# On Bohman's conjecture related to a sum packing problem of Erdős

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**Abstract**

Motivated by a sum packing problem of Erdős [2] Bohman [1] discussed an extremal geometric problem which seems to have an independent interest. Let $H$ be a hyperplane in $\mathbb{R}^n$ such that $H \cap \{0, \pm 1\}^n = \{0^n\}$. The problem is to determine

$$f(n) \triangleq \max_H |H \cap \{0, \pm 1, \pm 2\}^n|.$$

Bohman [1] conjectured that

$$f(n) = 1/2(1+\sqrt{2})^n + 1/2(1-\sqrt{2})^n.$$

We show that for some constants $c_1, c_2$ we have $c_1(2,538)^n < f(n) < c_2(2,723)^n$ — disproving the conjecture. We also consider a more general question of estimation of $|H \cap \{0, \pm 1, \dots, \pm m\}|$, when $H \cap \{0, \pm 1, \dots, \pm k\} = \{0^n\}$, $m > k > 1$.

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# 1 Introduction and Statement of the Result

Let $H$ be a hyperplane in $\mathbb{R}^n$ so that $H \cap \{0, \pm 1\}^n = \{0^n\}$. Let

$$f(n) = \max_H |H \cap \{0, \pm 1, \pm 2\}^n|.$$

The problem (of determination of $f(n)$) was raised by Bohman [1] in connection with a subset sum problem of Erdős [2].

A set $S$ of positive integers $b_1 < b_2 < \cdots < b_n$ has distinct subset sums, if all sums of subsets are distinct. Erdős [2] has asked for the value of

$$g(n) \triangleq \min\{a_n : S \text{ has distinct subset sums, } |S| = n\}.$$

A long–standing conjecture of Erdős claims that $g(n) \geq c2^n$ for some constant $c$.

In [1] Bohman explained the relationship between functions $f(n)$ and $g(n)$, and noticed that the studying of the function $f(n)$ might be helpful for further investigation of the problem of Erdős.

Suppose a hyperplane $H$ defined by the equation

$$\sum_{i=0}^{n-1} a_i x_i = 0; \;\; a_0, \ldots, a_{n-1} \in \mathbb{N} \tag{1.1}$$

satisfies $H \cap \{0, \pm 1\}^n = \{0^n\}$. This clearly means that $\{a_0, \ldots, a_{n-1}\}$ has distinct subsets sums. A simplest example of such a set with $a_{n-1} \leq 2^{n-1}$ is $\{1, 2, 2^2, \ldots, 2^{n-1}\}$. For more complicated examples see [1], [3].

For $f(n)$ Bohman [1] conjectured that

$$f(n) = 1/2(1 + \sqrt{2})^n + 1/2(1 - \sqrt{2})^n,$$

showing that this number can be achieved, taking $a_i = 2^i$ $(i = 0, \ldots, n-1)$ in (1.1).

Let us consider now the hyperplanes defined by equation

$$\sum_{i=0}^{n-1} 2^i \lambda_i x_i = 0, \tag{1.2}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are odd integers.

One can easily see that the set $\{\lambda_0, 2\lambda, \ldots, 2^{n-1}\lambda_{n-1}\}$ has distinct subset sums.

Let $f^*(n)$ denote the maximum possible number of solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of equation (1.2) over all choices of odd integers $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$.

**Theorem 1.** *For some constants $c', c''$*

$$c'(2,538)^n < f^*(n) < c''(2,547)^n.$$

Clearly this means that $f(n) > c_1(2,538)^n$ and the conjecture of Bohman fails.

Our next goal is to give an upper bound for $f(n)$. A simple upper bound is

$$f(n) \le 3^n. \tag{1.3}$$

Indeed, let $X$ be the set of solutions $x^n \in \{0, \pm1, \pm2\}^n$ of equation (1.1). Then observe that for any $u^n, v^n \in \{0,1\}^n$, $u^n \ne v^n$, we have $(X + u^n) \cap (X + v^n) = \varnothing$. This implies that $|X + \{0,1\}^n| = |X||\{0,1\}^n| = |X|2^n$. On the other hand $\big\{X + \{0,1\}^n\big\} \subset \{0, \pm1, \pm2, 3\}^n$. Hence $|X|2^n \le 6^n$ and thus (1.3). The next result improves bound (1.3).

**Theorem 2.** *For some constant $c$*

$$f(n) < c(2,723)^n.$$

**Conjecture 1.** For some constant $c$

$$f(n) \sim c\beta^n,$$

where $\beta$ is the biggest real root of the equation $z^8 - 8z^6 + 10z^4 + 1 = 0$ ($\beta = 2,5386\dots$). The construction attaining this number is given in section 2.

We also consider a more general problem. Let $Q \subset \mathbb{Z}$ be finite and $F = \{0, \pm1, \dots, \pm k\}$, then

$$f(n, Q, F) \triangleq \max\big\{|H \cap Q^n| : H \text{ is a hyperplane and } H \cap F^n = \{0^n\}\big\}.$$

In some cases we succeed to give the exact answer.

**Theorem 3.**

*(i) Let $Q = \{0, \pm1, \dots, \pm m\}$, $F = \{0, \pm1, \dots \pm k\}$ and $k+1 | 2m+1$. Then*

$$f(n, Q, F) = \left(\frac{2m+1}{k+1}\right)^{n-1}.$$

*(ii) Let $Q = \{0, \pm1, \dots, \pm(m-1), m\}$, $F = \{0, \pm1, \dots, \pm k\}$ and $k+1 | 2m$. Then*

$$f(n, Q, F) = \left(\frac{2m}{k+1}\right)^{n-1}.$$

An interesting case is

$$Q = \{0, \pm1, \dots, \pm(k+1)\}, F = \{0, \pm1, \dots, \pm k\}, k \ge 1.$$

Note that for $k = 1$ we have Bohman's problem. It can be shown that

$$(1 + \sqrt{2})^n \le f(n, Q, F) \le 3^n.$$

The upper bound is derived exactly as we did above for $k = 1$. For the lower bound consider the equation

$$x_0 + (k+1)x_1 + \cdots + (k+1)^{n-1}x_{n-1} = 0. \tag{1.4}$$

Let $X \subset Q^n$ denote the set of solutions of (1.4). Clearly $X \cap F^n = \{0^n\}$. On the other hand one can show that $|X| = 1/2(1+\sqrt{2})^n + 1/2(1-\sqrt{2})^n$ (like for $k = 1$). We believe that Bohman's conjecture is true for $k \geq 2$, that is

**Conjecture 2.** For $Q = \{0, \pm 1, \ldots, \pm(k+1)\}$, $F = \{0, \pm 1, \ldots, \pm k\}$ and $k \geq 2$ (or a weaker condition: for $k > k_0$) one has

$$f(n, Q, F) = 1/2(1+\sqrt{2})^n + 1/2(1-\sqrt{2})^n.$$

# 2 Proof of Theorem 1

We start with an auxiliary statement. Let $f^*_\lambda(n)$ denote the maximum number of solutions $x^n = (x_0, \ldots, x_{n-1}) \in \{0, \pm 1, \pm 2\}^n$ of the equation

$$\sum_{i=0}^{n-1} 2^i \lambda_i x_i = \lambda \tag{2.1}$$

over all choices of odd integers $\lambda_0, \ldots, \lambda_{n-1}$ and given integer $\lambda$. Remember that $f^*_0(n) = f^*(n)$.

**Lemma 1.**

$$f^*(n) \geq \frac{1}{25} f^*_\lambda(n).$$

**Proof:** Suppose we have an optimal equation (2.1). That is for the solutions of (2.1) $X \subset \{0, \pm 1, \pm 2\}^n$ one has $|X| = f^*_\lambda(n)$.

For an integer $\mu$ consider the equation

$$(2\mu + 1)y + 2z + 4\lambda_0 x_0 + \cdots + 2^{n+1}\lambda_{n-1}x_{n-1} = 0.$$

Then taking $y = -2$, $z = 1$ we come to equation $\sum\limits_{i=0}^{n-1} 2^i \lambda_i x_i = \mu$, which implies that $f^*(n+2) \geq \max\limits_{\mu} f_\mu(n)$. On the other hand clearly

$$\max_{\mu} f_\mu(n) \geq \frac{1}{25} f^*_\lambda(n+2).$$

$\square$

Consider an equation

$$x_0 + 2x_1 + \cdots + 2^{n-1}x_{n-1} = \lambda. \tag{2.2}$$

Let $X(\lambda)$ be the set of all solutions (from $\{0, \pm 1, \pm 2\}^n$) of (2.2). With the help of this lemma we can get a lower bound using an average argument. There are $5^n$ vectors $(x_0, \ldots, x_{n-1}) \in$

$\{0, \pm 1, \pm 2\}^n$. On the other hand there are $4(2^n - 1) + 1$ possible values for $\lambda$ for which equation (2.2) has solutions. Hence there exists a $\lambda$ such that

$$|X(\lambda)| \geq \frac{5^n}{4(2^n - 1) + 1}.$$

This together with Lemma 1 implies that $f(n) \geq c(2,5)^n$ for some constant $c$, which actually disproves the conjecture of Bohman. However we can improve this bound constructively.

**Lower bound.**

As above let $X(\lambda) = H \cap \{0, \pm 1, \pm 2\}^n$, where $H$ is the hyperplane defined by (2.2).

Let also $h_\lambda(n)$ denote the number of solutions of (2.2), that is $h_\lambda(n) = |X(\lambda)|$.

Suppose that $\lambda = 2s$, where $s$ is an integer. Then observe that

$$h_{2s}(n) = h_{s-1}(n-1) + h_s(n-1) + h_{s+1}(n-1). \tag{2.3}$$

Correspondingly, if $\lambda = 2s + 1$, then

$$h_{2s+1}(n) = h_s(n-1) + h_{s+1}(n-1). \tag{2.4}$$

For a positive integer $n$ define

$$S_n = \begin{cases} 2^{n-1} + 2^{n-3} + \cdots + 2^3 + 2, & \text{if } 2 \mid n \\ 2^{n-1} + 2^{n-3} + \cdots + 2^2 + 1, & \text{if } 2 \nmid n. \end{cases} \tag{2.5}$$

**Claim:** For $2 \mid n$ and some constant $c$

$$h_{S_n}(n) > c(2,538)^n. \tag{2.6}$$

**Proof:** In view of (2.3) we have

$$h_{S_n}(n) = h_{S_{n-1}-1}(n-1) + h_{S_{n-1}}(n-1) + h_{S_{n-1}+1}(n-1). \tag{2.7}$$

Correspondingly

$$\begin{aligned} h_{S_{n-1}-1}(n-1) &= h_{S_{n-2}-1}(n-2) + h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2), \\ h_{S_{n-1}}(n-1) &= h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2), \\ h_{S_{n-1}+1}(n-1) &= h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2) + h_{S_{n-2}+2}(n-2). \end{aligned} \tag{2.8}$$

It is easy to see that $h_{S_n}(n)$ can be represented by linear combinations of the functions $h_{S_{n-i}-1}(n-i), h_{S_{n-i}}(n-i), h_{S_{n-i}+1}(n-i), h_{S_{n-i}+2}(n-i)$.

In view of (2.7) and (2.8) we can write

$$\begin{aligned}
h_{s_n}(n) &= h_{S_{n-1}+1}(n-1) + h_{S_{n-1}}(n-1) + h_{S_{n-1}}(n-1) \\
&= h_{S_{n-2}-1}(n-2) + 3h_{S_{n-2}}(n-2) + 3h_{S_{n-2}+1}(n-2) + h_{S_{n-2}+2}(n-2) \\
&= 4h_{S_{n-3}-1}(n-3) + 8h_{S_{n-3}}(n-3) + 7h_{S_{n-3}+1}(n-3) + h_{S_{n-3}+2}(n-3) \\
&= 4h_{S_{n-4}-1}(n-4) + 19h_{S_{n-4}}(n-4) + 20h_{S_{n-4}+1}(n-4) + 8h_{S_{n-4}+2}(n-4) \\
&\quad \dots \\
&= a_i h_{S_{n-i}-1}(n-i) + b_i h_{S_{n-i}}(n-i) + c_i h_{S_{n-i}+1}(n-i) + d_i h_{S_{n-i}+2}(n-i) \\
&\quad \dots \\
&= a_{n-1}h_{S_1-1}(1) + b_{n-1}h_{S_1}(1) + c_{n-1}h_{S_1+1}(1) + d_{n-1}h_{S_1+2}(1) \\
&= a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}. \qquad (2.9)
\end{aligned}$$

From (2.7), (2.8) and (2.9) we obtain the following recurrences for the coefficients $a_i, b_i, c_i, d_i$ in (2.9)

$$\begin{aligned}
a_{2i} &= a_{2i-1}, \\
b_{2i} &= a_{2i-1} + b_{2i-1} + c_{2i-1}, \\
c_{2i} &= a_{2i-1} + b_{2i-1} + c_{2i-1} + d_{2i-1}, \\
d_{2i} &= c_{2i-1} + d_{2i-1}, \qquad (2.10)
\end{aligned}$$

$$\begin{aligned}
a_{2i+1} &= a_{2i} + b_{2i}, \\
b_{2i+1} &= a_{2i} + b_{2i} + c_{2i} + d_{2i}, \\
c_{2i+1} &= b_{2i} + c_{2i} + d_{2i}, \\
d_{2i+1} &= d_{2i} \ (i = 1, 2, \dots). \qquad (2.11)
\end{aligned}$$

Here are the first ten values of $a_i$, $b_i$, $c_i$, $d_i$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_i$: | 1 | 1 | 4 | 4 | 23 | 23 | 144 | 144 | 921 | 921 |
| $b_i$: | 1 | 3 | 8 | 19 | 51 | 121 | 328 | 777 | 2113 | 5003 |
| $c_i$: | 1 | 3 | 7 | 20 | 47 | 129 | 305 | 832 | 1969 | 5363 |
| $d_i$: | 0 | 1 | 1 | 8 | 8 | 55 | 55 | 360 | 360 | 2329 |

From (2.10) and (2.11) we obtain by elementary algebraic transformations the following recurrences:

$$t_{i+8} = 8t_{i+6} - 10t_{i+4} - t_i \text{ for } t_i \in \{a_i, b_i, c_i, d_i\},\ i = 1, 2, \dots .$$

In particular we have

$$c_{2i+8} = 8c_{2i+6} - 10c_{2i+4} - c_{2i} \qquad (2.12)$$

with initial values $c_2 = 3$, $c_4 = 20$, $c_6 = 129$, $c_8 = 832$.

The characteristic equation of (2.12)

$$z^8 - 8z^6 + 10z^4 + 1 = 0 \qquad (2.13)$$

has the biggest real root $\beta = 2,5386\dots$ .

Thus $c_{2i}$ can be estimated from below by $c_{2i} \geq c\beta^{2i} > c(2.538)^{2i}$, for some constant $c$ definable from the initial values of $c_{2i}$.

Further in view of (2.9) and (2.10) for $n = 2k$ we have

$$h_{S_n}(n) = a_{2k-1} + b_{2k-1} + c_{2k-1} + d_{2k-1} = c_n,$$

which implies that $h_{S_n}(n) > c(2,538)^n$.

□

Thus we have proved that $f^*_{S_n}(n) > (2,538)^n$. This with Lemma 1 completes the proof of the lower bound.

**Upper bound.**

Consider the equation

$$\lambda_0 x_0 + 2\lambda_1 x_1 + \cdots + 2^{n-1}\lambda_{n-1}x_{n-1} = \lambda. \tag{2.14}$$

We distinguish the three cases

- ($\alpha$) $\lambda \equiv 2(\mod 4)$: then denote by $h_\alpha(n)$ the maximum possible number of solutions (from $\{0, \pm 1, \pm 2\}^n$ of equation (2.14)),
- ($\beta$) $\lambda \equiv 0(\mod 4)$: the corresponding notation for this case is $h_\beta(n)$,
- ($\gamma$) $\lambda \equiv 1$ or $3(\mod 4)$: the corresponding notation for this case is $h_\gamma(n)$.

Then one can easily observe that the following reccurrence relations hold

$$\begin{aligned} h_\alpha(n) &\leq h_\alpha(n-1) + h_\beta(n-1) + h_\gamma(n-1), \\ h_\beta(n) &\leq \max\big\{h_\alpha(n-1), h_\beta(n-1)\big\} + 2h_\gamma(n-1), \\ h_\gamma(n) &\leq \max\big\{h_\alpha(n-1), h_\beta(n-1)\big\} + h_\gamma(n-1). \end{aligned} \tag{2.15}$$

We have also that $h_\alpha(1) = h_\beta(1) = h_\gamma(1) = 1$.

Introduce now function $g_\alpha(n)$, $g_\beta(n)$, and $g_\gamma(n)$ so that $g_\alpha(1) = g_\beta(1) = g_\gamma(1) = 1$ and

$$\begin{aligned} g_\alpha(n) &= g_\alpha(n-1) + g_\beta(n-1) + g_\gamma(n-1), \\ g_\beta(n) &= \max\big\{g_\alpha(n-1), g_\beta(n-1)\big\}, +2g_\gamma(n-1), \\ g_\gamma(n) &= \max\big\{g_\alpha(n-1), g_\beta(n-1)\big\} + g_\gamma(n-1). \end{aligned}$$

Clearly we have that $g_\alpha(n) \geq h_\alpha(n), g_\beta(n) \geq h_\beta(n), g_\gamma(n) \geq h_\gamma(n)$.

Observe also that for $n \geq 3$ we have $g_\alpha(n) > g_\beta(n) > g_\gamma(n)$.

Hence finally we come to the reccurrences

$$\begin{aligned} g_\alpha(n) &= g_\alpha(n-1) + g_\beta(n-1) + g_\gamma(n-1), \\ g_\beta(n) &= g_\alpha(n-1) + 2g_\gamma(n-1), \\ g_\gamma(n) &= g_\alpha(n-1) + g_\gamma(n-1). \end{aligned} \tag{2.16}$$

From (2.16) we obtain the following reccurence

$$g_\alpha(n) = 2g_\alpha(n-1) + g_\alpha(n-2) + g_\alpha(n-3) \tag{2.17}$$

with initial values $g_\alpha(1) = 1, g_\alpha(2) = 3, g_\alpha(3) = 8$.

Now to estimate the function $f^*(n)$ it remains to solve reccurrence (2.17), since $f^*(n) \leq g_\alpha(n)$. The latter gives the estimation

$$g_\alpha(n) \leq c''(2,547)^n$$

for some constant $c''$ definable from the initial values. This completes the proof of Theorem 1.

□

# 3 Proof of Theorem 2

Suppose that $\{a_1, \dots, a_n\} \subset \mathbb{N}$ has distinct subset sums. Let $X$ denote the set of all solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of the equation $\sum\limits_{i=1}^{n} a_i x_i = \lambda$.

Consider two mappings $\varphi_0$ and $\varphi_1$ from $\{0, \pm 1, \pm 2\}$ to $\{0, \pm 1\}$

$\varphi_0(-2) = \varphi_1(-2) = -1$, $\varphi_0(2) = \varphi_1(2) = 1$, $\varphi_0(\pm 1) = \varphi_1(\pm 1) = 0$, and $\varphi_0(0) = -1$, $\varphi_1(0) = 1$.

Next for $x^n \in X$ define

$$\varphi(x^n) = \big\{(\varphi_{\varepsilon_1}(x_1), \dots, \varphi_{\varepsilon_n}(x_n)) : \varepsilon_i \in \{0,1\}, i = 1, \dots, n\big\}.$$

**Claim 1.** For $x^n, y^n \in X$, $x^n \neq y^n$

$$\varphi(x^n) \cap \varphi(y^n) = \varnothing.$$

**Proof:** Suppose the opposite. Then it is not hard to verify that $x^n - y^n \in \{0, \pm 2\}^n \setminus \{0^n\}$, a contradiction.

□

Let us define

$$\alpha(x^n) = \text{ the number of zero coordinates in } x^n.$$

**Claim 2.** For any $x^n \in X$

$$|\varphi(x^n)| = 2^{\alpha(x^n)}.$$

**Proof:** This immediately follows from the definition of $\varphi(x^n)$.

$\square$

Combining Claims 1 and 2 we conclude that

$$\sum_{x^n \in X} 2^{\alpha(x^n)} \leq 3^n. \tag{3.1}$$

Now consider the mapping $\Psi : X \to \{0, \pm 1\}^n$, defined by $\Psi(x^n) = \big(\Psi_0(x_1), \ldots, \Psi_0(x_n)\big)$, where

$$\Psi_0(x_i) = \begin{cases} -1, & \text{if } x_i = -2, -1 \\ 1, & \text{if } x_i = 2, 1 \\ 0, & \text{if } x_i = 0; i = 1, \ldots, n. \end{cases}$$

**Claim 3.** For $x^n, z^n \in X$, $x^n \neq z^n$ holds $\Psi(x^n) \neq \Psi(z^n)$.

**Proof:** Assuming the opposite we will get $x^n - z^n \in \{0, \pm 1\}^n \smallsetminus \{0^n\}$, a contradiction.

$\square$

Note (and this is important for us) that $\Psi$ leaves the zero coordinates fixed. This with (3.1) implies that

$$\sum_{y^n \in \Psi(X)} 2^{\alpha(y^n)} \leq 3^n.$$

Since $|X| = |\Psi(X)|$ we can bound $|X|$ by the maximum cardinality of a set $Y \subset \{0, \pm 1\}^n$ satisfying

$$\sum_{y^n \in Y} 2^{\alpha(y^n)} \leq 3^n. \tag{3.2}$$

Define

$$Y_i = \big\{y^n \in Y : \alpha(y^n) = i\big\}, i = 0, 1, \ldots, n.$$

Note that $|Y_i| \leq 2^{n-i}\binom{n}{i}$.

Now (3.2) can be rewritten in the form

$$\sum_{i=0}^{n} |Y_i| 2^i \leq 3^n. \tag{3.3}$$

Observe that to maximize $|Y| = \sum\limits_{i=0}^{n} |Y_i|$ we have to take $|Y_i| = \begin{cases} \binom{n}{i} 2^{n-i}, & \text{if } i \leq \ell(n) \\ 0, & \text{if } i > \ell(n) \end{cases}$

where $\ell(n)$ is the maximal index for which one has

$$\sum_{i=0}^{\ell(n)} 2^{n-i} \binom{n}{i} 2^i \leq 3^n.$$

This gives (using standard technique) that $\ell(n) \geq \lfloor 0,1402\ n \rfloor$. Correspondingly we get an estimation for $|Y|$ and consequently for $|X|$:

$$|X| \leq |Y| < c\frac{3^n}{2^{0,14n}} < c(2,723)^n$$

for some constant $c$.

□

# 4 Proof of Theorem 3

Let $Q = \{0, \pm 1, \ldots, \pm m\}$, $F = \{0, \pm 1, \ldots, \pm k\}$ with $\alpha = (2m+1)/(k+1)$.

(a) First we will show that $f(n, Q, F) \leq \alpha^{n-1}$. Let $H$ be defined by an equation

$$\sum_{i=1}^{n} a_i x_i = 0. \tag{4.1}$$

Let also $H \cap F^n = \{0^n\}$ and $H \cap Q^n = X$ with $|X| = f(n, Q, F)$.

Define $Q_j = \{a \in Q : a \equiv j(\mod \alpha)\}$, $j = 0, 1, \ldots, \alpha - 1$.

Then consider the mapping $\varphi : X \to \mathbb{Z}_\alpha^n$, defined by the following transformation of coordinates.

$\varphi(x_1, \ldots, x_n) = (\varphi_0(x_1), \ldots, \varphi_0(x_n))$, where $\varphi_0(x_i) = j$, $(i = 1, \ldots, n)$ if $x_i \in Q_j$; $j \in \{0, \ldots, \alpha - 1\}$. Observe that $\varphi$ is an injection. Hence $|X| = |\varphi(X)|$.

Note now that

$$\dim(\mathrm{span}\varphi(X)) \leq \dim(\mathrm{span}(X)) = n - 1.$$

This implies that

$$|X| = |\varphi(X)| \leq \alpha^{n-1}. \tag{4.2}$$

□

(b) Next we will show that bound (4.2) can be achieved by taking the hyperplane $H$ defined by

$$x_0 + (k+1)x_1 + \cdots + (k+1)^{n-1}x_{n-1} = 0. \tag{4.3}$$

In fact $H \cap F^n = \{0^n\}$. Moreover we claim that for any $-m \leq \lambda \leq m$ the equation

$$\sum_{i=0}^{n-1} x_i (k+1)^i = \lambda \tag{4.4}$$

has exactly $\alpha^{n-1}$ solutions $x^n \in Q^n$. This can be shown using induction on $n$.

The case $n = 1$ is trivial.

**Induction step from $n-1$ to $n$:** Clearly $x_0 \in \big\{a : -m \le a \le m, a \equiv \lambda \mod (k+1)\big\}$. Thus $x_0$ can take $\alpha$ many values $x_0 \in [-m, m]$. For each $x_0$ we come to an equation

$$x_1 + (k+1)x_2 + \cdots + (k+1)^{n-2}x_{n-1} = \frac{\lambda - x_0}{k+1}$$

with $\left|\frac{\lambda - x_0}{k+1}\right| \le \frac{2m}{k+1} \le m$. Hence we get the result by induction hypothesis. This completes the proof of Theorem 3 in the case (i). The case (ii) can be proved similarly.

□

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