COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **24** (4) (2004) 555–566

ON SHADOWS OF INTERSECTING FAMILIES

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Received December 11, 2001
Revised November 14, 2003

The shadow minimization problem for t-intersecting systems of finite sets is considered. Let $\mathcal{A}$ be a family of k-subsets of $\mathbb{N}$. The ℓ-shadow of $\mathcal{A}$ is the set of all $(k-\ell)$-subsets $\partial_\ell\mathcal{A}$ contained in the members of $\mathcal{A}$. Let $\mathcal{A}$ be a t-intersecting family (any two members have at least t elements in common) with $|\mathcal{A}|=m$. Given k,t,m the problem is to minimize $|\partial_\ell\mathcal{A}|$ (over all choices of $\mathcal{A}$). In this paper we solve this problem when m is big enough.

1. Introduction and result

$\mathbb{N}$ denotes the set of positive integers and the set $\{1,\dots,n\}$ is abbreviated as $[n]$. Given $n,k\in\mathbb{N}$ and $X\subset\mathbb{N}$ denote

$$2^{[n]}=\{F:F\subset[n]\},\qquad \binom{X}{k}=\{F\subset X:|F|=k\}.$$

A finite family $\mathcal{A}\subset\binom{X}{k}$ is called t-intersecting if $|A\cap B|\ge t$ for all $A,B\in\mathcal{A}$. In the sequel this definition is used for $X=[n]$ resp. $X=\mathbb{N}$. $I(n,k,t)$ resp. $I(\infty,k,t)$ denote the sets of all such families.

We use the notation $\|\mathcal{A}\|=|\bigcup_{A\in\mathcal{A}}A|$.

The ℓ-shadow of $\mathcal{A}\subset\binom{X}{k}$ is defined by $\partial_\ell\mathcal{A}=\left\{F\in\binom{X}{k-\ell}:\exists A\in\mathcal{A}:F\subset A\right\}$. When $\ell=1$ we write $\partial\mathcal{A}$. Define the colex order for the elements $A,B\in\binom{\mathbb{N}}{k}$ as follows:

$$A<B\Leftrightarrow \max\big((A\setminus B)\cup(B\setminus A)\big)\in B.$$

Mathematics Subject Classification (2000): 05C35, 05C65, 05D05

For a family $\mathcal{F}\subset\binom{\mathbb{N}}{k}$ we denote by $L_m\mathcal{F}$ the set of the first m elements of $\mathcal{F}$ ($m\leq|\mathcal{F}|$) in colex order.

Let $\mathcal{A}\subset\binom{[n]}{k}$ (or $\mathcal{A}\subset\binom{\mathbb{N}}{k}$) with $|\mathcal{A}|=m$. How small can $|\partial_\ell\mathcal{A}|$ be?

The well known Kruskal–Katona Theorem (proved by Kruskal [8], by Katona [6] and by Lindström and Zetterström [9]) solves the (shadows minimization) problem for any parameters n,k,m,ℓ.

Theorem KK.

$$|\partial_\ell\mathcal{A}|\geq\left|\partial_\ell\left(L_m\binom{[n]}{k}\right)\right|. \tag{1.1}$$

Let now $\mathcal{A}\in I(n,k,t)$. What can we say about $|\partial_\ell\mathcal{A}|$? (Can we have a result like Theorem KK?) An important result of Katona [7] is the following

Theorem Ka. *For integers $1\leq\ell\leq t$ and $t\leq k\leq n$, $\mathcal{A}\in I(n,k,t)$*

$$|\partial_\ell\mathcal{A}|\geq|\mathcal{A}|\frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}. \tag{1.2}$$

For extensions and analogues of this inequality see Frankl [4].

In the lemma below we characterize equality in (1.2). The inequality actually is not valid for $\ell>t$: for example for $t=1$, $\ell=2$, $k=3$ and $n\geq7$ the 1-intersecting EKR family satisfies $\frac{|\partial_2(\mathcal{A})|}{|\mathcal{A}|}=\frac{n}{\binom{n-1}{2}}<\frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}=\frac{1}{2}$.

Thus in general finding the exact lower bound for $|\partial_\ell\mathcal{A}|$ for any given parameters n,k,t,ℓ,m is an open (and seemingly difficult) problem.

We state now our result, which solves the problem when the ground set is $\mathbb{N}$ and m is big enough. For integers $1\leq r\leq s$, $k\geq r$ define

$$\mathcal{B}(k,s,r)=\left\{B\in\binom{\mathbb{N}}{k}:|B\cap[s]|\geq r\right\}.$$

Theorem. *Let $\mathcal{A}\in I(\infty,k,t)$.*

(a) For $1\leq\ell\leq t<k$ and $|\mathcal{A}|=m>m_1(k,t,\ell)$ (suitable) we have

$$|\partial_\ell\mathcal{A}|\geq|\partial_\ell L_m\mathcal{B}(k,2k-2-t,k-1)|.$$

(b) For $1\leq t<\ell<k$ and $|\mathcal{A}|=m>m_2(k,t,\ell)$ (suitable) we have

$$|\partial_\ell\mathcal{A}|\geq|\partial_\ell L_m\mathcal{B}(k,t,t)|.$$

2. An auxiliary result

Lemma. *Under the conditions of Theorem Ka equality in (1.2) holds iff* $\mathcal{A}=\binom{[2k-t]}{k}$.

Proof. Actually for any $1\le \ell,t\le k$ counting edges, which are defined by containment, in the bipartite graph $(\mathcal{A},\partial_\ell\mathcal{A})$ in two ways one gets for $s=\|\mathcal{A}\|$

$$|\partial_\ell(\mathcal{A})|\ge|\mathcal{A}|\frac{\binom{k}{\ell}}{\binom{s-k+\ell}{\ell}}. \tag{2.1}$$

Now in the

Case 1: $s\le 2k-t$

we continue with

$$|\partial_\ell(\mathcal{A})|\ge|\mathcal{A}|\frac{\binom{k}{\ell}}{\binom{k-t+\ell}{\ell}}=|\mathcal{A}|\frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}$$

with equality iff $s=2k-t$.

(Notice that in this case the assumption $\ell\le t$ is not used.)

Case 2: $s>2k-t$.

Recall the well known shifting operation S_{ij} defined for any $F\in 2^{[n]}$ and for any family $\mathcal{F}\subset 2^{[n]}$. For integers $1\le i<j\le n$

$$S_{ij}(F)=\begin{cases}((F\smallsetminus\{j\})\cup\{i\}), & \text{if } i\notin F, j\in F, ((F\smallsetminus\{j\})\cup\{i\})\notin\mathcal{F};\\ F, & \text{otherwise.}\end{cases}$$

$$S_{ij}(\mathcal{F})=\{S_{ij}(F):F\in\mathcal{F}\}.$$

We say that $\mathcal{F}$ is shifted, if $S_{ij}(\mathcal{F})=\mathcal{F}$ for all $1\le i<j\le n$.

The following properties of $S_{ij}(\mathcal{F})$ are well known (see e.g. [2] or [3])

P_1. $|S_{ij}(\mathcal{F})|=|\mathcal{F}|$,
P_2. $\partial_\ell S_{ij}(\mathcal{F})\subset S_{ij}(\partial_\ell\mathcal{F})$ (hence $|\partial_\ell S_{ij}(\mathcal{F})|\le|\partial_\ell\mathcal{F}|$),
P_3. $\mathcal{F}\in I(n,k,t)\Rightarrow S_{ij}(\mathcal{F})\in I(n,k,t)$.

We also need the following result due to Mörs [10] and Füredi and Griggs [5].

Theorem MFG. *Let* $\mathcal{F}\subset\binom{[n]}{k}$ *have minimal* ℓ*-shadow, then its* $(\ell+1)$*-shadow is minimal as well.*

In particular the theorem implies that for an optimal family $\mathcal{F}$ the shifting operation does not decrease $\|\mathcal{F}\|$. In particular if $\mathcal{F}$ is a t-intersecting family with minimal ℓ-shadow and $|\mathcal{F}|\le\binom{2k-t}{k}$ then $\|\mathcal{F}\|\le 2k-t$.

Combining all these facts we conclude that w.l.o.g. we *can assume that $\mathcal{A}$ is shifted.*

Define now

$$\mathcal{A}_1 = \{A \in \mathcal{A} : s \in A\}, \mathcal{A}_0 = \mathcal{A} \setminus \mathcal{A}_1, \quad \mathcal{A}_1' = \{A \setminus \{s\} : A \in \mathcal{A}_1\}.$$

Proposition 1. $|\partial_\ell \mathcal{A}| = |\partial_\ell \mathcal{A}_0| + |\partial_\ell \mathcal{A}_1'|$.

Proof. Define

$$\mathcal{B}_1 = \{B \subset \partial_\ell \mathcal{A}_1 : s \in B\}, \ \mathcal{B}_0 = \partial_\ell \mathcal{A}_1 \setminus \mathcal{B}_1.$$

It is not hard to see that the shiftedness of $\mathcal{A}$ implies that $\mathcal{B}_0 \subset \partial_\ell \mathcal{A}_0$. Also it is clear that $\mathcal{B}_1 \cap \partial_\ell \mathcal{A}_0 = \varnothing$. ∎

Proposition 2. For $s \geq 2k-t+1$ $\quad \mathcal{A}_1' \subset \binom{[s-1]}{k-1}$ is a t-intersecting family.

Proof. Suppose $\mathcal{A}_1'$ is $(t-1)$-intersecting, that is there are two elements $A, B \in \mathcal{A}_1$ with $|A \cap B| = t$. Then in view of the shiftedness we must have $|A \cup B| = s$. This is clear, because otherwise there exists $i \in [s] \setminus (A \cup B)$ such that $S_{is}(A) \triangleq A' \in \mathcal{A} \setminus \{A\}$, and hence $|A' \cap B| = t-1$ a contradiction. Note however that for $s \geq 2k-t+1$ the conditions $|A \cup B| = s$ and $|A \cap B| \geq t$ are contradictory. ∎

We are prepared now to complete the proof of the lemma.

We proceed by induction on $s \geq 2k-t$. The induction beginning $s = 2k-t$ is already done by Case 1.

For the induction $s \to s+1$ we first show that for $s \geq 2k-t+1$

$$\mathcal{A}_0 \neq \binom{[2k-t]}{k}. \tag{2.2}$$

For this we distinguish two subcases of case 2.

Subcase $s = 2k-t+1$: Observe that $|\mathcal{A}_0| < \binom{2k-t}{k}$, because otherwise by the t-intersecting property of $\mathcal{A}$ $\quad \mathcal{A}_1$ would be empty in contradiction to $s = 2k-t+1$.

Subcase $s \geq 2k-t+2$: Here by shiftedness $\|\mathcal{A}_0\| \geq 2k-t+1$ and again (2.2) holds.

For $\|\mathcal{A}\| = s+1$ we have $\|\mathcal{A}_0\| = s$ and the induction hypothesis yields

$$|\partial_\ell \mathcal{A}_0| > |\mathcal{A}_0| \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}. \tag{2.3}$$

Further in view of Proposition 2 and (1.2) and the fact $|\mathcal{A}_1'|=|\mathcal{A}_1|$ we have

$$|\partial_\ell \mathcal{A}_1'| \geq |\mathcal{A}_1'| \frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}} = |\mathcal{A}_1| \frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}}. \tag{2.4}$$

Finally by Proposition 1, (2.2), (2.3) and the inequality $\frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}} \geq \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}$ (for $t \geq \ell$) we get

$$|\partial_\ell \mathcal{A}| > (|\mathcal{A}_1| + |\mathcal{A}_0|) \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}} = |\mathcal{A}| \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}, \tag{2.5}$$

concluding the proof. ∎

3. Proof of the theorem in case (a): $1 \leq \ell \leq t < k$

Given $m > \binom{2k-t}{k}$ consider the family $\mathcal{A}^* \triangleq L_m \mathcal{B}(k, 2k-2-t, k-1)$.

Observe that $\mathcal{A}^*$ is t-intersecting.

Let us write $m = |\mathcal{A}^*|$ in the form

$$m = \binom{2k-2-t}{k} + n\binom{2k-2-t}{k-1} + r, \tag{3.1}$$

where $n \in \mathbb{N}$ and $0 \leq r < \binom{2k-2-t}{k-1}$.

Note then that $\mathcal{A}^* \setminus \binom{[2k-2-t]}{k}$ can be partitioned into $n+1$ (or n, if $r=0$) classes, where each class consists of sets containing a fixed element $j \in [2k-1-t, 2k-1-t+n]$ (or $[2k-1-t, 2k-1-t+(n-1)]$, if $r=0$).

We can observe now that

$$|\partial_\ell \mathcal{A}^*| = \binom{2k-2-t}{k-\ell} + n\binom{2k-2-t}{k-1-\ell} + \partial_\ell(r), \tag{3.2}$$

where $\partial_\ell(r) \triangleq \left|\partial_\ell\left(L_r\binom{[2k-2-t]}{k-1}\right)\right|$.

Now we show first that the ratio $|\partial_\ell \mathcal{A}^*||\mathcal{A}^*|^{-1}$ can be approximated like in (1.2) of Katona's Theorem from above but also from below by passing from k to $k-1$. By elementary calculations of binomial coefficients we get Claim 1 below. The lower bound is not needed in this paper, but perhaps useful elsewhere.

In the second part of this proof of (a) we show that for any family $\mathcal{A} \not\subset \mathcal{B}(k, 2k-2-t, k-1)$, $|\mathcal{A}| = m$, we can establish a lower bound for $|\partial_\ell \mathcal{A}|$ which exceeds the upper bound in Claim 1 for m large. This contradiction will complete the proof of (a).

Claim 1.

$$m\frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}}+1 \le |\partial_\ell \mathcal{A}^*| \le m\frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}}+\alpha, \tag{3.3}$$

for some $1 \le \alpha < \binom{2k-1-t}{k-\ell}$.

Proof. Let us abbreviate $\frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}}$ by λ. We want to see how much $|\partial_\ell \mathcal{A}^*|$ deviates from $m\lambda$.

The following identity can be easily verified using (3.1) and (3.2).

$$|\partial_\ell \mathcal{A}^*| = m\lambda + \binom{2k-2-t}{k-\ell} - \binom{2k-2-t}{k}\lambda + \partial_\ell(r) - r\lambda. \tag{3.4}$$

Since $\partial_\ell(r) < \binom{2k-2-t}{k-\ell-1}$ in view of (3.4) we have the desired upperbound

$$|\partial_\ell \mathcal{A}^*| < m\lambda + \binom{2k-2-t}{k-\ell} + \binom{2k-2-t}{k-\ell-1} = m\lambda + \binom{2k-1-t}{k-\ell},$$

(which can be improved, but is good enough for our purposes).

Furthermore by (1.2) we have $\partial_\ell(r) \ge r\lambda$. Also one can check that

$$\binom{2k-2-t}{k-\ell} - \binom{2k-2-t}{k}\lambda = \frac{\ell}{k}\binom{2k-1-t}{k-\ell} \ge 1.$$

Hence by (3.4) we get our lower bound

$$|\partial_\ell \mathcal{A}_\ell^*| \ge m\lambda + 1. \tag{3.5}$$

Note that the constant 1 cannot be improved in general. For example for $k=t+1$, $\ell=1$ and $r=0$ we have equality in (3.5). ∎

Suppose now $\mathcal{A} \in I(\infty,k,t)$ is an optimal family (has minimal ℓ-shadow) with $|\mathcal{A}| = m > \binom{2k-t}{k}$ and $\|\mathcal{A}\| = u$. Suppose also again that $\mathcal{A}$ is shifted.

Let us partition $\mathcal{A}$ into $s+1$ disjoint classes $\mathcal{A} = \bigcup_{i=1}^{s+1} \mathcal{A}_i$, $s \triangleq u-(2k-t)$, defined by

$$\begin{aligned}
\mathcal{A}_1 &= \{A \in \mathcal{A} : u \in A\},\\
\mathcal{A}_2 &= \{A \in \mathcal{A} \setminus \mathcal{A}_1 : u-1 \in A\},\\
&\vdots\\
\mathcal{A}_s &= \{A \in \mathcal{A} \setminus (\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{s-1}) : u-s+1 \in A\},\\
\mathcal{A}_{s+1} &= \mathcal{A} \setminus (\mathcal{A}_1 \cup \dots \cup \mathcal{A}_s).
\end{aligned}$$

Note that $\mathcal{A}_{s+1} \subset \binom{[2k-t]}{k}$.
Define also

$$\mathcal{A}'_i = \{A \smallsetminus \{u-i+1\} : A \in \mathcal{A}_i\},\ i = 1, \ldots, s.$$

Since $\mathcal{A}$ is shifted, from Proposition 1 we infer that

$$|\partial_\ell \mathcal{A}| = |\partial_\ell \mathcal{A}_{s+1}| + |\partial_\ell \mathcal{A}'_s| + \cdots + |\partial_\ell \mathcal{A}'_1|. \tag{3.6}$$

We distinguish now between two cases.

Case 1: $\mathcal{A}'_i \neq \binom{[2(k-1)-t]}{k-1}$, $i = 1, \ldots, s$.

By Proposition 2 each class $\mathcal{A}'_i$, $i = 1, \ldots, s$, is t-intersecting. Note also that $|\mathcal{A}'_i| = |\mathcal{A}_i|$. Therefore by the Lemma we have

$$|\partial_\ell \mathcal{A}'_i| \geq |\mathcal{A}_i| \frac{\binom{2(k-1)-t}{k-1-\ell}}{\binom{2(k-1)-t}{k-1}} + 1,\ i = 1, \ldots, s.$$

This with (3.6) implies

$$\begin{aligned}|\partial_\ell \mathcal{A}| &\geq |\partial_\ell \mathcal{A}_{s+1}| + \lambda(|\mathcal{A}_1| + \cdots + |\mathcal{A}_s|) + s = |\partial_\ell \mathcal{A}_{s+1}| + \lambda|\mathcal{A}| - \lambda|\mathcal{A}_{s+1}| + s \\ &> \lambda|\mathcal{A}| + s - \lambda|\mathcal{A}_{s+1}|.\end{aligned}$$

Since $|\mathcal{A}_{s+1}| < \binom{2k-t}{k}$ we have $\beta = \beta(k,t,\ell) \triangleq \lambda|\mathcal{A}_{s+1}| < \lambda\binom{2k-t}{k}$ and a fortiori $|\partial_\ell \mathcal{A}| > m \frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}} + s - \beta$, where $\beta < \binom{2k-t}{k} \frac{\binom{2k-2-t}{k-1-\ell}}{\binom{2k-2-t}{k-1}}$.

If now m is big enough, such that $s \geq \beta + \binom{2k-t-1}{k-\ell}$, then in view of (3.3) we get $|\partial_\ell \mathcal{A}| > |\partial_\ell \mathcal{A}^*|$, a contradiction with the optimality of $\mathcal{A}$.

Case 2: There exists a family $\mathcal{A}_i$, $i \in \{1, \ldots, s\}$ such that

$$\mathcal{A}'_i = \binom{[2k-2-t]}{k-1}. \tag{3.7}$$

Claim 2.

$$\mathcal{A} \subset \mathcal{B}(k, 2k-2-t, k-1).$$

Proof. By definition each member of $\mathcal{A}_i$ contains the element $u-i+1$ and $u-i+1 \geq 2k-t+1$, since $i \leq s = u-2k+t$. Also in view of (3.7) $\mathcal{A}_i$ contains the set $B_1 \triangleq \{k-t, \ldots, 2k-2-t, u-i+1\}$. Moreover by the shiftedness (and the fact that $u-i-1 > 2k-2-t$) $\mathcal{A}$ contains also $B_2 \triangleq \{k-t, \ldots, 2k-2-t, u-i\}$ and $B_3 \triangleq \{k-t, \ldots, 2k-2-t, u-i-1\}$.

Suppose now $F \in \mathcal{A}$ and $F \notin \mathcal{B}(k, 2k-2-t, k-1)$, that is $|F \cap [2k-2-t]| \leq k-2$. In view of the shiftedness we can take $F = \{1, \ldots, k'\} \cup E$, where $\{1, \ldots, k'\} = F \cap [2k-2-t]$, $k' \leq k-2$.

If $k' < k-2$, then observe that $|F \cap B_1| < t$, a contradiction. Let now $k' = k-2$. Then clearly $|\{1,\dots,k'\} \cap B_j| = t-1$ $(j=1,2,3)$. Hence (to provide t-intersection with B_1, B_2, B_3) F must contain the elements $u-i+1$, $u-i$ and $u-i-1$, a contradiction with $|F|=k$. ∎

To complete the proof of case (a) we use a special case of the result in [1].

Theorem AAK. *For $s,r,k \in \mathbb{N}$, $1 \le r \le s, \ell \le k$ let $\mathcal{F} \subset \mathcal{B}(k,s,r)$ with $|\mathcal{F}|=m$. Then*

$$|\partial_\ell \mathcal{F}| \ge |\partial_\ell L_m \mathcal{B}(k,s,r)|. \quad ∎$$

4. Proof of the theorem in case (b): $1 \le t < \ell < k$

Claim. It is sufficient to consider the case $\ell = t+1$.

Proof. Let's explain first that (for $\ell \ge t$)

$$\partial_\ell L_m \mathcal{B}(k,t,t) = L_{M^*} \binom{\mathbb{N}}{k-\ell},$$

where $M^* \triangleq |\partial_\ell L_m \mathcal{B}(k,t,t)|$.

By definition of $\mathcal{B}_m(k,t,t)$

$$L_m \mathcal{B}(k,t,t) = \left\{ [t] \cup E : E \in L_m \binom{\mathbb{N} \setminus [t]}{k-t} \right\}.$$

Since $\ell \ge t$ the largest element F of $\partial_\ell L_m \mathcal{B}(k,t,t)$ (in colex order) satisfies $F \in \partial_{\ell-t} L_m \binom{\mathbb{N}\setminus[t]}{k-t}$. Therefore for every $(k-\ell)$-set $F' < F$ there exists $B \in L_m \mathcal{B}(k,t,t)$ such that $F' \subset B$ and the identity follows.

Suppose now we have proved the theorem for some $\ell \ge t+1$. That is for any family $\mathcal{A} \in I(\infty,k,t)$ with $|\mathcal{A}|=m$ $(m > m_2(k,t,\ell))$ we have

$$M \triangleq |\partial_\ell \mathcal{A}| \ge |\partial_\ell L_m \mathcal{B}(k,t,t)| = M^*.$$

Then using Theorem KK we can write

$$|\partial_{\ell+1}\mathcal{A}| = |\partial(\partial_\ell \mathcal{A})| \ge \left| \partial L_M \binom{\mathbb{N}}{k-\ell} \right| \ge \left| \partial L_{M^*} \binom{\mathbb{N}}{k-\ell} \right| = |\partial_{\ell+1} L_m \mathcal{B}(k,t,t)|. \quad ∎$$

Let now $\ell = t+1$ and let $\mathcal{A} \in I(\infty,k,t)$ be an optimal, shifted family with $|\mathcal{A}|=m$.

Define next (in new notation)

$$\mathcal{A}_1 = \{A \in \mathcal{A} : 1 \in A\}, \quad \mathcal{A}'_1 = \{A \setminus \{1\} : A \in \mathcal{A}_1\}, \quad \mathcal{A}_0 = \mathcal{A} \setminus \mathcal{A}_1,$$

and denote $|\mathcal{A}_0| = m_0$, $|\mathcal{A}_1| = m_1 = m - m_0$.

We consider two cases.

Case 1: $\mathcal{A}_0 = \varnothing$

Clearly we have

$$|\partial_{t+1}\mathcal{A}| = |\partial_t \mathcal{A}_1'| + |\partial_{t+1}\mathcal{A}_1'|.$$

Since $\mathcal{A}_1' \in I(\infty, k-1, t-1)$ we can use induction on t (the case $t=1$ can be easily derived). That is we have

$$\begin{aligned}|\partial_{t+1}\mathcal{A}| &\geq |\partial_t L_m \mathcal{B}(k-1, t-1, t-1)| + |\partial_{t+1} L_m \mathcal{B}(k-1, t-1, t-1)| \\ &= |\partial_{t+1} L_m \mathcal{B}(k, t, t)|.\end{aligned}$$

Case 2: $\mathcal{A}_0 \neq \varnothing$.

Let us note first that $\mathcal{A}_0$ is $(t+1)$-intersecting. This easily follows from the shiftedness of $\mathcal{A}$.

Hence by (1.2) we have

$$|\partial_{t+1}\mathcal{A}_0| \geq |\mathcal{A}_0|. \tag{4.1}$$

Also in view of the shiftedness $\{2, \ldots, k+1\} \in \mathcal{A}_0 \neq \varnothing$.

This implies that

$$\mathcal{A}_1' \subset \mathcal{F} \triangleq \left\{ F \in \binom{\mathbb{N} \setminus \{1\}}{k-1} : |F \cap \{2, \ldots, k+1\}| \geq t \right\}.$$

We apply now Theorem AAK to $\mathcal{A}_1'$ identifying $\mathcal{F}$ with $\mathcal{B}(k-1, k, t)$. We get then

$$|\partial_{t+1}\mathcal{A}_1'| \geq |\partial_{t+1} L_{m_1} \mathcal{B}(k-1, k, t)|, \quad |\partial_t \mathcal{A}_1'| \geq |\partial_t L_{m_1} \mathcal{B}(k-1, k, t)|. \tag{4.2}$$

Obviously we also have

$$\sum_{i=0}^{k-t-1} \binom{k}{t+i} \binom{x-k}{k-1-t-i} < |\mathcal{A}_1'| = m_1 \leq \sum_{i=0}^{k-t-1} \binom{k}{t+i} \binom{x-k+1}{k-1-t-i}, \tag{4.3}$$

where $x = \|L_{m_1}\mathcal{B}(k-1, k, t)\| - 1$.

Therefore for a positive constant $c_1 = c_1(k, t)$

$$m_1 \sim c_1 x^{k-1-t}. \tag{4.4}$$

On the other hand

$$|\partial_{t+1}\mathcal{A}_1| = |\partial_{t+1}\mathcal{A}_1'| + |\partial_t \mathcal{A}_1'|.$$

This with (4.2) and (4.3) gives for another positive constant $c_2 = c_2(k,t)$ the estimation

$$|\partial_{t+1}\mathcal{A}_1| \geq |\partial_t \mathcal{A}_1'| \geq \binom{x}{k-t-1} \geq c_2 m_1. \tag{4.5}$$

Assuming now, for a contradiction, that $|\partial_{t+1}\mathcal{A}| \leq |\partial_{t+1} L_m \mathcal{B}(k,t,t)|$ we observe that for any $\varepsilon > 0$, if $m \geq m(\varepsilon,t,k)$, suitable, we have

$$\frac{|\partial_{t+1}\mathcal{A}|}{|\mathcal{A}|} = \frac{|\partial_{t+1}\mathcal{A}|}{m} \leq \frac{|\partial_{t+1} L_m \mathcal{B}(k,t,t)|}{m} < \varepsilon.$$

This with (4.1) implies for $m > m(\varepsilon,t,k)$ also

$$\varepsilon > \frac{|\partial_{t+1}\mathcal{A}|}{|\mathcal{A}|} \geq \frac{|\partial_{t+1}\mathcal{A}_0|}{|\mathcal{A}|} \geq \frac{|\mathcal{A}_0|}{|\mathcal{A}|} = \frac{m_0}{m} = 1 - \frac{m_1}{m}. \tag{4.6}$$

Together with (4.5) we get

$$\varepsilon > \frac{|\partial_{t+1}\mathcal{A}|}{|\mathcal{A}|} \geq \frac{|\partial_{t+1}\mathcal{A}_1|}{|\mathcal{A}|} \geq \frac{c_2 m_1}{m} \text{ or } \frac{m_1}{m} \leq \frac{\varepsilon}{c_2}.$$

On the other hand (4.6) implies that $\frac{m_1}{m} > 1 - \varepsilon$ and thus $\varepsilon > (1-\varepsilon)c_2$. This contradiction completes the proof of (b). ∎

5. Remarks

1. We give a numerical version for case (a) of our theorem. Note first that any integer $m > \binom{2k-t}{k}$ can be uniquely represented in the form (3.1). Also we can uniquely represent r in the $(k-1)$-cascade form (see [2] or [3])

$$r = \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s},$$

where $a_{k-1} > a_{k-2} > \cdots > a_s \geq s \geq 1$.
Then in view of our theorem and Theorem KK we have the following: For $1 \leq \ell \leq t < k$ and $m > m_1(k,t,\ell)$ holds

$$|\partial_\ell \mathcal{A}| \geq \binom{2k-2-t}{k-\ell} + n\binom{2k-2-t}{k-\ell-1} + \binom{a_{k-1}}{k-1-\ell} + \cdots + \binom{a_s}{s-\ell}.$$

Note that for applications one can use also the lower estimate in (3.3) (for $m > m_1(k,t,\ell)$).

2. As an improvement of the theorem it would be interesting to find minimal values of $m_1(k,\ell,t)$ and $m_2(k,\ell,t)$ for which the result holds. In fact for the case (a) the proof gives also an upper bound for $m_1(k,\ell,t)$. However this estimation seems to be rough.
   More generally one should decide whether t-intersecting $\mathcal{B}(k,s,r)$ sets are extremal.
   On the other hand note that our theorem is not valid for all $m > \binom{2k-t}{k}$. Here are examples for cases (a) and (b).

   (a) $1 \le \ell < t$.
   Let $k>3$, $t=\ell=1$, $m=\binom{2k-1}{k}+1$. Then we can write

$$m = \binom{2k-3}{k} + 3\binom{2k-3}{k-1} + 1.$$

   Hence

$$\begin{aligned} \Delta_1 \triangleq |\partial(L_m\mathcal{B}(k,2k-3,k-1))| &= \binom{2k-3}{k-1} + 3\binom{2k-3}{k-2} + k-1 \\ &= 4\binom{2k-3}{k-1} + k - 1. \end{aligned}$$

   Define now the following intersecting family

$$\mathcal{F} = \left(\binom{[2k-1]}{k} \smallsetminus \{k,\dots,2k-1\}\right) \cup \{1,\dots,k-1,2k\} \cup \{1,\dots,k-1,2k+1\}.$$

   Clearly $|\mathcal{F}| = \binom{2k-1}{k}+1$ and $|\partial\mathcal{F}| = \binom{2k-1}{k-1} + 2(k-2) = 3\binom{2k-3}{k-1} + \binom{2k-3}{k-3} + 2(k-2)$.
   Thus $|\partial\mathcal{F}| < \Delta_1$.

   (b) $1 \le t < \ell$.
   For the same m,k and $t=1$, let now $\ell=2$. Then

$$\Delta_2 \triangleq |\partial_2(L_m\mathcal{B}(k,1,1))| = \binom{2k}{k-2} + \binom{k-1}{2},$$

   while

$$|\partial_2\mathcal{F}| = \binom{2k-1}{k-2} + 2\binom{k-1}{2} < \Delta_2.$$

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