More about shifting techniques

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Abstract

We discovered a new and simple shifting technique. It makes it possible to prove results on shadows like the Kruskal–Katona Theorem without any additional arguments.

As another application we obtain the following new result. For $s, d, k \in \mathbb{N}$, $1 \leq d \leq s$, $d \leq k$ define the subclass of $\binom{\mathbb{N}}{k}$ (the k–subsets of $\mathbb{N}$) $\mathcal{B}(k,s,d) = \left\{ B \in \binom{\mathbb{N}}{k} : |B \cap [1,s]| \geq d \right\}$. Let $\mathcal{A} \subset \mathcal{B}(k,s,d)$ and $|\mathcal{A}| = m$. Then the cardinality of the ℓ–shadow of $\mathcal{A}$ is minimal if $\mathcal{A}$ consists of the first m elements of $\mathcal{B}(k,s,d)$ in colexicographic order. A more general form of this result is given as well. Other applications are to be expected.

Keywords: shifting, shadow, Kruskal–Katona Theorem.

1 Introduction

$\mathbb{N}$ denotes the set of positive integers and the set $\{1, \ldots, n\}$ is abbreviated as $[n]$. Given $k \in \mathbb{N}$ and $X \subset \mathbb{N}$ we denote

$$2^X = \{F : F \subset X\}, \binom{X}{k} = \{F \subset X : |F| = k\}.$$

Recall the well known exchange or shifting operation S_{ij} which was introduced by Erdős, Ko and Rado [2]. For a family $\mathcal{B} \subset 2^{[n]}$ and $B \in \mathcal{B}$ set

$$S_{ij}(B) = \begin{cases} \{i\} \cup (B \smallsetminus \{j\}), & \text{if } i \notin B, j \in B, \{i\} \cup (B \smallsetminus \{j\}) \notin \mathcal{B}, \\ B, & \text{otherwise} \end{cases}$$
$$S_{ij}(\mathcal{B}) = \{S_{ij}(B) : B \in \mathcal{B}\}.$$

Although the shifting operation was introduced in [2] to prove intersection theorems, it turned out to be a powerful tool to obtain many other important results in Extremal Set Theory. An excellent survey on it is given by Frankl [3].

Later on we will distinguish between left shifting, if $i < j$, and right shifting, if $i > j$.

We say that $\mathcal{B}$ is left shifted (right shifted) if $S_{ij}(\mathcal{B}) = \mathcal{B}$ for all $1 \le i < j \le n$ (for all $1 \le j < i$).

We also say that $\mathcal{B}$ is left shifted with respect to an element $u \in [n]$ if $S_{iu}(\mathcal{B}) = \mathcal{B}$ for all $1 \le i < u$.

The following simple properties of the shifting operation are well known (see e.g. [3]).

Proposition.

(i) $|S_{ij}(\mathcal{B})| = |\mathcal{B}|$

(ii) Any family $\mathcal{B} \subset \binom{[n]}{k}$ *can be brought to a left shifted (right shifted) family by repeatedly applying left (right) shifts.*

For any $1 \le \ell \le k$ the ℓ–shadow of a family $\mathcal{A} \subset \binom{X}{k}$ is defined by

$\partial_\ell(\mathcal{A}) = \{F \in \binom{X}{\ell} : \exists A \in \mathcal{A} : F \subset A\}$.

Define the colexicographic (colex) order for the elements $A, B \in \binom{\mathbb{N}}{k}$ as follows:

$A \prec B \Leftrightarrow \max\big((A \smallsetminus B) \cup (B \smallsetminus A)\big) \in B$, where the operation "max" is taken in the natural order on $\mathbb{N}$.

We denote by $L(k, m)$ the initial m members of $\binom{\mathbb{N}}{k}$ in the colex order.

The well–known Kruskal–Katona Theorem was discovered in 1963 by Kruskal [5], in 1966 by Katona [4], and in 1967 by Lindström and Zetterström [6].

Theorem KK (Kruskal–Katona). *Let* $\mathcal{A} \subset \binom{\mathbb{N}}{k}$, $|\mathcal{A}| = m$, *then*

$$|\partial_\ell(\mathcal{A})| \ge |\partial_\ell(L(k, m))|.$$

Let us mention the following important property of the shifting operation (see [3]).

Lemma 1.1. *Let $\mathcal{B} \subset \binom{[n]}{k}$, then $\partial_\ell\big(S_{ij}(\mathcal{B})\big) \subseteq S_{ij}\big(\partial_\ell(\mathcal{B})\big)$, i.e. $|\partial_\ell\big(S_{ij}(\mathcal{B})\big)| \le |\partial_\ell(\mathcal{B})|$.*

There is an elegant proof of Theorem KK due to Frankl [3] where Lemma 1.1, induction (on m and k), and the cascade representation of m are used. (For a short proof see also Daykin [1]).

In this paper we introduce a new shifting operation which makes it possible to prove results like Theorem KK using only shifting and nothing in addition. In particular we prove that any finite family $\mathcal{A} \subset \binom{\mathbb{N}}{k}$ can be brought to $L(k, |\mathcal{A}|)$ (applying the new shifting) with nonincreasing size of its shadow.

2 The main tool: new shifting

For $\mathcal{B} \subset \binom{\mathbb{N}}{k}$ and $u \in \mathbb{N}$ define the families

$$\mathcal{B}_u = \{B \in \mathcal{B} : u \in B\}, \mathcal{B}_{\bar{u}} = \mathcal{B} \setminus \mathcal{B}_u.$$

We introduce now an operation which we call right–left shifting (RL–shifting). Given a family $\mathcal{A} \subset \binom{[n]}{k}$ and integers $1 \le j \le i < u$ the RL–shift $S_{ij|u}(\mathcal{A})$ consists of two parts

P1. First we apply the right shift S_{ij} to $\mathcal{A}_u$.

P2. Next we apply iteratively left shifts S_{ru}, $r = 1, \dots, u-1$, to the family $S_{ij}(\mathcal{A}_u) \cup \mathcal{A}_{\bar{u}}$.

More formally one can write

$$S_{ij|u}(\mathcal{A}) \triangleq S_{u-1u}\big(\dots S_{2u}\big(S_{1u}(S_{ij}(\mathcal{A}_u) \cup \mathcal{A}_{\bar{u}})\big) \dots\big).$$

The idea behind this operation is to get from family $\mathcal{A}$ a family with fewer sets containing u. Whereas in part P1 "place is made at the left" for replacements of u in part P2 the left shifting of u is actually done.

Clearly if $u \notin \bigcup_{A \in \mathcal{A}} A$, then $\mathcal{A}_u = \varnothing$ and $S_{ij|u}(\mathcal{A}) = \mathcal{A}$. In this case the RL–shift $S_{ij|u}$ leaves $\mathcal{A}$ unchanged. It is important that we included RL–shifts with $i = j$. Here for every $1 \le i < u$ $S_{ii|u}$ makes no changes on a considered $\mathcal{A}$ in part P1. However, in part P2 $\mathcal{A}$ is transformed into $S_{u-1u}\big(\dots S_{2u}\big(S_{1u}(\mathcal{A})\big) \dots\big)$, left shifted with respect to u. With such operations we can obtain a left shifted family.

Given $\mathcal{A} \subset \binom{[n]}{k}$ and $u \in \bigcup_{A \in \mathcal{A}} A$ let $RL_u(\mathcal{A})$ be the set of all families which can be obtained from $\mathcal{A}$ by iteratively applying RL–shifts $S_{ij|u}$. Then we say that $\mathcal{A}$ is RL_u–stable if for every $\mathcal{A}' \in RL_u(\mathcal{A})$ we have $|\mathcal{A}'_u| = |\mathcal{A}_u|$ (equivalently $\mathcal{A}'_{\bar{u}} = \mathcal{A}_{\bar{u}}$).

We also say that $\mathcal{A}$ is RL–stable if $\mathcal{A}$ is RL_u–stable for all $u \in \bigcup_{A \in \mathcal{A}} A$.

Lemma 2.1. *Suppose a family $\mathcal{A} \subset \binom{[n]}{k}$ with $|\mathcal{A}| = m$ is RL–stable, then $\mathcal{A} = L(k, m)$.*

Proof: Note first that $\mathcal{A}$ is left shifted, since in particular we have for all $1 \le r < u \le n$ and any $1 \le i < u$ $S_{ru}(\mathcal{A}) = S_{ii|u}(\mathcal{A}) = \mathcal{A}$.

Let $A = \{a_1, \ldots, a_k\} \in \mathcal{A}$, $a_1 < \cdots < a_k$. Given element $a_t \in A$ with $t < a_t$ observe that the RL_{a_t}–stability implies that $\mathcal{A}$ contains the set $\{a_t - t, \ldots, a_t - 1, a_{t+1}, \ldots, a_k\}$. Hence by left shiftedness $\mathcal{A}$ contains all sets $B = \{b_1, \ldots, b_k\} \prec A$ with $b_t < a_t$, $b_{t+1} = a_{t+1}, \ldots, b_k = a_k$. For $a_t = t$ this is obvious since there is no such B. Since $\mathcal{A}$ is RL_{a_t}–stable for all a_t, $t = 1, \ldots, k$ we infer that $\mathcal{A}$ contains every set $B \in \binom{[n]}{k}$ which precedes A in the colex order.
□

Lemma 2.2. *Any family $\mathcal{A} \subset \binom{[n]}{k}$ can be brought to an RL–stable family, i.e. to $L(k, |\mathcal{A}|)$, by repeatedly applying RL–shifts.*

Proof: Let $\mathcal{A} \subset \binom{\mathbb{N}}{k}$ be a finite family with $|\mathcal{A}| = m$ and let $r(\mathcal{A})$ denote the maximal element of $\bigcup\limits_{A \in \mathcal{A}} \mathcal{A}$. Also let $\mathcal{A}$ be already left shifted. We apply now an RL–shift $S_{ij|r_0}$ with $r_0 \triangleq r(\mathcal{A})$. Clearly for the resulting family $\mathcal{A}' = S_{ij|r_0}(\mathcal{A})$ with $r_1 \triangleq r(\mathcal{A}')$ we have $r_0 - 1 \le r_1 \le r_0$. We consider two cases.

(i) $|\mathcal{A}'_{r_0}| < |\mathcal{A}_{r_0}|$. In this case we apply left shifts to $\mathcal{A}'$ reducing it to a left shifted family.

(ii) $|\mathcal{A}'_{r_0}| = |\mathcal{A}_{r_0}|$ (correspondingly $r_0 = r_1$ and $\mathcal{A}_{\bar{r}_0} = \mathcal{A}'_{\bar{r}_1}$). By definition of the RL–shift $\mathcal{A}'$ is left shifted with respect to the element r_1. Moreover $\mathcal{A}'_{\bar{r}_1}$ is left–shifted since $\mathcal{A}_{\bar{r}_0}$ is left shifted.

Thus in both cases w.l.o.g. we may assume that $\mathcal{A}'$ is left shifted with respect to r_1, and $\mathcal{A}'_{\bar{r}_1}$ is left shifted. However note that $\mathcal{A}'$ is not necessarily a left shifted family. Next we apply an RL–shift $S_{ij|r_1}(\mathcal{A}')$ for some $1 \le j < i < r_1$ transforming $\mathcal{A}'$ to a new family $\mathcal{A}''$ which is left shifted with respect to the biggest element $r_2 \triangleq r(\mathcal{A}'') \le r_1$ and $\mathcal{A}''_{\bar{r}_2}$ is left shifted, etc.

The described procedure cannot be continued indefinitely. After finitely many RL–shifts we will come to a family $\mathcal{A}^*$ with a biggest element r such that r cannot be decreased anymore by RL–shifts. Since each RL–shift $S_{ij|r}$ does not increase $|\mathcal{A}^*_r|$ (which is lower bounded) we finally end up with an RL_r–stable family $\mathcal{B}$. Note that this with the left shiftedness of $\mathcal{B}_{\bar{r}}$ implies (as we observed in the proof of Lemma 2.1) that $\mathcal{B}_{\bar{r}} = \binom{[r-1]}{k}$. Further we repeat the described procedure, applying now RL–shifts $S_{ij|r-1}$ and assuming that $\mathcal{B}$ is left shifted. Note that since $S_{ij|u}(\mathcal{B}_{\bar{r}}) = \mathcal{B}_{\bar{r}}$ for all $1 \le j \le i < u \le n$ we may proceed only for $\mathcal{B}_r$ applying RL–shifts $S_{ij|u}$ for $u = \max\left(\bigcup\limits_{B \in \mathcal{B}_r} B \setminus \{r\}\right)$. Continuing this procedure we finally obtain an RL–stable family $\mathcal{F}$, or equivalently $\mathcal{F} = L(k, |\mathcal{A}|)$. □

3 Shadows and RL–shifting

In addition to Lemma 1.1 for shadows we have the following property of shifting.

Lemma 3.1. *Let $\mathcal{A} \subset \binom{\mathbb{N}}{k}$ be left shifted with respect to element u, (i.e. $S_{iu}(\mathcal{A}) = \mathcal{A}$ for all $1 \le i < u$) then for any $1 \le j < i < u$ one has*

$$|\partial_\ell\big(S_{ij}(\mathcal{A}_u) \cup \mathcal{A}_{\bar{u}}\big)| \le |\partial_\ell(\mathcal{A})|. \tag{3.1}$$

Proof: We have

$$\partial_\ell(\mathcal{A}) = \partial_\ell(\mathcal{A}_u) \cup \partial_\ell(\mathcal{A}_{\bar{u}}). \tag{3.2}$$

We can assume that $\mathcal{A}_u, \mathcal{A}_{\bar{u}} \neq \varnothing$, since if $\mathcal{A}_u = \varnothing$ (3.1) is trivial and if $\mathcal{A}_{\bar{u}} = \varnothing$ we apply Lemma 1.1.

Let us denote $\mathcal{B} = \partial_\ell(\mathcal{A}_u)$. Then we can write $\mathcal{B} = \mathcal{B}_u \,\dot\cup\, \mathcal{B}_{\bar{u}}$ ($\mathcal{B}_u \cap \mathcal{B}_{\bar{u}} = \varnothing$).

For a set $A \in \mathcal{A}_u$ let $1 \le s < u$ be an element such that $s \notin A$. Since $\mathcal{A}$ is left shifted with respect to u we have $A' \triangleq \big((A \setminus \{u\}) \cup \{s\}\big) \in \mathcal{A}_{\bar{u}}$.

Therefore $A \setminus \{u\} = A' \setminus \{s\}$ which implies that for any ℓ–subset (ℓ–shadow) $E \subset A$ with $u \notin E$ one has $E \in \partial_\ell\big(\{A' \setminus \{s\}\}\big) = \partial_\ell\big(\{A \setminus \{u\}\}\big) \subset \partial_\ell(\mathcal{A}_{\bar{u}})$.

This implies that $\mathcal{B}_{\bar{u}} \subset \partial_\ell(\mathcal{A}_{\bar{u}})$ and hence with (3.2) and the definition of $\mathcal{B}$

$$\partial_\ell(\mathcal{A}) = \mathcal{B}_u \,\dot\cup\, \partial_\ell(\mathcal{A}_{\bar{u}}). \tag{3.3}$$

Consider now a right shift $S_{ij}(\mathcal{A}_u)$ for some $1 \le j < i < u$, and denote $\mathcal{D} = \partial_\ell\big(S_{ij}(\mathcal{A}_u)\big)$.

We have

$$\partial_\ell\big(S_{ij}(\mathcal{A}_u) \cup \mathcal{A}_{\bar{u}}\big) = \partial_\ell\big(S_{ij}(\mathcal{A}_u)\big) \cup \partial_\ell(\mathcal{A}_{\bar{u}}) = \mathcal{D} \cup \partial_\ell(\mathcal{A}_{\bar{u}}) = (\mathcal{D}_u \,\dot\cup\, \mathcal{D}_{\bar{u}}) \cup \partial_\ell(\mathcal{A}_{\bar{u}}). \tag{3.4}$$

Suppose now $B \in \mathcal{A}_u$ so that $j \in B$ and $i \notin B$. Then clearly $B' \triangleq \big((B \setminus \{u\}) \cup \{i\}\big) \in \mathcal{A}_{\bar{u}}$ and $B' \setminus \{j\} = S_{ij}(B) \setminus \{u\}$. This implies that for any ℓ–subset $F \subset S_{ij}(B)$ with $u \notin F$ we have

$$F \in \partial_\ell\big(\{B' \setminus \{j\}\}\big) = \partial_\ell\big(\{S_{ij}(B) \setminus \{u\}\}\big) \subset \partial_\ell(\mathcal{A}_{\bar{u}}).$$

Thus $\mathcal{D}_{\bar{u}} \subset \partial_\ell(\mathcal{A}_{\bar{u}})$ and with (3.4)

$$\partial_\ell\big(S_{ij}(\mathcal{A}_u) \cup \mathcal{A}_{\bar{u}}\big) = \mathcal{D}_u \,\dot\cup\, \partial_\ell(\mathcal{A}_{\bar{u}}). \tag{3.5}$$

Further by (3.3) $|\partial_\ell(\mathcal{A})| = |\mathcal{B}_u| + |\partial_\ell(\mathcal{A}_{\bar{u}})|$, and by (3.5) $|\partial_\ell\big(S_{ij}(\mathcal{A}_u) \cup \mathcal{A}_{\bar{u}}\big)| = |\mathcal{D}_u| + |\partial_\ell(\mathcal{A}_{\bar{u}})|$.

But $|\mathcal{D}_u| \le |\mathcal{B}_u|$ by Lemma 1.1, which completes the proof. $\square$

Clearly Lemmas 3.1 and 1.1 imply

Lemma 3.2. *Suppose $\mathcal{A} \subset \binom{[n]}{k}$ is left shifted with respect to element u, then for any $1 \le j < i < u$ one has*

$$|\partial_\ell\big(S_{ij|u}(\mathcal{A})\big)| \le |\partial_\ell(\mathcal{A})|.$$

$\square$

4 A proof of an improved Kruskal–Katona Theorem

Theorem 4.1. *Any family $\mathcal{A} \subset \binom{\mathbb{N}}{k}$ with $|\mathcal{A}| = m$ can be brought by RL–shifts (with monotonically decreasing size of the ℓ–shadow in each step) to the initial segment of size m in the colex order.*

Proof: To prove the theorem we just note that at each step of the procedure described in the proof of Lemma 2.2 we apply an RL–shift $S_{ij|u}$ to a family which is left–shifted with respect to the element u. This with Lemma 3.2 gives the result. □

5 A new result

For $s, d, k \in \mathbb{N}$, $1 \le d \le s$, $d \le k$ define the following subclass of $\binom{\mathbb{N}}{k}$:

$$\mathcal{B}(k, s, d) = \left\{ B \subset \binom{\mathbb{N}}{k} : |B \cap [1, s]| \ge d \right\}.$$

Denote by $L_m\mathcal{B}(k, s, d)$ the first m elements of $\mathcal{B}(k, s, d)$ in the colex order.

Theorem 5.1. *Let $\mathcal{A} \subset \mathcal{B}(k, s, d)$ with $|\mathcal{A}| = m$, then for $\ell \le k$*

$$|\partial_\ell(\mathcal{A})| \ge |\partial_\ell\big(L_m\mathcal{B}(k, s, d)\big)|.$$

Proof: We may assume again that $\mathcal{A}$ is left shifted. We want to show now that applying certain type of RL–shifts $\mathcal{A}$ can be brought to the initial segment of $\mathcal{B}(k, s, d)$ in the colex order.

Note first that if $|A \cap [1, s]| > d$ for all $A \in \mathcal{A}$ we can apply RL–shifts proceeding as in the proof of Lemma 2.2. Thus suppose there exists an $A \in \mathcal{A}$ with $|A \cap [1, s]| = d$. We apply now only RL–shifts of type

RL1: $S_{ij|u}(\mathcal{A})$ for any $1 \le j \le i \le s$ and $i, j < u \le r$ $\big(r = r(\mathcal{A})\big)$,

RL2: $S_{ij|u}(\mathcal{A})$ for any $s + 1 \le j \le i < u \le r$.
Note that the obtained families are still in $\mathcal{B}(k, s, d)$.

Using the same arguments as in the proof of Lemma 2.2 we infer that $\mathcal{A}$ can be brought to an RL–stable family with nonincreasing size of the shadow. Note that the stability here is defined with respect to RL–shifts of type RL1 or RL2. But now we can easily see that $\mathcal{A}$ is nothing else but the first m members of $\mathcal{B}(k, s, d)$ in the colex order. This is clear because the RL–stability with respect to RL1 and RL2 implies that if $A \in \mathcal{A}$, $B \prec A$ and $B \in \mathcal{B}(k, s, d)$ then $B \in \mathcal{A}$. □

One can prove a more general statement using the same approach.

Let $\mathbb{N} = [1, s_1] \cup [s_1 + 1, s_2] \cup \cdots \cup [s_{t-1} + 1, s_t] \cup \{s_t + 1, \dots, \}$, $d_1 \le d_2 \le \cdots \le d_t \le k$, $d_i \le s_i$ $(i = 1, \dots, t)$.

Define

$$\mathcal{B} = \left\{ B \in \binom{\mathbb{N}}{k} : |B \cap [1, s_i]| \geq d_i,\ i = 1, \ldots, t \right\}.$$

Let also $L_m\mathcal{B}$ be the first m elements of $\mathcal{B}$ in the colex order.

Theorem 5.2. *Let $\mathcal{A} \subset \mathcal{B}$ and $|\mathcal{A}| = m$, then*

$$|\partial_\ell(\mathcal{A})| \geq |\partial_\ell(L_m\mathcal{B})|.$$

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