLAIM. *For* $i = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$ *we have monotonicity in the RHS of (4) with respect to* i, *that is,*

$$\binom{it}{i}\binom{(k-i)t}{k-i} > \binom{(i+1)t}{i+1}\binom{(k-i-1)t}{k-i-1}. \tag{5}$$

Proof. (5) is equivalent to

$$\frac{t(k-i)(t(k-i)-1)\cdots(t(k-i-1)+1)}{(k-i)((t-1)(k-i))((t-1)(k-i)-1)\cdots((t-1)(w-i-1)+1)}$$
$$> \frac{t(i+1)(t(i+1)-1)\cdots(ti+1)}{(i+1)(t-1)(i+1)((t-1)(k-i)-1)\cdots((t-1)i+1)}.$$

If now for $1 \le a \le t-1$ holds

$$\frac{(k-i)t-a}{(k-i)(t-1)-a+1} > \frac{t(i+1)-a}{(i+1)(t-1)-a+1}, \tag{6}$$

then clearly we are done.

But (6) is equivalent to

$$t(k-2i-1) > a(w-2i-1)$$

and the latter is true because $i < \frac{k}{2}$. This completes the case $k_1 = 0$. ■

Induction Step $k_1 - 1 \to k_1$

We have

$$\binom{t}{1}\binom{(k-1)t+k_1}{k-1} = \binom{t}{1}\binom{(k-1)t+k_1-1}{k-1}\frac{(k-1)t+k_1}{(k-1)(t-1)+k_1}$$

$$\binom{it+\alpha}{i}\binom{(k-i)t+k_1-\alpha}{k-i} = \binom{it+\alpha}{i}\binom{(k-i)t+k_1-\alpha}{k-i}\frac{it+\alpha}{i(t-1)+\alpha}$$
$$= \binom{it+\alpha}{i}\binom{(k-i)t+k_1-\alpha-1}{k-i}\frac{(k-i)t+k_1-\alpha}{(k-i)(t-1)+k_1-\alpha}.$$

Case 1.

$$\frac{(k-1)t+k_1}{(k-1)(t-1)+k_1} \geq \frac{it+\alpha}{i(t-1)+\alpha} \Leftrightarrow \frac{(k-1)}{(k-1)(t-1)+k_1} \geq \frac{i}{i(t-1)+\alpha} \Leftrightarrow \alpha \geq \frac{ik_1}{k-1}. \tag{7}$$

Then we are done by induction hypothesis.

Case 2.

$$\frac{(k-1)t+k_1}{(k-1)(t-1)+k_1} \geq \frac{(k-i)t+k_1-\alpha}{(k-i)(t-1)+k_1-\alpha} \Leftrightarrow \frac{k-1}{(k-1)(t-1)+k_1} \geq \frac{k-i}{(k-i)(t-1)+k_1-\alpha} \Leftrightarrow \alpha \leq \frac{k_1(i-1)}{k-1}. \tag{8}$$

Then again we are done by the same reason.

Thus it remains to consider the

Case 3.

$$\frac{(i-1)k_1}{k-1} < \alpha < \frac{ik_1}{k-1}. \tag{9}$$

We have

$$\binom{t}{1}\binom{(k-1)t+k_1}{k-1} = \binom{t}{1}\binom{(k-2)t+k_1}{k-2}\lambda_1,$$

where

$$\lambda_1 = \frac{((k-1)t+k_1)((k-1)t+k_1-1)\cdots((k-2)t+k_1+1)}{(k-1)\cdot((k-1)(t-1)+k_1)\cdots((k-2)(t-1)+k_1+1)}$$

and

$$\binom{it+\alpha}{i}\binom{(k-i)t+k_1-\alpha}{k-i} = \binom{t(i-1)+\alpha}{i-1}\binom{(k-i)t+k_1-\alpha}{k-i}\lambda_2,$$

where

$$\lambda_2 = \frac{(it+\alpha)(it+\alpha-1)\cdots((i-1)t+\alpha+1)}{i(i(t-1)+\alpha)(i(t-1)+\alpha)\cdots((i-1)(t-1)+\alpha+1)}\ .$$

If $\lambda_1 \geq \lambda_2$ we are done by induction hypothesis.

First show that

$$\frac{(k-1)t+k_1}{k-1} > \frac{it+\alpha}{i},$$

or equivalently

$$\alpha < \frac{ik_1}{k-1}.$$

But this is true in view of (9).

Further show that for $0 \leq a \leq t-2$ holds

$$\frac{(k-1)t + k_1 - 1 - a}{(k-1)(t-1) + k_1 - a} \geq \frac{it + \alpha - 1 - a}{i(t-1) + \alpha - a}$$

$\Leftrightarrow (k-i-1)(t-1-a) + \alpha(k-2) \geq (i-1)k_1$
and for $a = t-2$
$\Leftrightarrow (k-i-1) + \alpha(k-2) \geq (i-1)k_1$
$\Leftrightarrow \alpha(k-2) \geq (i-1)(k_1-1)$ (since $k-i-1 \geq i-1$)
$\Leftrightarrow \alpha \geq \frac{(i-1)(k_1-1)}{k-2}$.

By (9) we have

$$\alpha > \frac{(i-1)k_1}{k-1}$$

and clearly

$$\frac{(i-1)k_1}{k-1} > \frac{(i-1)(k_1-1)}{k-2},$$

since

$$k \geq k_1 + 1. \qquad \blacksquare$$

Remark 2. Note that for $n \geq k^2/2$ statement (c) in Lemma 2 can be sharpened as follows

(c′) For $n = tk + k_1 > k^2/2$, $1 \leq r \leq k/2$

$$\max_{\substack{1 \leq \ell < n \\ r \leq i \leq k-r}} \binom{\ell}{i}\binom{n-\ell}{k-i} = \binom{rt+\alpha}{r}\binom{n-rt-\alpha}{k-r},$$

where

$$\alpha = \left\lfloor \frac{r(k_1+1)}{k} \right\rfloor. \tag{10}$$

The proof is somewhat tedious and we omit it.

Note also that in general (10) does not hold. For example take $n = 76$, $k = 33$. Then $t = 2$, $k_1 = 10$. In view of (1) we get $\ell_{15} = 35$, $\ell_{16} = 37$.

Now we can verify that (10) fails, that is

$$\binom{35}{15}\binom{41}{18} < \binom{37}{16}\binom{39}{17}.$$

The next statement directly follows from Lemmas 1 and 2.

LEMMA 3. *Given $a_1, \ldots, a_n \in \mathbb{R}^+$ and integer $0 < \ell < n$. Let X be the (0,1)-solutions of the equation*

$$\sum_{i=1}^{\ell} a_i x_i - \sum_{j-\ell+1}^{n} a_j x_j = 0 \tag{11}$$

with $\sum_{i=1}^{n} x_i = k$ (that is $X \subset E(n,k)$).

Then

$$\text{(i)} \qquad |X| \leq \max_{1\leq i\leq k} \binom{\ell}{i}\binom{n-\ell}{k-i} = \binom{\ell}{i_\ell}\binom{n-\ell}{k-i_\ell} \quad \left(i_\ell = \left\lceil \frac{\ell k}{n+1} \right\rceil\right). \tag{12}$$

(ii) The equality in (12) holds iff $a_1 = \cdots = a_\ell = 1$, $a_{\ell+1} = \cdots = a_n = \frac{i_\ell}{k-i_\ell}$ *(we take* $a_1 = 1$*), that is an optimal* X *is unique up to permutations of coordinates.*

One can also easily obtain a slightly sharpened version of Lemma 3.

LEMMA 3′. *Given* $a_1, \ldots, a_n \in \mathbb{R}^+$ *and* $0 < \ell < n$. *Let* X *be the (0,1)-solutions of (11) with* $k-1 \leq \sum_{i=1}^{n} x_i \leq k$, $k \leq n/2$. *Then the statements (i) and (ii) in Lemma 3 hold as well.*

Proof. To prove the lemma we just note that the (0,1)-solutions (of any equation) of two consecutive weights form an antichain. This together with Lemma 1 gives the result. ■

3. Old Related Results Used

The following result is due to Erdős [8].

THEOREM E. *Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *and* $\mathcal{F}$ *contains no* s *pairwise disjoint sets. Then for* $n > n_0(w, s)$ *holds*

$$|\mathcal{F}| \leq \binom{n}{k} - \binom{n-s+1}{k}.$$

The bound is achieved by taking

$$\mathcal{F}_s = \left\{ A \in \binom{n}{k} : A \cap [1, s-1] \neq \varnothing \right\}.$$

A family $\mathcal{A} \subset 2^{[n]}$ is called intersecting, if $A_1 \cap A_2 \neq \varnothing$ holds for all $A_1, A_2 \in \mathcal{A}$. An intersecting family $\mathcal{A}$ is called nontrivial intersecting system if $\bigcap_{A\in\mathcal{A}} = \varnothing$.

Hilton and Milner proved in [11].

THEOREM HM. *Let* $\mathcal{A} \subset \binom{[n]}{k}$ *be a nontrivial intersecting system with* $n > 2k$. *Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Remark 1. The complete solution of the nontrivial t-intersecting problem is given in [5] (see also predecessors [10], [11] and the book [7]). Bollobas, Daykin and Erdős [6] generalized Theorem HM as follows.

THEOREM BDE. *Let* $\mathcal{A} \subset \binom{[n]}{k}$ *contain no* s $(s \geq 2)$ *pairwise disjoint sets and* $\mathcal{A} \not\subseteq \mathcal{F}_s$. *Then for* $n > n_0(k, s)$

(i) $|\mathcal{A}| \leq \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-s+1-k}{k-1} + 1.$

(ii) The unique (up to permutations) family achieving the bound is

$$\mathcal{A} = \left(\mathcal{F}_s \Big\backslash \left\{B \in \binom{[n]}{k} : (s-1) \in B,\ B \cap [s, s+k-1] = \varnothing\right\}\right) \cup [s, s+k-1].$$

4. Proof of Theorem 1

(i) Let $w = sm + r$, $0 < r < m$ and let V be defined by

$$\sum_{i=0}^{n} a_i x_i = 0. \tag{13}$$

W.l.o.g. suppose that $a_1, \ldots, a_\ell > 0$ $(1 \leq \ell < n)$, $a_{\ell+1}, \ldots, a_k < 0 (k > \ell)$ and $a_{k+1} = \cdots = a_n = 0$.

Consider two cases:

(a) Case. $\boldsymbol{n - k < r \leq n - 1}$

In this case for any solution $(x_1, \ldots, x_n) \in E(n, m)$ of equation (13) we have $x_1 + \cdots + x_\ell \geq 1$ and $x_{\ell+1} + \cdots + x_n \geq 1$.

Hence in view of Lemma 3 the number of solutions $X \subset E(n, m)$ of (13) is bounded by

$$|X| \leq \max_{1 \leq i \leq m-1} \binom{\ell}{i}\binom{n-\ell}{m-i}.$$

Combining this with Lemma 2 we get the desired result.

(b) Case. $\boldsymbol{r \leq n - k \leq n - 2}$

Partition the set of solutions X of (13) into two disjoint sets

$$X_0 \triangleq \left\{(x_1, \ldots, x_n) \in X : \sum_{i=1}^{n-r} x_i = m\right\} \quad \text{and} \quad X_1 = X \setminus X_0.$$

The set X_0 has the property: (turning to the set theoretical language) no s vectors of X_0 are pairwise disjoint. This is clear, because otherwise we would have a vector of weight sm (in the first $n - r$ coordinates) and consequently a vector $x \in X$ of weight $sw + r_1$.

Theorem E says that for large n we have

$$|X_0| \leq \binom{n-r}{m} - \binom{n-r-s}{m}.$$

On the other hand by definition of X_1 we have

$$|X_1| \leq \binom{n}{m} - \binom{n-r}{m}.$$

Therefore

$$|X| = |X_0| + |X_1| < \binom{n}{m} - \binom{n-r-s}{m} = O(n^{m-1}).$$

But $\binom{t}{1}\binom{n-t}{m-1} \sim cn^m$ as $n \to \infty$, a contradiction which shows that

$$F(n, w, m) \leq \binom{t}{1}\binom{n-1}{m-1}, \text{ for } n > n_0(w, m).$$

To show that $F(n, w, m) \geq \binom{t}{1}\binom{n-1}{m-1}$ partition the coordinate set $[1, n]$ into two parts $[1, t] \cup [t+1, n]$ and consider all vectors of weight m with weight one in part $[1, t]$. That is consider the set

$$X = \left\{ (x_1, \dots, x_n) \in E(n, m) : \sum_{i=1}^{t} x_i = 1 \right\}.$$

This set can be described as the set of (0,1)-solutions of weight m of the equation

$$\sum_{i=1}^{t}(m-1)x_i - \sum_{j=t+1}^{n} x_j = 0. \tag{14}$$

Observe that if the hyperplane defined by (14) contains a vector of weight w, then one has

$$s(m-1) = w - s \qquad \text{(for some } 1 \leq s \leq w-1),$$

which implies that $m \mid w$, a contradiction. This completes the proof of part (i).

(ii) Consider now the case $m > w$.

Again suppose an optimal subspace with the required properties is defined by (13), where $a_1, \dots, a_\ell > 0$ $(1 \leq \ell < n)$; $a_{\ell+1}, \dots, a_k < 0$ $(k > \ell)$ and $a_{k+1} = \cdots = a_n = 0$.

Clearly $n - k \leq w - 1$ and therefore for any $(x_1, \dots, x_n) \in X$ one has $\sum_{i=1}^{\ell} x_i \geq 1$, $\sum_{j=\ell+1}^{n} x_j \geq 1$.

This together with Lemma 3 implies

$$|X| \leq \max_{1 \leq i \leq m-1} \binom{\ell}{i}\binom{n-\ell}{m-i}. \qquad \blacksquare$$

5. Proof of Theorem 2

We prove the identity by first showing that $F(n, w, m) \geq (s-1)\binom{n-s+1}{m-1}$. This can be seen by taking the hyperplane H defined by

$$(m-1)\sum_{i=1}^{s-1} x_i - \sum_{j=s}^{n} x_j = 0.$$

Indeed obviously $H \cap E(n, w) = \varnothing$ and $|H \cap E(n, m)| = (s-1)\binom{n-s+1}{m-1}$, which gives the desired inequality.

The inverse inequality is more difficult to establish. In fact we proceed by establishing 3 claims, which then yield the result.

For n large let X be an optimal family, that is $|X| = F(n, w, m)$.

CLAIM 1. $X \subset \chi(\mathcal{F}_s)$.

Proof. Assume that $X \not\subset \chi(\mathcal{F}_s)$, then by Theorem BDE for large n we have

$$\begin{aligned}|X| &\le \binom{n}{m} - \binom{n-s+1}{m} - \binom{n-m-1}{m-1} + 1 = \frac{(s-2)n^{m-1}}{(m-1)!} + 0(n^{m-2}) \\ &< \frac{(s-2)n^{m-1}}{(m-1)!} + 0(n^{m-2}) = (s-1)\binom{n-s+1}{m-1} \le F(n, w, m).\end{aligned}$$

This contradicts the optimality of X and hence $X \subset \chi(\mathcal{F}_s)$. ■

CLAIM 2. *Suppose that X is from a hyperplane defined by $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$, then necessarily $\alpha_1 = \cdots = \alpha_{s-1}$.*

Proof. Assume $\alpha_1 \neq \alpha_2$.

Then clearly for any $(x_1, x_2, \ldots, x_n) \in X \quad (1 - x_1, 1 - x_2, x_3, \ldots, x_n) \notin X$.

This implies that

$$\begin{aligned}|X| &\le |F_s| - \binom{n-2}{m-1} = \binom{n}{m} - \binom{n-s+1}{m} - \binom{n-2}{m-1} \text{ and as } n \to \infty \\ &< F(n, w, m), \text{ for } n \text{ sufficiently large as we observed above, a contradiction.}\end{aligned}$$

■

Thus $\alpha_1 = \cdots = \alpha_{s-1}$ and w.l.o.g. we can assume that $\alpha_1, \ldots, \alpha_{s-1+\ell} > 0$, $(\ell \ge 0)$, $\alpha_{s+\ell}$, $\alpha_k < 0$ $(s + \ell \le k \le n)$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$ $(0 \le n - k \le m - 1)$. ■

CLAIM 3. $\alpha_1 \neq \alpha_{s+i}$, $i = 0, \ldots, \ell$.

Proof. Suppose $\alpha_1 = \alpha_s$.

Then clearly

$$x = (1, 0, \ldots, 0, x_{s+1}, \ldots, x_n) \notin X,$$

because otherwise $y = (0, \ldots, 0, 1, x_{s+1}, \ldots, x_n) \in X$ (note that y cannot be excluded from X), a contradiction with $X \subset F_{s-1} = \chi(\mathcal{F}_{s-1})$.

This implies that for any $x \in X$ with $\sum_{i=1}^{s-1} x_i = 1$ we have $x_s = 1$. Hence

$$|X| \le \binom{n-s}{m-2}(s-1) + O(n^{m-2}) < F(n, w, m), \text{ a contradiction.}$$

■

In view of these claims we can describe now the set X as the (0,1)-solutions of the equations

$$\begin{cases} x_1 + \cdots + x_{s-1} + \beta_s x_2 + \cdots + \beta_{s-1+\ell} x_{s-1+\ell} - \beta_{s+\ell} x_{s+\ell} - \cdots - x_n \beta_n = 0 \\ x_1 + \cdots + x_n = m \end{cases}, \tag{15}$$

where $\beta_s, \ldots, \beta_{s+\ell} > 0$, $\beta_1 = \cdots = \beta_{s-1} = 1$, $\beta_i \neq 1$ for $i = s, \ldots, s+\ell$ and $\beta_{s+\ell}, \ldots, \beta_n \geq 0$.

Further we can reduce equation (15) to the following equivalent ones

$$b_s x_s + \cdots + b_n x_n = m \quad \text{and} \quad \sum_{i=1}^{n} x_i = m,$$

where $b_i \neq 0$ for $i = s, \ldots, n$ and $b_j > 0$ for $j = s+\ell, \ldots, n$, $\sum_{i=1}^{s-1} x_i \geq 1$, $\sum_{j=s+\ell}^{n} x_i \geq 1$.

Now we are going to show that for big n's we must have $\ell = 0$. Suppose for a contradiction that $\ell \geq 1$. Let Y be the restriction of X on coordinates $s, \ldots, n$. That is

$$Y = \{(x_s, \ldots, x_n) : (x_1, \ldots, x_{s-1}, x_s, \ldots, x_n) \in X\}.$$

Define

$$Y_i = \left\{ (x_s, \ldots, x_n) \in Y : \sum_{j=s}^{n} x_j = i \right\} \quad \text{for } i = 1, \ldots, m-1.$$

Then in view of Lemma 3 we have

$$W_i = |Y_i| \leq \binom{\ell}{k_i}\binom{n-s+1-\ell}{m-i-k_i} \quad \text{for some } 0 \leq k_i \leq m-i.$$

Thus

$$|X| \leq \sum_{i=1}^{m-1} \binom{s-1}{m-i} W_i \leq \sum_{i=1}^{m-1} \binom{s-1}{m-i}\binom{\ell}{k_i}\binom{n-s+1-\ell}{m-i-k_i}.$$

As we mentioned above $Y_{m-1} \cup Y_{m-2}$ forms an antichain. Therefore by Lemma 1 we can write

$$\sum_{m-2 \leq i+j \leq m-1} \frac{\alpha_{ij}}{\binom{\ell}{i}\binom{n-s+1-\ell}{j}} \leq 1. \tag{16}$$

Further clearly

$$\begin{aligned} 1 \geq \text{LHS (16)} &= \sum_{i+j=m-1} \frac{\alpha_{ij}\binom{s-1}{m-i-j}}{\binom{\ell}{i}\binom{n-s+1-\ell}{j}\binom{s-1}{m-i-j}} + \sum_{i+j=m-2} \frac{\alpha_{ij}\binom{s-1}{m-i-j}}{\binom{\ell}{i}\binom{n-s+1-\ell}{j}\binom{s-1}{m-i-j}} \\ &\geq \sum_{i+j=m-1} \frac{\alpha_{ij}\binom{s-1}{1}}{\binom{s-1}{1}\max_{0\leq i\leq m-1}\binom{\ell}{i}\binom{n-s+1-\ell}{m-1-i}} + \sum_{i+j=m-2} \frac{\alpha_{ij}\binom{s-1}{2}}{\binom{s-1}{1}\max_{0\leq i\leq m-2}\binom{\ell}{i}\binom{n-s+1-\ell}{m-2-i}}. \end{aligned}$$

This implies that

$$\binom{s-1}{1} W_{m-1} + \binom{s-1}{2} W_{m-2} \leq (s-1) \max_{0\leq i\leq m-1} \binom{\ell}{i}\binom{n-\ell-s+1}{m-1-i}.$$

One can easily observe that

$$\max_{\substack{0\leq i\leq m-1\\ 1\leq \ell\leq n-s}} \binom{\ell}{i}\binom{n-s-\ell+1}{m-1-i} = \binom{1}{0}\binom{n-s}{m-1}.$$

Hence

$$|X| \le (s-1)\binom{n-s}{m-1} + \sum_{i=3}^{m-1}\binom{s-1}{m-i}W_i.$$

But

$$\sum_{i=3}^{m-1}\binom{s-1}{m-i}W_i < \sum_{i=3}^{m-1}\binom{s-1}{m-i}\binom{n-s+1-\ell}{i} < (s-1)\binom{n-s}{m-2} \text{ for } n \text{ large.}$$

Finally we get

$$|X| < (s-1)\binom{n-s}{m-1} + (s-1)\binom{n-s}{m-2} = (s-1)\binom{n-s+1}{m-1} \le F(n, sm, m),$$

a contradiction which yields $\ell = 0$.

This clearly completes the proof of Theorem 2 since for $\ell = 0$ we get

$$|X| \le \max_{1\le i\le m-1}\binom{s-1}{i}\binom{n-s+1}{m-i} = \binom{s-1}{1}\binom{n-s+1}{m-1}. \qquad \blacksquare$$

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