Designs, Codes and Cryptography, 29, 29–40, 2003

Cone Dependence—A Basic Combinatorial Concept

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Abstract. We call $A \subset \mathbb{E}^n$ *cone independent* of $B \subset \mathbb{E}^n$, the euclidean n-space, if no $a = (a_1, \dots, a_n) \in A$ equals a linear combination of $B \setminus \{a\}$ with non-negative coefficients. If A is cone independent of A we call A a *cone independent set*. We begin the analysis of this concept for the sets $P(n) = \{A \subset \{0,1\}^n \subset \mathbb{E}^n : A \text{ is cone independent}\}$ and their maximal cardinalities $c(n) \triangleq \max\{|A| : A \in P(n)\}$.

We show that $\lim_{n\to\infty} \frac{c(n)}{2^n} > \frac{1}{2}$, but can't decide whether the limit equals 1.

Furthermore, for integers $1 < k < \ell \le n$ we prove first results about $c_n(k,\ell) \triangleq \max\{|A| : A \in P_n(k,\ell)\}$, where $P_n(k,\ell) = \{A : A \subset V_k^n \text{ and } V_\ell^n \text{ is cone independent of } A\}$ and V_k^n equals the set of binary sequences of length n and Hamming weight k. Finding $c_n(k,\ell)$ is in general a very hard problem with relations to finding Turan numbers.

Keywords: combinatorial extremal problems, Turan problem, positive linear combinations, binary sequences

Mathematics Subject Classification: 05C35, 05B30, 52C99

1. Introduction

We begin with our notation. $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{E}^n$ is the Euclidean space of dimension n. For $i, j \in \mathbb{N}$, $i < j$, the set $\{i, i+1, \dots, j\}$ is abbreviated as $[i, j]$, and $[n]$ stands for $[1, n]$. For $k, n \in \mathbb{N}$, we set

$$2^{[n]} = \{E : E \subset [n]\}, \quad \binom{[n]}{k} = \left\{E \in 2^{[n]} : |E| = k\right\}.$$

There is a natural bijection T between $2^{[n]}$ and $\{0,1\}^n$—the set of binary sequences of length n: for any $E \in 2^{[n]}$ $T(E) = (v_1, \dots, v_n) = v \in \{0,1\}^n$, where $v_i = \begin{cases} 1 & \text{if } i \in E \\ 0 & \text{if } i \notin E \end{cases}$.

More generally, for $\mathcal{E} \subset 2^{[n]}$ (resp. $H \subset \{0,1\}^n$) define

$$T(\mathcal{E}) = \{T(E) : E \in \mathcal{E}\} (\text{resp. } T^{-1}(H)).$$

In particular $T(2^{[n]}) = \{0,1\}^n$ and $T\binom{[n]}{k} = V_k^n$—the set of binary sequences of length n and Hamming weight k.

Now new concepts and questions follow.

New Definitions

Definition 1. $A \subset \mathbb{E}^n$ is *cone independent* of $B \subset \mathbb{E}^n$ if no $a = (a_1, \dots, a_n) \in A$ equals a linear combination of $B \setminus \{a\}$ with non-negative coefficients.

Definition 2. If A is cone independent of A we call A a *cone independent set*.

Definition 3. *We study the case* $A, B \subset \{0, 1\}^n \subset \mathbb{E}^n$ *and in particular consider* $P(n) = \{A \subset \{0, 1\}^n : A \text{ is cone independent}\}$.

Problems

PROBLEM 1. *Find*

$$c(n) \triangleq \max\{|A| : A \in P(n)\}$$

PROBLEM 2. *For integers* $1 < k < \ell \leq n$ *find*

$$c_n(k, \ell) \triangleq \max\{|A| : A \in P_n(k, \ell)\},$$

where $P_n(k, \ell) = \{A : A \subset V_k^n \text{ and } V_\ell^n \text{ is cone independent of } A\}$

Remark. Finding $c_n(k, \ell)$ is in general a very hard problem.
We have

$$c_n(k, k+1) = \tau_n(k, k+1)$$

where $\tau_n(k, \ell) \triangleq$ Turan number $\triangleq \max\{|\mathcal{A}| : \mathcal{A} \subset \binom{[n]}{k}$, no $B \in \binom{[n]}{\ell}$ contains more than $\binom{\ell}{k} - 1$ members of $\mathcal{A}\}$.

We begin with a bound and a conjecture for Problem 1 in Section 2.

Section 3 contains classical results for graphs and hypergraphs, which are used in the analysis of Problem 2.

The results on this problem are stated as Theorems 1, 2 in Section 4, where also further conjectures about $c_n(k, \ell)$ are stated.

The rest of the paper is devoted to proofs of the theorems, auxiliary results needed are with their proofs in Section 5, Theorem 2 is proved in Section 6, and finally Theorem 2 is proved in Section 7.

2. A Bound for Problem 1

Consider the set

$$C = \{v^n = (v_1, \dots, v_n) \in \{0, 1\}^n : v_1 = 1\}.$$

Clearly $|C| = 2^{n-1}$ and it is easy to see that $C \in P(n)$.

One more naive construction is

$$D = \{10, 01\} \times \{0, 1\}^{n-2} = \{v^n = (v_1, \dots, v_n) \in \{0, 1\}^n : (v_1, v_2) \in \{(0, 1), (1, 0)\}\}.$$

Again we have $|D| = 2^{n-1}$ and $D \in P(n)$.

PROPOSITION

(i) $c(n+1) \geq 2c(n)$

(ii) If an $A \in P(n)$ *and* $1^n = (1, \dots, 1) \in A$, *then* $|A| \leq 2^{n-1}$.

Proof. (i) For an $A \in P(n)$ consider $A' = A \times \{0, 1\} = \{v^{n+1} = (v_1, \dots, v_n, v_{n+1}) \in \{0, 1\}^{n+1} : (v_1, \dots, v_n) \in A\}$.
We have $|A'| = 2|A|$ and verify that $A' \in P(n+1)$.

(ii) follows from the observation that from every complemented pair $(v^n, 1^n - v^n)$ at most one can be in A. ■

Can we beat the naive bound 2^{n-1}? The following construction shows that this is the case for $n \geq 5$.

CONSTRUCTION. *Let* $C \in P(n)$ *and* $1^n \notin C$. *Take an* $m \in \mathbb{N}$ *with* $m > |C|$.
Consider

$$C' = \{C \times \{\{0, 1\}^m \setminus \{0\}^m\}\} \cup \{1^n \times \{e_1, \dots, e_m\}\},$$

where $e_1, \dots, e_m$ *are unit vectors in the ground set* $[n+1, n+m]$.
It can be easily proved that $C' \in P(n+m)$. *We have*

$$|C'| = |C| \cdot (2^m - 1) + m = |C| \cdot 2^m + m - |C| > |C| \cdot 2^m.$$

Now choose $n = 2$, $C = \{(1, 0), (0, 1)\}$, $m = 3$, $(3 > 2 = |C|)$. *Since* $C \in P(2)$ *and* $(1, 1) = 1^2 \notin C$ *we can apply the construction to get*

$$C' = \{(10100), (10010), (10001), (10110), (10101), (10011), (10111), (01100), (01010), (01001), (01110), (01101), (01011), (01111), (11100), (11010), (11001)\}$$

with $C' \in P(5)$, $|C'| = 17$.

It is convenient to introduce the parameter $\beta(n) = \frac{c(n)}{2^n}$.

LEMMA 1.

(i) $\beta = \lim_{n\to\infty} \beta(n)$ *exists.*

(ii) β *is never assumed, i.e.,* $\beta > \beta(n)$ *for all* $n \in \mathbb{N}$.

Proof. (i) directly follows from (i) in the proposition.

(ii) We know that $\beta(n) \geq \frac{17}{32}$, $n \geq 5$ and hence by the proposition ((ii)) an optimal $A \in P(n)$ does not contain the vector 1^n. Consequently we can apply the construction to get $\beta(n+m) > \beta(n)$ (for a suitable m). ■

How far can we go with the construction? A simple calculation shows that we can have only $\beta > 0, 55$. We **conjecture** that $\beta < 1$.

3. Some Classical Results

THEOREM (Mantel [6]). *Let $G = (\mathcal{V}, \mathcal{E})$ be a graph on n vertices not containing triangles. Then*

$$|\mathcal{E}| \leq M_n \triangleq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \tag{1}$$

THEOREM (Erdős–Gallai [3]). *Let $G = (\mathcal{V}, \mathcal{E})$ be a graph on n vertices not containing s pairwise disjoint edges. Then for $s \geq 2$, $n \geq 2s$*

$$|\mathcal{E}| \leq g_n(2, s) \triangleq \max\left(\binom{2s-1}{s}, \binom{s-1}{2} + (s-1)(n-s+1) \right). \tag{2}$$

Moreover, equality holds here iff—up to permutation—

$$\mathcal{E} = \binom{[2s-1]}{2} \text{ or } \left\{ A \in \binom{[n]}{2} : |A \cap [1, s-1]| \neq 0 \right\}.$$

CONJECTURE (Erdős [2]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ not contain pairwise disjoint sets. Then for $n \geq ks$*

$$|\mathcal{F}| \leq g_n(k, s) \triangleq \max\left(\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k} \right). \tag{3}$$

1965 Erdős proved (3) for $n > n_0(k, s)$.
1976 Bollobas, Daykin, Erdős proved (3) for $n > 2k^3 s$.
1987 Frankl, Füredi proved (3) for $n > 100ks^3$.

THEOREM (Frankl [5]).

$$g_n(k, s) \leq (s-1)\binom{n-1}{k-1}.$$

In particular for $n = ks$

$$g_{ks}(k, s) = \binom{ks-1}{k}.$$

It is convenient to write $g_n(s)$ instead of $g_n(2, s)$.

4. Results and Conjectures for Problem 2

We succeeded in settling two special cases.

The case $\ell = n$. Clearly $c_n(k, n) \geq \binom{n-1}{k}$, because $1^n = (1, 1, \ldots, 1\}$ is cone independent of $V_k^{n-1} \times \{0\}$ and $|V_k^{n-1}| = \binom{n-1}{k}$.

In case $k \mid n$ any $A \subset V_k^n$ cone independent of 1^n does not contain $\frac{n}{k}$ pairwise disjoint elements and hence by Theorem F we get

$$c_n(k, n) = \binom{n-1}{k}.$$

Thus we have proved part (a) of the following theorem. The main work consists in proving part (b) in Sections 5, 6.

THEOREM 1.

$$c_n(k, n) = \binom{n-1}{k}, \text{ if } \begin{cases} (a) & k \mid n \\ (b) & k \nmid n \text{ and } n > n_0(k). \end{cases}$$

The case $k = 2$. Recall the numbers $g_n(s)$ (Theorem EG) and M_n (Theorem M).

THEOREM 2.

$$c_n(2, \ell) = \begin{cases} g_n\left(\frac{\ell}{2}\right), & \text{if } 2 \mid \ell \\ \max\left\{M_n, g_n\left(\frac{\ell+1}{2}\right)\right\}, & \text{if } 2 \nmid \ell. \end{cases} \tag{4}$$

Conjectures

For $1 \leq s \leq k$ define $n_s = \lceil \frac{n \cdot s}{k} \rceil - 1$ and the set

$$H_s = \left\{ v = (v_1, \ldots, v_n) \in V_k^n : \sum_{i=1}^{n_s} v_i \geq s \right\}, \quad |H_s| = \sum_{i=0}^{k-s} \binom{n_s}{s+i} \binom{n-n_s}{k-s-i}.$$

It can be easily verified that $H_s \in P_n(k, n)$ for all $1 \leq s \leq k$.

CONJECTURE 1.

$$c_n(k, n) = \max_s |H_s|.$$

Theorem 1 proves this conjecture for $n > n(k)$. For big n $\max_s |H_s| = |H_k| = \binom{n-1}{k}$.

Clearly, cone dependence is a stronger concept than linear dependence. The difference seems to be smaller for very different parameters k, ℓ, n.

CONJECTURE 2. *For $k \ll \ell \ll n$ $c_n(k, \ell)$ behaves like in the case where positive independence is replaced by linear independence.*

5. Auxiliary Results: Left–Compression

The following method was introduced in [4] (see [5] for a nice survey). For integers $1 \leq i < j \leq n$ and a family $\mathcal{F} \subset 2^{[n]}$ define the (i, j)-shift S_{ij} as follows:

$$S_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\} = F_1 & \text{if } i \notin F,\ j \in F,\ F_1 \notin \mathcal{F} \\ F & \text{otherwise} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}.$$

Now, for $\mathcal{F} \subset 2^{[n]}$ $T(\mathcal{F}) = A \subset \{0, 1\}^n$, and the (i, j)-shift is defined in a natural way:

$$S_{ij}(A) = T(S_{ij}(T^{-1}(A))).$$

For a $v \in \{0, 1\}^n$, $i, j \in \mathbb{N}$, we also define $E_{ij}(v)$, which is a vector obtained from v by exchanging the ith and jth coordinates, and for $B \subset \{0, 1\}^n$ define

$$E_{ij}(B) = \{E_{ij}(v) : v \in B\}.$$

LEMMA 2.

(i) $|S_{ij}(A)| = |A|$

(ii) *if* $A \subset V_k^n$, *then* $S_{ij}(A) \subset V_k^n$ *as well.*

(iii) *if* $A \in P_n(k, n)$, *then* $S_{ij}(A) \in P_n(k, n)$ *as well.*

Proof. (i) and (ii) are trivial. To prove (iii), assume to the opposite, for some $A \in P_n(k, n)$ and $1 \leq i < j \leq n$, $S_{ij}(A) \notin P_n(k, n)$ holds, that is, there is a subset $V \subset S_{ij}(A)$ and positive numbers $\{\lambda_v : v \in V\}$ such that

$$(1, \ldots, 1) = 1^n = \sum_{v \in V} \lambda_v \cdot v. \tag{5}$$

Let

$$V = V_{00} \dot{\cup} V_{10} \dot{\cup} V_{01} \dot{\cup} V_{11},$$

where $V_{\varepsilon\delta}$ is the set of vectors of V having ε in the position i and δ in the position j.
By the definition of the (i, j)-shift we have

$$(V \setminus A) = V'_{10} \subset V_{10} \tag{6}$$

and that for every

$$v \in V_{01}, \quad v \in A \quad \text{and} \quad E_{ij}(v) \in A. \tag{7}$$

Denote $E_{ij}(V_{01})$ by W. We look at the equality (5) for the ith and jth components. We have

$$\sum_{v \in V_{10} \cup V_{11}} \lambda_v = 1 \quad \text{and} \quad \sum_{v \in V_{01} \cup V_{11}} \lambda_v = 1. \tag{8}$$

It follows from (8) that

$$\sum_{v\in V_{10}} \lambda_v = \sum_{v\in V_{01}} \lambda_v \tag{9}$$

and by (6) and the positivity of λ_v's we get

$$\sum_{v\in V'_{10}} \lambda_v \le \sum_{v\in V_{01}} \lambda_v. \tag{10}$$

Let $U \subset A$ be the image of V'_{10}, that is $S_{ij}(U) = V'_{10}$. Clearly, also $U = E_{ij}(V'_{10})$.

Consider the set

$$V^* = U \cup (V \setminus V'_{10}) \cup W.$$

We have $V^* \subset A$. By (10) we can split the coefficients λ_v, $v \in V_{01}$, in such a way, that

$$\lambda_v = \lambda'_v + \lambda''_v,\ \lambda'_v,\ \lambda''_v \ge 0 \quad \text{for every } v \in V_{01}$$

and

$$\sum_{v\in V_{01}} \lambda_v = \sum_{v\in V_{01}} (\lambda'_v + \lambda''_v) = \sum_{v\in W} \lambda'_v + \sum_{v\in V_{01}} \lambda''_v = \sum_{v\in V'_{10}} \lambda_v + \sum_{v\in V_{01}} \lambda''_v. \tag{11}$$

Finally from (5)–(11) we have

$$\begin{aligned} 1^n &= \sum_{v\in V} \lambda_v \cdot v = \sum_{v\in V\setminus(V'_{10}\cup V_{01})} \lambda_v \cdot v + \sum_{v\in V'_{10}} \lambda_v \cdot v + \sum_{v\in V_{01}} \lambda_v \cdot v \\ &= \sum_{v\in V\setminus(V'_{10}\cup V_{01})} \lambda_v \cdot v + \sum_{\substack{u=E_{ij}(v)\\ v\in V'_{10}}} \lambda_v \cdot u + \sum_{\substack{w=E_{ij}(v)\\ v\in V_{01}}} \lambda'_v \cdot w + \sum_{v\in V_{01}} \lambda''_v v \end{aligned}$$

i.e., 1^n is positively dependent on $V^* \subset A$, a contradiction to $A \in P_n(k, n)$. ■

Definition 4. A $\mathcal{B} \subset 2^{[n]}$ (resp. $B \subset \{0, 1\}^n$) is said to be stable or left-compressed if $S_{ij}(\mathcal{B}) = \mathcal{B}$ for all $1 \le i < j \le n$ (resp. $S_{ij}(B) = B$). Denote by $LP_n(k, n)$ the set of all stable systems of $P_n(k, n)$.

By Lemma 2 (after finitely many shifts) we get

$$c_n(k, n) = \max_{A\in P_n(k,n)} |A| = \max_{A\in LP_n(k,n)} |A|. \tag{12}$$

Definition 5. A vector $v = (v_1, \dots, v_n) \in \mathbb{E}^n$, $v_i \ge 0$ is called "good" if there exists an $s \in \mathbb{N}$, $1 \le s \le n - 1$, such that

$$\frac{\sum_{i=1}^{s} v_i}{s} > \frac{\sum_{i=s+1}^{n} v_i}{n-s}.$$

Otherwise, it is called "bad."

We observe that a positive, linear combination of any "bad" vectors is again "bad," but the similar statement with respect to "good" vectors, in general, is false.

We also observe that for any $\alpha > 0$ $\alpha \cdot v$ is "good" (resp. "bad") whenever v is "good" (resp. "bad"). We note that clearly 1^n is a "bad" vector.

LEMMA 3. *Let $A \subset V_k^n$ be left-compressed. Then $A \in P_n(k, n)$ (and hence $A \in LP_n(k, n)$) if and only if any non-negative, nonzero combination of A produces a "good" vector. In particular, if $A \in LP_n(k, n)$, then necessarily all vectors of A are "good."*

Proof. Since the vector 1^n is a "bad" vector, the "if" part of the lemma is trivially true.

To prove the part "only if" we assume to the opposite, that $A \in LP_n(k, n)$ but there exists a nonempty subset $A' \subset A$ and positive coefficients $\lambda_v > 0 : v \in A'$ such that $\sum_{v \in A'} \lambda_v \cdot v$ is a "bad" vector.

Clearly, we can assume that all coefficients λ_v are rational numbers, and consequently (multiplying all coefficients by a suitable integer) we can assume

$$\lambda_v \in \mathbb{N};\ v \in A', \quad \sum_{v \in A'} \lambda_v \cdot v = v^* = (a_1, \ldots, a_n),$$

$$\sum_{i=1}^{n} a_i = m \cdot n \quad \text{for some } m \in \mathbb{N} \quad \text{and} \quad v^* \text{ is a "bad" vector.} \tag{13}$$

In other words, v^* is a sum of vectors of A' (possibly taken with multiplicity).
By the definition of "bad" vectors for v^* we have

$$a_1 \leq \frac{a_2 + \cdots + a_n}{n-1}, \quad \frac{a_1 + a_2}{2} \leq \frac{a_3 + \cdots + a_n}{n-2} \cdots \frac{a_1 + \cdots + a_{n-1}}{n-1} \leq a_n.$$

The last inequality together with (13) implies $a_n \geq m$. If $a_n > m$, then we build a new "bad" vector as follows:

Let i, $1 \leq i \leq n-1$ be the largest index for which $a_i < m$ (such an index always exists by (13)). Consider the vector $u = (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{n-1}, a_n - 1)$. It is easy to verify that u is a "bad" vector. Moreover, since $a_n > m$, $a_i < m$, then in A' there exists a vector (call it w), which has 1 in the nth component and 0 in the ith component. Since A is a left-compressed set, then $E_{ij}(w) \in A$ as well, and consequently the vector u also can be positively produced from A.

The sum of coordinates of u is still $m \cdot n$. Continuing, we get a "bad" vector where the last component equals m.

Now we follow the same procedure with respect to the $(n-1)$th component and so on. Finally, we produce the vector $(m, m, \ldots, m)$, equivalently, the vector $(1, 1, \ldots, 1) = 1^n$, a contradiction. ∎

Remark. In the proof we did not use the weight of vectors in A. With it Lemma 3 can be formulated in a more general form.

6. Proof of Theorem 1

Let $A \in P_n(k, n)$ and $|A| = c_n(k, n)$. By (12) we can assume that $A \in LP_n(k, n)$. We partition A by the last component: $A = A_0 \cup A_1$, where

$$A_0 = \{A = (a_1, \ldots, a_n) \in A : a_n = 0\},\ A_1 = A \setminus A_0.$$

We want to prove, and this is equivalent to the statement (b) in Theorem 1, that $A_1 = \emptyset$ if n is big enough. Assume to the opposite, that $A_1 \neq \emptyset$ for infinitely many $n, k \nmid n$. Write

$$n = mk + r, \quad \text{where } 1 \leq r < k.$$

Since A is a left-compressed set and by assumption $A_1 \neq \emptyset$, then clearly $v^* = (v_1, \ldots, v_n) \in A_1$, where

$$v_1 = \cdots = v_{k-1} = 1, \quad v_k = \cdots = v_{n-1} = 0, \quad v_n = 1. \tag{14}$$

CLAIM. *Assume $A_1 \neq \emptyset$, then*

$$|A_0| \leq \binom{mk + r - 1}{k} - \binom{mk - k - 1}{k - 1}. \tag{15}$$

Proof of the claim. Let $B \subset A_0$ be the set of all vectors having all k ones in the interval $[k + r, mk + r - 1]$ (of length $k(m - 1)$). If $|B| \leq \binom{k(m-1)}{k} - \binom{k(m-1)-1}{k-1}$ then (15) trivially holds. Otherwise, since k divides the length of the interval, by the part (b) of the Theorem, the vector $u = (u_1, \ldots, u_{mk+r})$, where

$$u_1 = u_2 = \cdots = u_{k+r-1} = 0, \quad u_{k+r} = \cdots = u_{mk+r-1} = 1, \quad u_{mk+r} = 0$$

can be positively built using vectors of B.

The vector u is a "bad" vector in the ground set $[k, mk + r - 1]$, and A is left-compressed. Hence by Lemma 3, we can positively build, from vectors of A, also the vector $u^* = (u_1^*, \ldots, u_{mk+r}^*)$, where

$$u_1^* = \cdots = u_{k-1}^* = 0, \quad u_k^* = \cdots = u_{mk+r-1}^* = 1, \quad u_{mk+r}^* = 0.$$

Now

$$v^* + u^* = (1, 1, \ldots, 1) = 1^n,$$

where $v^* \in A_1$ is the vector in (14), a contradiction.

By Lemma 3 all vectors of A_1 must be "good." We estimate from below (very roughly) the number of "bad" vectors: consider the partition of the ground set

$$[1, mk + r] = [1, m + r - 1] \cup [m + r, 2m + r - 1] \cup \ldots$$
$$[(k - 1)m + r, km + r - 1] \cup \{mk + r\}$$

and the set $W \subset V_k^{mk+r}$ consisting of the vectors having all 0-s in the first part and single 1-s in every remaining part. It is easy to verify that all vectors of W are "bad" and $|W| = m^{k-1}$. Hence

$$|A_1| \leq \binom{mk + r - 1}{k - 1} - m^{k-1}. \tag{16}$$

The combination of (15) and (16) gives

$$|A| = |A_0| + |A_1| \leq \binom{mk + r - 1}{k} - \binom{mk - k - 1}{k - 1} + \binom{mk + r - 1}{k - 1} - m^{k-1}.$$

It is easily seen that $RHS < \binom{mk+r-1}{k}$ if $m > m_0(k)$ (hence $n > n_0(k)$), because by the binomial formula $\binom{mk-k-1}{k-1} = \frac{(mk)^{k-1}}{(k-1)!} + 0(mk)^{k-2}$, $\binom{mk+r-1}{k-1} = \frac{(mk)^{k-1}}{(k-1)!}, + 0(mk)^{k-2}$ and therefore their difference is smaller than $0(m^{k-1})$. Therefore $|A| < \binom{n-1}{k}$ if $n > n_0(k)$, a contradiction. Hence $A_1 = \varnothing$ for $m > n_0(k)$, $|A| = c_n(k, m) = \binom{n-1}{k}$ and the optimal set is unique up to permutation. ■

Remarks.

1. It is easy to calculate $c_n(2, n)$ $(k = 2)$. Moreover, this is a special case of Theorem 2. We have

$$c_n(2, n) = \binom{n-1}{2} \quad \text{for all } 2 \mid n,\ c_n(2, 3) = 2,\ c_n(2, 5) = 7$$

and the optimal set is

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1. \end{matrix}$$

Let $n = 2\ell + 1$. Look at the sets A_0, A_1 in the proof of Theorem 1. It is easily seen that there are ℓ "bad" vectors and consequently $|A_1| \leq \ell$. Since $|A_0| \leq \binom{2\ell}{2} - \binom{2\ell-3}{1}$ (by claim), then

$$|A| = |A_0| + |A_1| \leq \binom{2\ell}{2} + \ell - (2\ell - 3) = \binom{2\ell}{2} - \ell + 3.$$

Hence $A \leq \binom{2\ell}{2} = c_n(2, n)$ for $\ell \geq 3$.

Note, that in the case $n = 7$ $(\ell = 3)$ we have the second optimal set: Take

$$A = \{v = (v_1, \dots, v_7) \in V_2^7 : v_1 + v_2 + v_3 \geq 1\}.$$

It is easy to verify that $A \in P_7(2, 7)$. We have $|A| = \binom{3}{2} + \binom{3}{1} \cdot \binom{4}{1} = 15 = \binom{6}{2}$.

2. The estimation (16) used in the proof of Theorem 1 is very rough, and of course can be greatly improved.[1]

7. Proof of Theorem 2

At first we show, that the bound in (4) can be achieved. For this we just take the 0, 1 images of optimal graphs in Theorem M (only for odd values of ℓ) and in Theorem EG. It can be easily shown that these sets belong to $P_n(2, \ell)$.

Now, the case $2 \mid \ell$ is trivial, since having $\frac{\ell}{2}$ pairwise disjoint 2-sets, we just sum the corresponding vectors and get a vector of weight ℓ, a contradiction.

Let $\ell = 2\ell_1 + 1$, $A \in P_n(2, \ell)$ be with $|A| = c_n(2, \ell)$.

If $T^{-1}(A) \subset \binom{[n]}{2}$ does not contain $\frac{\ell+1}{2} = \ell_1 + 1$ pairwise disjoint edges (2-sets), then

$$|T^{-1}(A)| = |A| \le g_n\left(\frac{\ell+1}{2}\right)$$

proving the Theorem in this case.

CLAIM. *Assume $T^{-1}(A)$ contains $\ell_1 + 2$ pairwise disjoint edges. Then $T^{-1}(A)$ does not contain triangles, and hence $|T^{-1}(A)| = |A| \le M_n$.*

Proof of the claim. Assume to the opposite, that the graph with $\mathbb{E} = T^{-1}(A)$ contains a triangle, say $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and we denote by v_1^n, v_2^n, v_3^n the corresponding vectors in A.

By assumption $T^{-1}(A)$ contains $\ell_1 + 2$ pairwise disjoint edges and at most 3 of them can intersect (have a common vertex) with the triangle. Hence in the ground set $[4, n]$ one can find $(\ell_1 - 1)$ from these edges, say $\{4, 5\}, \{6, 7\}, \ldots, \{2\ell_1, 2\ell_1 + 1\}$, and let $v_4^n, \ldots, v_{\ell_1+2}^n$ be the corresponding vectors in A.

Now we just observe, that

$$\frac{1}{2}v_1^n + \frac{1}{2}v_2^n + \frac{1}{2}v_3^n + v_4^n + \cdots + v_{\ell_1+2}^n = (11\ldots10\ldots0) \in V_\ell^n,$$

a contradiction.

So, it remains to treat the case, when $T^{-1}(A)$ contains exactly $\ell_1 + 1$ pairwise disjoint edges, say

$$\{1, 2\}, \{3, 4\}, \ldots, \{2\ell_1 + 1, 2\ell_1 + 2\}.] \tag{17}$$

We observe that

(i) in $T^{-1}(A)$ there are no edges $\{i, j\}$ with $2\ell_1 + 2 < i < j \le n$, otherwise we would have $\ell_1 + 2$ pairwise disjoint edges.

(ii) There are no triangles involving edges from (17), otherwise if, say $\{1, 2\}, \{1, 3\}, \{2, 3\} \in T^{-1}(A)$, then as in the claim, the positive combination of images of these and $(\ell_1 - 1)$ disjoint edges $\{5, 6\}, \ldots, \{2\ell_1 + 1, 2\ell_1 + 2\}$ produces a vector from V_ℓ^n, a contradiction. The case $\{1, 2\}, \{1, i\}, \{2, i\}$ for $i \in [2\ell_1 + 3, n]$ is excluded by the same reason.
We note, that actually we can have triangles in this case, say $\{1, 3\}, \{1, 5\}, \{3, 5\}$.
Now we estimate $|A| = |T^{-1}(A)|$ from above.
By the observation we have
- at most 2 edges between any two edges of (17), and consequently at most $2\binom{\ell_1+1}{2}$ edges in $[1, 2\ell_2 + 2]$ except the $(\ell_1 + 1)$ edges of (17).
- at most $(\ell_1 + 1)(n - 2\ell_2 - 1)$ edges $\{i, j\}$, with $1 \le i \le 2\ell_1 + 2$, $2\ell_1 + 2 < j \le n$.

Hence

$$|T^{-1}(A)| = |A| \le (\ell_1 + 1) + 2\binom{\ell_1 + 1}{2} + (\ell_1 + 1)(n - 2\ell_1 - 2)$$

$$= (\ell_1 + 1)(n - \ell_1 - 1) \le \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil = M_n.$$

■

Note

1. A referee suggested the following improvements: A weight k vector ending in 1 has in its orbit under the permutations on $[n-1]$ at least one bad vector. Therefore (16) can be improved to $|A_1| \leq (1 - \frac{1}{(k-1)!})\binom{km+r-1}{k-1}$. Actually it can even be shown that in each orbit under rotations there is at least one bad vector. Therefore the term $\frac{1}{(k-1)!}$ can also be replaced by $\frac{1}{k-1}$.

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