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Forbidden (0,1)-vectors in Hyperplanes of $\mathbb{R}^n$: The unrestricted case

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Abstract. In this paper, we continue our investigation on "Extremal problems under dimension constraints" introduced [1]. The general problem we deal with in this paper can be formulated as follows. Let $\mathscr{U}$ be an affine plane of dimension k in $\mathbb{R}^n$. Given $F \subset E(n) \triangleq \{0,1\}^n \subset \mathbb{R}^n$ determine or estimate $\max\{|\mathscr{U} \cap E(n)| : \mathscr{U} \cap F = \varnothing\}$.

Here we consider and solve the problem in the special case where $\mathscr{U}$ is a hyperplane in $\mathbb{R}^n$ and the "forbidden set" $F = E(n,k) \triangleq \{x^n \in E(n) : x^n \text{ has } k \text{ ones}\}$. The same problem is considered for the case, where $\mathscr{U}$ is a hyperplane passing through the origin, which surprisingly turns out to be more difficult. For this case we have only partial results.

Keywords: combinatorial extremal theory, (0,1)-vectors, dimension constraints, forbidden weights

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers. For the set $\{i, i+1, \ldots, j\}(i, j \in \mathbb{N})$ we use the notation $[i, j]$ and for $[1, j]$ we simply write $[j]$. For $k, n \in \mathbb{N}$, $k \le n$ we set

$$2^{[n]} = \{A : A \subset [n]\}, \quad \binom{[n]}{k} = \{A \in 2^{[n]} : |A| = k\}.$$

For a subset $A \subset [n]$ its characteristic vector is defined by $\chi(A) = (x_1, \ldots, x_n)$, where $x_i = 1$ if $i \in A$ and $x_i = 0$, if $i \notin A$. The set of (0,1)-vectors in $\mathbb{R}^n$ is denoted by $E(n) = \{0,1\}^n$. Correspondingly for the vectors of weight k we use the notation $E(n,k) = \{x^n \in E(n) : x^n \text{ has } k \text{ ones}\}$. We are interested in the following geometrical extremal problem.

Let $\mathscr{U}$ be a k–dimensional affine plane in $\mathbb{R}^n$. Given a "forbidden set" $F \subset E(n)$ determine or estimate $\max\{|\mathscr{U} \cap E(n)| : \mathscr{U} \cap F = \varnothing\}$.

In this paper, we consider the special case of this problem where $\mathscr{U}$ is a hyperplane and forbidden sets are the (0,1)-vectors of certain weight. We also consider the problem when a hyperplane contains (0,1)-vectors of only even or odd weight.

For our purposes, we need some well-known notions and results from extremal set theory. The reader can find all this for instance in the textbooks [6] and [7].

A family $\mathscr{A}=\{A_1,\dots,A_m\}\subset 2^{[n]}$ is called a chain of size m if $A_1\subset\cdots\subset A_m$. If $|A_i|=|A_{i+1}|-1$ for $i=1,\dots,m-1$ and $|A_1|+|A_m|=n$ then $\mathscr{A}$ is called a symmetric chain.

A family $\mathscr{A}\subset 2^{[n]}$ is called an antichain if $A_1\not\subset A_2$ holds for all $A_1,A_2\in\mathscr{A}$.

A family $\mathscr{A}\subset 2^{[n]}$ is called intersecting if $A_1\cap A_2\neq\varnothing$ holds for all $A_1,A_2\in\mathscr{A}$.

For integers $1\le\ell\le k\le n$ and a family $\mathscr{A}\subset\binom{[n]}{k}$ the ℓ–shadow of $\mathscr{A}$ is defined by $\partial_\ell\mathscr{A}=\left\{B\in\binom{[n]}{k-\ell}:\exists A\in\mathscr{A}:B\subset A\right\}$. The colex order for elements $A,B\in\binom{[n]}{k}$ is defined as follows: $A\prec B\Leftrightarrow\max\big((A-B)\cup(B-A)\big)\in B$. We denote by $L(k,m)$ the initial m members of $\binom{[n]}{k}$ in colex order.

Theorem S (Sperner). *Let $\mathscr{A}\subset 2^{[n]}$ be an antichain, then $|\mathscr{A}|\le\binom{n}{\lfloor\frac{n}{2}\rfloor}$ and the maximum is assumed only for $\mathscr{A}=\binom{[n]}{\lfloor\frac{n}{2}\rfloor}$ or $\binom{[n]}{\lceil\frac{n}{2}\rceil}$.*

Theorem BTK (de Bruijn–Tengbergen–Kruyswik). *There exists a partition of $2^{[n]}$ into symmetric chains.*

Theorem EKR (Erdős–Ko–Rado). *Let $\mathscr{A}\subset\binom{[n]}{k}$ be an intersecting family and $2k\le n$, then $|\mathscr{A}|\le\binom{n-1}{k-1}$.*

Theorem KK (Kruskal–Katona). *Let $\mathscr{A}\subset\binom{[n]}{k}$ with $|\mathscr{A}|=m$, then $|\partial_\ell\mathscr{A}|\ge|\partial_\ell L(k,m)|$.*

Representing a family $\mathscr{A}\subset 2^{[n]}$ as the set of its characteristic vectors $\chi(\mathscr{A})\subset E(n)$ we extend the notions of antichain, intersecting system and shadow to (0,1)-vectors in a natural way.

2. Forbidden Weights in Hyperplanes

Let H be a hyperplane in $\mathbb{R}^n$. Given integers $0\le w\le n$, $n\ge 1$ define

$$F(n,w)=\max\big\{|H\cap E(n)|:H\cap E(n,w)=\varnothing\big\}.$$

The next result determines $F(n,w)$ for all parameters.

Theorem 1.
(i) $F(n,w)=F(n,n-w)$

(ii) $F(n,w)=\begin{cases}\binom{2w+1}{w+1}2^{n-2w-1}, & \text{if } n\geq 2w+1\\ \binom{2w}{w}, & \text{if } n=2w.\end{cases}$

The main auxiliary result we use to prove Theorem 1 is

Theorem 2. *Given integers $0\leq t\leq w-1$, $n=2w-t$, and $a_1,\ldots,a_n\in\mathbb{R}\setminus\{0\}$, $b\in\mathbb{R}$, let X be the set of (0,1)-solutions of the equation*

$$\sum_{i=1}^{n}a_i x_i=b \tag{2.1}$$

such that $\sum_{i=1}^{n}x_i\neq w,\ w-1,\ldots,w-t-1.$

Then

$$|X|\leq\binom{n}{w+1}, \tag{2.2}$$

and equality holds if $a_1=a_2=\cdots=a_n=1,\ b=w+1.$

As a consequence of Theorem 2 we have,

Corollary 1. *Given $a_1,\ldots,a_{2w-t}\in\mathbb{R}\setminus\{0\}$ let H be a hyperplane defined by the equation*

$$\sum_{i=1}^{2w-t}a_i y_i=b,$$

so that $H\cap E(2w-t,w-i)=\varnothing,\ i=0,\ldots,t+1.$

Then

$$|H\cap E(2w-t)|\leq\binom{2w-t}{w+1}.$$

Remark 1. Note the difference between the set X (in Theorem 2)

$$X=Z\setminus\big\{E(n,w)\cup\cdots\cup E(n,w-t-1)\big\}\ \text{and}\ H\cap E(n),$$

where Z is the set of all (0,1)-solutions of (2.1). Clearly $|X|\geq|H\cap E(n)|$.

Proof of Theorem 1.

Let H be a hyperplane such that $H\cap E(n,w)=\varnothing$ and $|H\cap E(n)|=F(n,w)$.

To prove the part (i) we just note that for the hyperplane $(1^n-H)\triangleq\{1^n-\mathbf{v}:\mathbf{v}\in H\}\big(1^n\triangleq(1,\ldots,1)\big)$ we have

$$(1^n-H)\cap E(n,n-w)=\varnothing,\qquad |(1^n-H)\cap E(n)|=|H\cap E(n)|.$$

Let H be defined by

$$H = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b \right\}, \tag{2.3}$$

where $a_i \neq 0$; $i = 1, \ldots, \ell$ $(\ell \leq n)$ and $a_{\ell+1} = \cdots = a_n = 0$.
Then

$$H \cap E(n) = \big(H^* \cap E(\ell)\big) \times E(n-\ell),$$

where $H^* \subset \mathbb{R}^\ell$ is defined by

$$H^* = \left\{ (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell : \sum_{i=1}^{\ell} a_i x_i = b \right\}.$$

Hence

$$|H \cap E(n)| = |H^* \cap E(\ell)| 2^{n-\ell}.$$

Clearly taking $\ell = 2w+1$ and $b = w+1$ with $a_1 = \cdots = a_\ell = 1$ in (2.3) we guarantee the lower bound $|H \cap E(n)| \geq \binom{2w+1}{w+1} 2^{n-2w-1}$ for the case $n \geq 2w+1$. To see that $F(2w, w) \geq \binom{2w}{w}$ we take $a_1 = -1$, $a_2 = \cdots = a_{2w}$, $b = w-1$.

Next we show that this lower bound is also an upper bound.

Case $n \geq 2w+1$.

Claim: $2w+1 \leq \ell \leq 2w+2$

Proof. To prove the claim we need the following simple fact (which can be proved using Sperner's Theorem).

LEMMA 1. *Let* $a_1, \ldots, a_\ell \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$. *Then the number of* $(0,1)$*-solutions of the equation* $\sum_{i=1}^{\ell} a_i x_i = b$ *is at most* $\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}$, *(for a more general form of this statement see* [2]*).*

If $\ell = 2w+2$ or $2w+1$, then by Lemma 1 we have

$$|H \cap E(n)| \leq \binom{2w+2}{w+1} 2^{n-2w-2}$$

and equality can be achieved for the hyperplane (2.3) with $a_1 = \cdots = a_\ell = 1$, $a_{\ell+1} = \cdots = a_n = 0$, $b = \lceil \frac{\ell}{2} \rceil$.

Assuming $\ell \geq 2w+3$ and using Lemma 1 we get

$$|H \cap E(n)| \leq \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor} 2^{n-\ell} < \binom{2w+2}{w+1} 2^{n-2w-2} \leq F(n, w),$$

a contradiction to the optimality of H.

Suppose now that $\ell < 2w+1$. Since $H \cap E(n) = \big(H^* \cap E(\ell)\big) \times E(n-\ell)$ and $H \cap E(n,w) = \varnothing$, we should have $H^* \cap \big[E(\ell,w) \cup \cdots \cup E(\ell, w-s)\big] = \varnothing$, where $s = \min\{w, n-\ell\}$.

For convenience we set $\ell = 2w - t$, where $t \geq 0$. Let us show that

$$|H^* \cap E(2w-t)| \leq \binom{2w-t}{w+1}. \tag{2.4}$$

Note that it suffices to show (2.4) for $n = 2w+1$. This is clear because for $n > 2w+1$ we get new forbidden weights in H^* besides those arising for $n = 2w+1$. In this case the forbidden weights in H^* are $w, w-1, \ldots, w-t-1$. Now (2.4) follows in view of Corollary 1. Consequently

$$|H \cap E(n)| \leq \binom{2w-t}{w+1} 2^{n-2w+t} < \binom{2w+1}{w+1} 2^{n-2w-1}.$$

This completes the proof of the claim and consequently of the case $n \geq 2w+1$. ■

Case $n = 2w$. If $\ell = n$ then we are done by Lemma 1, therefore let $1 \leq \ell \leq n-1$. In this case we note that

$$|H \cap E(n)| \leq 2F(2w-1, w) = 2F(2w-1, w-1).$$

This gives the desired result since we already proved that

$$F(2w-1, w-1) = \binom{2w-1}{w-1}.$$ ■

An auxiliary result for Theorem 2.

Let the ground set $[2w-t]$ be partitioned into $[1,k] \cup [k+1, 2w-t]$, $0 < k < 2w-t$.

Let $A_1 \subset A_2 \subset \cdots \subset A_m$ and $B_1 \subset B_2 \subset \cdots \subset B_r$ be any symmetric chains in $[1,k]$ and $[k+1, 2w-t]$, resp. By definition of a symmetric chain

$$\begin{aligned} |A_1| &= \frac{k-m+1}{2}, \ldots, |A_m| = \frac{k+m-1}{2}, \\ |B_1| &= \frac{2w-t-k-r+1}{2}, \ldots, |B_r| = \frac{2w-t-k+r-1}{2}. \end{aligned} \tag{2.5}$$

Consider the "product" of these chains, defined as $S = \{A_i \cup B_j : i = 1, \ldots, m;\ j = 1, \ldots, r\}$.

Let now $S' \subset S$ be a subset with the properties

(a) For any $(A_i \cup B_j) \in S'$

$$|A_i \cup B_j| \neq w, w-1, \ldots, w-t-1$$

(b) For any $(A_i \cup B_j)$, $(A_{i_1} \cup B_{j_1}) \in S'$

$$A_i \subseteq A_{i_1} \Rightarrow B_j \not\supset B_{j_1}.$$

Define also $S_{w+1} = \{A_i \cup B_j \in S : |A_i \cup B_j| = w+1\}$.
Then we have the following

LEMMA 2.

$$|S'| \leq |S_{w+1}|. \tag{2.6}$$

Proof. W.l.o.g. we may assume that $m \geq r$. It follows from the definitions of S' and S_{w+1} that $|S'| \leq r$ and $|S_{w+1}| \leq r$. We can also assume that $s \triangleq |S_{w+1}| \leq r-1$ for otherwise (2.6) trivially holds.

Next consider two cases:

Case (i): $s > 0$. For $i = 1, \dots, m$; $j = 1, \dots, r$ by (2.5) we have

$$\frac{2w - t - m - r + 2}{2} \leq |A_i \cup B_j| \leq \frac{2w - t + m + r - 2}{2}. \tag{2.7}$$

In view of assumption $s \leq r-1$ with $m \geq r$ there exists a minimal integer $1 \leq \ell \leq r$ such that $|A_m| + |B_\ell| = w+1$. Then clearly we also have $|A_{m-i+1}| + |B_{\ell+i-1}| = w+1$; $i = 1, \dots, s$ and $\ell + s - 1 = r$. This implies that $|A_m| + |B_r| = \frac{2w-t+m+r-2}{2} = w+s$, or equivalently $t = m + r - 2s - 2$. Consequently by (2.7) we get $w + s - m - r + 2 \leq |A_i \cup B_j| \leq w + s$ and condition (a) gives

$$|A_i \cup B_j| \neq w, w-1, \dots, w-m-r+2s+1. \tag{2.8}$$

Hence if $(A_i \cup B_j) \in S'$ then $|A_i \cup B_j| \in I_1 \cup I_2$, where $I_1 = [w-m-r+s+2, w-m-r+2s]$, $I_2 = [w+1, w+s]$.

Partition now S' into two sets $S' = S'_1 \cup S'_2$ so that $S'_1 = \{(A_i \cup B_j) \in S' : |A_i \cup B_j| \in I_1\}$ and $S'_2 = S' \setminus S'_1$.

Note that condition (b) in particular says that S' is a chain with the restriction

$$\big||A_i \cup B_j| - |A_{i_1} \cup B_{j_1}|\big| \geq 2$$

for any two distinct members $A_i \cup B_j$ and $A_{i_1} \cup B_{j_1}$ of S'. Since $|I_1| = s-1$, $|I_2| = s$ we conclude that $|S'_1| \leq \left\lceil \frac{s-1}{2} \right\rceil$, $|S'_2| \leq \left\lceil \frac{s}{2} \right\rceil$, and whence $|S'| \leq \left\lceil \frac{s-1}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil = s$, thus proving the lemma for case (i).

Case (ii): $s = 0$. By (2.7) we have $\frac{2w-t+m+r-2}{2} \leq w$ or equivalently $t \geq m + r - 2$, which with (2.7) gives $w - t \leq A_i \cup B_j \leq w$.

Hence $S' = \varnothing$ by condition (a). ■

Proof of Theorem 2. W.l.o.g. we may rewrite equation (2.1) in the form

$$\sum_{i=1}^{k} a_i x_i - \sum_{j=k+1}^{2w-t} a_j x_j = b \tag{2.9}$$

where $a_i > 0$, $i = 1, \dots, 2w - t$ and $1 \leq k \leq 2w - t$.

Let now $\mathbf{u}, \mathbf{v} \in X$ be two distinct solutions of equation (2.1). Let also $(A_1 \cup B_1)$, $(A_2 \cup B_2) \subset [1, 2w - t]$ be the sets corresponding to $\mathbf{u}$ and $\mathbf{v}$ resp. (i.e. $\mathbf{u}$ and $\mathbf{v}$ are the incident vectors of these sets), where $A_1, A_2 \subset [1, k]$, $B_1, B_2 \subset [k+1, 2w-t]$. It is clear that $(A_1 \cup B_1)$ and $(A_2 \cup B_2)$ satisfy both conditions (a) and (b) in Lemma 2.

Consider now symmetric chain decompositions of $2^{[k]}$ and $2^{[k+1,2w-t]}$.

For every pair of symmetric chains $\mathscr{C}_1 \subset 2^{[k]}$, $\mathscr{C}_2 \subset 2^{[k+1,2w-t]}$ consider their "product" defined in the proof of Lemma 2. To conclude the proof we note that Lemma 2 implies that the number of (0,1)-solutions $|X|$ to equation (2.1) does not exceed the number of (0,1)-vectors of weight $w+1$. ∎

Remark 2. Note that Theorem 2 is not true if one allows vectors of weight $w-t-1$ as solutions of (2.1). This can be shown by taking the hyperplane defined by the equation

$$(t+1)x_1 - \sum_{i=2}^{2w-t} x_i = -w+t+1. \tag{2.10}$$

Indeed the (0,1)-solutions of (2.10) are $X = (\{1\} \times E(2w-t-1, w)) \cup (\{0\} \times E(2w-t-1, w-t-1))$, i.e. X contains only vectors of weights $w+1$ and $w-t-1$. Furthermore

$$|X| = \binom{2w-t-1}{w} + \binom{2w-t-1}{w-t-1} = 2\binom{2w-t-1}{w} > \binom{2w-t}{w+1}.$$

3. Forbidden Weights in Subspaces

Let V be a proper subspace of $\mathbb{R}^n$. Define

$$FS(n, w) = \max\{|V \cap E(n)| : V \cap E(n, w) = \varnothing\}.$$

We note that there is an essential difference between the functions $F(n, w)$ and $FS(n, w)$.

Clearly $F(n, w) \geq FS(n, w)$. However small examples show that $F(n, w)$ can be much bigger and optimal sets for these two problems have different structures. Note also that in general $FS(n, w) \neq FS(n, n-w)$ in contrast to $F(n, w) = F(n, n-w)$. For instance (by Theorem 1) we have $F(5, 1) = F(5, 4) = 12$, while (by the theorems below) we have $FS(5, 1) = 5$ and $FS(5, 4) = 8$.

For $FS(n, w)$ we have only partial results.

Remark 3. Note that the "restricted case" of this problem was considered in [3]. Namely the problem of determination of

$$FS(n, w, m) \triangleq \max\{|V \cap E(n, m)| : V \cap E(n, w) = \varnothing\}.$$

This problem was solved in [3] for all parameters $1 \leq m, w \leq n$ and $n > n_0(m, n)$. In the following we essentially use the following result

LEMMA 3 [4]. *Let* $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ *and* $|a_i| \neq |a_j|$ *for some* $i, j \in [1, n]$. *Let* X *be the* (0,1)-*solutions of the equation*

$$\sum_{i=1}^{n} a_i x_i = b.$$

Then

$$|X| \leq \begin{cases} 2\binom{n-1}{\frac{n-3}{2}}, & \text{if } 2 \nmid n \\ \binom{n}{\frac{n-2}{2}}, & \text{if } 2 \mid n. \end{cases} \tag{3.1}$$

The next observation is rather simple.

THEOREM 3.
(i) $FS(n, n) = 2^{n-1}$,

(ii) $FS(n, 1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$,

(iii) $FS(n, 3) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, *for* $n \geq 4$.

Proof. The case (i) is obvious. Suppose V is a subspace which does not contain a unit vector. W.l.o.g. we may assume that $\dim(V) = n - 1$. This is clear because otherwise we can embed V in an $(n-1)$-dimensional subspace V' such that $V \cap E(n) = V' \cap E(n)$.

Thus let V be defined by the set of solutions $(x_1, \ldots, x_n) \in \mathbb{R}^n$ of

$$\sum_{i=0}^{n} a_i x_i = 0. \tag{3.2}$$

Clearly $a_i \neq 0$; $i = 1, \ldots, n$, since otherwise we would have a unit vector satisfying (3.2).

But in this case by Lemma 1 the number of (0,1)-solutions of (3.2) is upper bounded by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

The case (iii) is also simple. Note that in this case we have not more than two zero coefficients in (3.2). Suppose first $a_1, \ldots, a_{n-1} \neq 0$, $a_n = 0$. Then $|V \cap E(n)| = 2|Y|$, where Y is the set of (0,1)-solutions of $\sum_{i=1}^{n-1} a_i x_i = 0$. Clearly $Y \cap E(n-1, 2) = \varnothing$ and this implies that for some $i, j \in [1, n-1]$ we have $a_i \neq a_j$. Applying now Lemma 3 we get

$$2|Y| \leq \begin{cases} 4\binom{n-2}{\frac{n-4}{2}}, & \text{if } 2 \mid n \\ 2\binom{n-1}{\frac{n-3}{2}}, & \text{if } 2 \nmid n. \end{cases} \tag{3.3}$$

In both cases RHS of (3.3) $< \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

By the same argument one can exclude the case with two zero coefficients. This together with Lemma 1 implies that $|V \cap E(n)| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

On the other hand this bound (for both cases (ii) and (iii)) can be achieved by taking $a_1 = \cdots = a_{\lfloor \frac{n}{2} \rfloor} = -1$, $a_{\lfloor \frac{n}{2} \rfloor + 1} = \cdots = a_n = 1$. Moreover Lemma 3 implies that the optimal subspace is unique up to the permutations of the coordinates. ■

The case $w = n-1$ requires more work.

THEOREM 4.

$$FS(n, n-1) = \begin{cases} 2^{n-2}, & \text{if } n \ge 9 \text{ or } n = 3, 5, 7 \\ \binom{n}{\lfloor \frac{n}{2} \rfloor}, & \text{otherwise.} \end{cases}$$

Proof. Let $n \ge 9$. Suppose an "optimal" space V is defined by (3.2) where $a_1, \ldots, a_\ell \neq 0$ and $a_{\ell+1} = \cdots = a_n = 0$. Then the number of (0,1)-solutions of (3.2) is bounded by $2^{n-\ell} \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor} < 2^{n-2}$ whenever $9 \le \ell \le n$. Thus it remains only to consider the case $\ell \le 8$.

Case $\ell = 8$. Suppose $|a_i| \neq |a_j|$ for some $i, j \in [1, 8]$. Then by Lemma 3 for the (0,1)-solutions X of (3.2) we have $|X| \le \binom{8}{3} 2^{n-8} < 2^{n-2}$.

Suppose now $|a_1| = \cdots = |a_8|$. Denote by ℓ_1 the number of positive a_i's. Observe that $\ell_1 \neq 4$, because otherwise we would have $\sum_{i=1}^{8} a_i = 0$ and consequently a solution of (3.2) of weight $n-1$. On the other hand if $\ell_1 < 4$ then $|X| \le \binom{8}{\ell_1} 2^{n-8} < 2^{n-2}$, and hence $\ell \neq 8$.

Similarly using Lemma 3 one can easily prove that $\ell \neq 7$ and 6.

Case $\ell = 5$. If $|a_i| \neq |a_j|$ for some i, j then $|X| \le 2\binom{4}{1} 2^{n-5} = 2^{n-2}$.

This bound can be achieved only with $a_1 = 2$, $a_2 = 1$, $a_3 = a_4 = a_5 = -1$ (up to permutations). But in this case $x_1 = \cdots = x_{n-1} = 1$, $x_n = 0$ is a solution to (3.2), a contradiction.

If now $|a_1| = \cdots = |a_5|$ then clearly $\ell_1 \neq 2, 3$ and therefore $|X| \le \binom{5}{1} \cdot 2^{n-5} < 2^{n-2}$. Hence $\ell \neq 5$.

Case $\ell = 4$. If $a_i \neq a_j$ for some i, j then $|X| \le \binom{4}{1} 2^{n-4} = 2^{n-2}$.

The only configuration achieving this bound is $a_1 = 2$, $a_2 = a_3 = a_4 = -1$. But in this case we will have a solution of weight $n-1$, a contradiction.

Let now $|a_1| = \cdots = |a_4|$. Then clearly $\ell_1 \le 1$. Taking $a_1 = 1$, $a_2 = a_3 = a_4 = -1$, we get $|X| = 4 \cdot 2^{n-4} = 2^{n-2}$. Moreover X does not contain a vector of weight $n-1$.

Thus in the case $\ell = 4$ we can achieve the claimed upper bound in Theorem 4.

Case $\ell = 3$ is impossible and this can be easily verified.

Case $\ell = 2$. We have $|X| \le 2^{n-2}$ and this bound can be achieved only by taking $a_1 \neq a_2$.

Let now $n \le 8$.

Case $n = 8$. It follows from the observations above that if $a_i = 0$ for some $i \in [1, 8]$ then $|X| \le 2^6$. On the other hand if $a_i \neq 0$, $i = 1, \ldots, 8$ then $|X| \le \binom{8}{4} > 2^6$ and this bound can be achieved by taking $a_1 = \cdots = a_4 = 1$, $a_5 = \cdots = a_8 = -1$. Thus

$$FS(8,7) = \binom{8}{4}.$$

Similarly one can easily show that

$$FS(6,5) = \binom{6}{3}, \quad FS(4,3) = \binom{4}{2}, \quad FS(2,1) = \binom{2}{1}.$$

Case $n = 7$. If $a_i = 0$ for some $i \in [1,7]$, then again by the observations above we get $|X| \leq 2^5$ and this bound can be achieved in two different ways

(a) $a_1 = 1$, $a_2 = a_3 = a_4 = -1$, $a_5 = a_6 = a_7 = 0$,

(b) a_1, $a_2 \neq 0$, $a_1 \neq a_2$, $a_3 = \cdots = a_7 = 0$.

On the other hand if $a_i \neq 0$; $i = 1, \ldots, 7$ then $|a_1| = \cdots = |a_7|$, because otherwise $|X| \leq 2\binom{6}{2} < 2^5$. But in this case $\ell_1 \leq 2$, avoiding weight 6 and therefore again $|X| \leq \binom{7}{2} < 2^5$. Hence $FS(7,6) = 2^5$.

Case $n = 5$. If $a_i = 0$ for some $i \in [1,5]$ then we know that $|X| \leq 2^3$. This bound can be achieved in two different ways:

(a) a_1, $a_2 \neq 0$, $a_1 \neq a_2$, $a_3 = a_4 = a_5 = 0$,

(b) $a_1 = 1$, $a_2 = a_3 = a_4 = -1$, $a_5 = 0$.

If $a_i \neq 0$; $i = 1, \ldots, 5$, then for some $i, j \in [1,5] |a_i| \neq |a_j|$. Hence in view of Lemma 3 we have $|X| \leq 2\binom{4}{1} = 8$, and this bound can be achieved with

(c) $a_1 = 2$, $a_2 = 1$, $a_3 = a_4 = a_5 = -1$.

Hence $FS(5,4) = 2^3$.

Case $n = 3$. We have $FS(3,2) = 2$ and the bound can be achieved in two different ways:

(a) $a_1 \neq a_2$; $a_1, a_2 \neq 0$, $a_3 = 0$,

(b) $a_1 = 2$, $a_2 = a_3 = -1$.

This completes the proof of Theorem 2. ■

Remark 4. Note that we have described all nonequivalent configurations attaining the bound. Indeed we have proved that for $n \geq 9$ or $n = 7, 3$ there are only two optimal nonequivalent configurations. For $n = 8, 6, 4, 2$ the optimal configurations are unique up to permutations of coordinates. For $n = 5$ there are three nonequivalent optimal configurations.

What can we say about other values for w? The simplest unsolved cases are $w=2$ and $w=n-2$. For these cases we have the following conjectures.

Conjecture 1. For $n=3\ell+r,\ 0\le r\le 2$

$$FS(n,2)=2\sum_{i=0}^{\ell}\binom{\ell}{i}\binom{2\ell+r-1}{2i}.$$

The corresponding $(n-1)$-dimensional subspace is defined by

$$2\sum_{i=1}^{\ell}x_i-\sum_{j=\ell+1}^{n-\ell-1}x_j=0.$$

Conjecture 2. For $n\ge 6$

$$FS(n,n-2)=11\cdot 2^{n-6}.$$

The corresponding subspace is defined by

$$2x_1-x_2-x_3-x_4-x_5-x_6=0.$$

The next partial result directly follows from Theorem 1 and the simple fact that $FS(n,w)\ge\binom{n}{\lfloor\frac{n}{2}\rfloor}$, if $2\nmid w$.

Proposition 1. For $n=2w,\ 2w\pm 1,\ 2w\pm 2$ and $2\nmid w$ we have

$$FS(n,w)=\binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

Note however that we do not know the answer if w is even. In general we have the following

Conjecture 3. For $2\nmid w$ and $n\ge 2w$

$$FS(n,w)=\binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

4. Forbidden Weights of Different Parity

Let $H\subset\mathbb{R}^n$ be a hyperplane which contains (0,1)-vectors of only even or only odd weight. How big can $|H\cap E(n)|$ be? The next result gives a complete answer to this question. Define

$$F(n,\varepsilon \bmod 2)=\max\left\{|H\cap E(n)|:\forall(x_1,\dots,x_n)\in\big(H\cap E(n)\big)\sum_{i=1}^{n}x_i\not\equiv\varepsilon \bmod 2\right\},\quad \varepsilon\in\{0,1\}.$$

THEOREM 5.

(i) $F(n, \varepsilon \ mod\ 2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$,

(ii) *All optimal hyperplanes, up to the permutations of the coordinates, are those that are defined by*

$$-\sum_{i=1}^{\ell} x_i + \sum_{j=\ell+1}^{n} x_j = \lambda, \tag{4.1}$$

where $\lambda = \lfloor \frac{n}{2} \rfloor - \ell$ *or* $\lceil \frac{n}{2} \rceil - \ell$, $0 \le \ell \le \lfloor \frac{n}{2} \rfloor$.

Proof. Consider the case where all (0,1)-vectors in a hyperplane H have even weights and let H be defined by

$$\sum_{i=1}^{n} a_i x_i = \lambda. \tag{4.2}$$

Clearly $a_i \neq 0$ $(i = 1, \ldots, n)$ because otherwise we would have an "odd" vector. This in view of Lemma 1 implies

$$|H \cap E(n)| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{4.3}$$

Let now X be the (0,1)-solutions of equation (4.1), i.e. $X \triangleq (E(n) \cap H)$. Observe first that $|X| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Note also that for any other value of λ we have $|X| < \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Moreover all vectors of X have the same parity, namely for every $(x_1, \ldots, x_n) \in X$ one has

$$\sum_{i=1}^{n} x_i \equiv \lambda \mod 2.$$

To complete the proof we apply Lemma 3 which in particular says that if $|a_i| \neq |a_j|$ (in (4.2)) for some $i, j \in [1, n]$ then we have strict inequality in (4.3).

The proof of the "odd" case is identical. ∎

Consider now the same problem in the case where H is a subspace of $\mathbb{R}^n$. Denote the corresponding function by $FS(n, \varepsilon \mod 2)$. Clearly

$$FS(n, \varepsilon \mod 2) \le F(n, \varepsilon \mod 2).$$

Moreover taking the hyperplane defined by

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} x_i - \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} x_j = 0,$$

we get $FS(n, 1 \mod 2) = F(n, 1 \mod 2) = \binom{n}{\lfloor n/2 \rfloor}$.

The "even" case is more complicated.

THEOREM 6.

$$FS(n, 0 \bmod 2) = \begin{cases} \binom{n-1}{\frac{n-1}{2}-1}, & \text{if } n \equiv 3 \bmod 4 \\ \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}, & \text{otherwise.} \end{cases}$$

To prove the theorem we need the following result from [5]. Let H be the hyperplane defined by the equation

$$\sum_{i=1}^{n} a_i x_i = 0 \tag{4.4}$$

and suppose also (w.l.o.g.) that $a_1, \dots, a_\ell > 0,\ a_{\ell+1}, \dots, a_n \le 0,\ 1 \le \ell \le n-1$.

THEOREM 7 [5]. *Let $\mathscr{A} \subset \big(E(n) \cap H\big)$ be an antichain. Then*

$$|\mathscr{A}| \le \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor} \le \max_{1 \le \ell \le n} \binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor} = \begin{cases} 2\binom{n-2}{\frac{n-2}{2}}, & \text{if } 2 \mid n \\ \binom{n-1}{\frac{n-1}{2}}, & \text{if } 2 \nmid n. \end{cases} \tag{4.5}$$

We will also use the following fact which can be easily verified.

PROPOSITION 2. *For integers $3 \le \ell \le \frac{n}{2}$ we have*
(a)

$$\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor} < \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} \tag{4.6}$$

except for the case $\ell = 7,\ n = 8$.
(b) if $n = 4k+3$ then

$$\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}\binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor} < \binom{n-1}{\frac{n-1}{2}-1} \tag{4.7}$$

except for cases $\ell = 3,\ n = 4$ and $\ell = 3$ or $4,\ n = 11$ (for this case we have equality in (4.7)).

Proof of Theorem 6. Let X be the set of (0,1)-solutions of (4.4), i.e. $X \triangleq \big(E(n) \cap H\big)$ and X does not contain "even" vectors. Clearly $a_i \ne 0,\ i = 1, \dots, n$ and w.l.o.g. we may assume that $a_1, \dots, a_\ell > 0,\ a_{\ell+1}, \dots, a_n < 0,\ 1 \le \ell \le \lfloor \frac{n}{2} \rfloor$. Let $n = 4k+r,\ 0 \le r \le 3$.

Observe first that the bound (4.3) can be achieved for the hyperplane H defined by

$$2kx_1 - \sum_{i=2}^{n} x_i = 0. \tag{4.8}$$

Then clearly $X=\{1\}\times E(n-1,2k)$. Note further that two different vectors $\mathbf{u},\mathbf{v}\in X$ form an antichain. This is clear because otherwise either $(\mathbf{u}-\mathbf{v})$ or $(\mathbf{v}-\mathbf{u})\in X$ has even weight, a contradiction.

Therefore by Theorem 7 we have

$$|X|\le\binom{\ell}{\lfloor\frac{\ell}{2}\rfloor}\binom{m-\ell}{\lfloor\frac{n-\ell}{2}\rfloor}. \tag{4.9}$$

In view of (4.5), (4.6) and (4.7) we infer that the main values for ℓ we have to consider are $\ell=1$ or $\ell=2$. Moreover observe that if $\ell=1$ then we are done. This is obvious for the cases $n=4k$, $4k+1$, $4k+2$. If $n=4k+3$ then $X=\{1\}\times X'$ where $X'\subset E(4k+2)$, $X'\cap E(4k+2,2k+1)=\varnothing$ and X' is an antichain. It is not hard to prove that under these conditions one has $|X'|\le\binom{4k+2}{2k}$ (and we leave it to the reader).

Case $n=4k+1$. Combining (4.5) with (4.6) we get

$$|X|\le\binom{n}{\frac{n-1}{2}}.$$

Case $4k+2$. Consider symmetric chain decompositions of power sets $2^{[2]}$ and $2^{[3,n]}$. This corresponds to the symmetric chain decompositions of $E(2)$ and $E(n-2)$. In $E(2)$ we have two symmetric chains $\mathscr{C}_1=\{(0,0),(0,1),(1,1)\}$ and $\mathscr{C}_2=\{(1,0)\}$.

For each symmetric chain $B=\{\mathbf{b}_1,\dots,\mathbf{b}_r\}\subset E(n-2)$ consider the "product chains" $\mathscr{C}_1\times B$ and $\mathscr{C}_2\times B$ (defined before), that is $\mathscr{C}_i\times B\{(\mathbf{c},\mathbf{b}):\mathbf{c}\in\mathscr{C}_i,\mathbf{b}\in B\}$, $i=1,2$ suppose first that $a_1\ne a_2$ (in 4.4). This with the antichain condition implies that X contains at most one vector from the products $\mathscr{C}_1\times B$ and $\mathscr{C}_2\times B$, for each symmetric chain $B\subset E(n-2)$.

Note also that in the symmetric chain decomposition of $E(n-2)$ (corresponding to $2^{[3,n]}$) we have $\binom{4k}{2k}-\binom{4k}{2k-1}$ "singles", that is chains of size one (and hence of weights $2k$). Since X contains only "odd" vectors these singles can be combined only with (0,1) or (1,0) in $\mathscr{C}_1$ and $\mathscr{C}_2$. The number of product chains is $2\binom{4k}{2k}$ therefore we can estimate

$$|X|\le 2\binom{4k}{2k}-\left(\binom{4k}{2k}-\binom{4k}{2k-1}\right)=\binom{4k+1}{2k}.$$

In fact one can show that $|X|<\binom{4k+1}{2k}$.

Suppose now $a_1=a_2$.

Note that in this case if $(1,0,x_3,\dots,x_2)\in X$ then $(1,0,1-x_3,\dots,1-x_n)\notin X$ since otherwise $(0,1,1-x_3,\dots,1-x_n)$ and consequently $(1,1,\dots,1)\in X$, a contradiction with the vector being "even".

Let us define $X'=\{(x_1,\dots,x_n)\in X: x_1+x_2=1\}$ and $X''=\{(x_1,\dots,x_n)\in X: x_1=x_2=1\}$. Clearly

$$X=X'\cup X'' \text{ and } |X''|\le\binom{4k}{2k-1}. \tag{4.10}$$

Suppose further $|X'| \le \binom{4k}{2k} - \binom{4k}{2k-1}$. Then with (4.10) we get

$$|X| = |X'| + |X''| \le \binom{4k}{2k}.$$

If conversely $|X'| > \binom{4k}{2k} - \binom{4k}{2k-1}$ then by observation above at least $|X'|$ product chains have not elements from X. Hence

$$|X| \le 2\binom{4k}{2k} - |X'| < \binom{4k+1}{2k}.$$

Case $n = 4k$. As above we consider all product chains $\mathscr{C}_1 \times B$, $\mathscr{C}_1 \times B$ where B is a chain from a symmetric chain decomposition of $E(4k-2)$.

The number of singles in a symmetric chain decomposition of $E(4k-2)$ is $\binom{4k-2}{2k-1} - \binom{4k-2}{2k-2}$ and these singles cannot be combined with $(1,1) \in \mathscr{C}_1$. Therefore

$$|X| \le 2\binom{4k-2}{2k-1} - \left(\binom{4k-2}{2k-1} - \binom{4k-2}{2k-2}\right) \le \binom{4k-1}{2k-1}.$$

The same argument can be used to analyse the case $\ell = 4$, $n = 8$.

Case $n = 4k+3$. We proceed as before. In a symmetric chain decomposition of $E(4k+1)$ (corresponding to $2^{[3,4k+1]}$) we have $m \triangleq \binom{4k+1}{2k} - \binom{4k}{2k-1}$ chains of size two, i.e. chains consisting of two vectors of weight $2k$ and $2k+1$.

Suppose $a_1 \ne a_2$. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\} \subset E(4k+1)$ be a symmetric chain where $\mathbf{b}_1$ and $\mathbf{b}_2$ have weights $2k$ and $2k+1$ resp. Then note that X contains at most one vector from the vectors $(1,0,\mathbf{b}_1)$ $(1,1,\mathbf{b}_2)$, $(0,1,\mathbf{b}_1)$. This implies that at least m product chains have not common vectors with X. Thus we get

$$|X| \le 2\binom{4k+1}{2k} - \left(\binom{4k+1}{2k} - \binom{4k+1}{2k-1}\right) = \binom{4k+2}{2k}.$$

Suppose now $a_1 = a_2$. Define three new sets

$X_{10} = \{\mathbf{b} \in E(4k+1) : (1,0,\mathbf{b}) \in X\}$, $X_{01} = \{\mathbf{b} \in E(4k+1) : (0,1,\mathbf{b}) \in X\}$, $X_{11} = \{\mathbf{b} \in E(4k+1) : (1,1,\mathbf{b}) \in X\}$.

Clearly $X_{10} = X_{01}$, $X_{10} \cap X_{11} = \varnothing$, $X_{10} \cap E(4k+1, 2k-1) = \varnothing$, and

$$|X| = |X_{10}| + |X_{01} \cup X_{11}|. \tag{4.11}$$

Claim.

$$|X_{10}| \le \binom{4k+1}{2k+2}. \tag{4.12}$$

Proof. First note that any two elements $\mathbf{u}, \mathbf{v} \in X_{10}$ are intersecting, since otherwise $(1,0,\mathbf{u}), (0,1,\mathbf{v}) \in X$ and consequently the even vector $(1,1,\mathbf{v}+\mathbf{u}) \in X$, a contradiction. Thus X is an intersecting antichain. We use now the approach which was used in Sperner's original proof of his theorem. The idea is as follows (see for details [6] or [7]).

Let $W_i = X_{10} \cap E(4k+1, i)$ be the vectors of minimal weight i and let $1 \le i \le 2k-1$. We replace then W_i by the set of all vectors $W'_{i+1} \subset E(4k+1, i+1)$ which "cover" (contain in the language of sets) the vectors of W_i.

One can easily see that $(X \setminus W_i) \cup W'_{i+1}$ is again an intersecting antichain. Moreover it can be shown that $|W'_{i+1}| \ge |W_i|$.

The described transformation can be iteratively applied to all levels of weight less than $2k$. The same procedure we apply to the set of vectors $W_j \subset X_{10}$ of maximum weight $2k+2 < j \le n$, replacing W_j by the set of all vectors $W'_{j-1} \subset E(4k+1, j-1)$ which are covered by vectors of W_j. In other words we replace W_j by its 1-shadow. It can be shown again that this transformation does not decrease the size of the family. Thus X_{10} can be brought to an intersecting antichain X^* with $|X^*| \ge |X_{10}|$ such that $X^* = W_{2k} \cup W_{2k+2}$ consists only of vectors W_{2k} of weight $2k$ and vectors W_{2k+2} of weight $2k+2$. In view of Theorem EKR we have $|W_{2k}| \le \binom{4k}{2k-1}$. If now $|W_{2k+2}| \le \binom{4k}{2k+2}$ then we are done since

$$|X^*| = |W_{2k}| + |W_{2k+2}| \le \binom{4k}{2k-1} + \binom{4k}{2k+2} = \binom{4k+1}{2k+2}.$$

Therefore assume

$$|W_{2k+2}| = \binom{4k}{2k+2} + s, s \ge 1. \tag{4.13}$$

Since X^* is an antichain we can write

$$|W_{2k}| \le \binom{4k+1}{2k} - |\partial_2 W_{2k+2}|.$$

Further using Theorem KK we get the estimation

$$|\partial_2 W_{2k+2}| \ge \binom{4k}{2k} + s.$$

Hence

$$|W_{2k}| \le \binom{4k+1}{2k} - \binom{4k}{2k} - s = \binom{4k}{2k+1} - s$$

which with (4.13) gives

$$|X^*| \le \binom{4k}{2k+2} + \binom{4k}{2k+1} = \binom{4k+1}{2k+2}.$$

■

Note now that $X_{10} \cup X_{11}$ is an antichain and therefore

$$|X_{10} \cup X_{11}| \le \binom{4k+1}{2k+1}.$$

Hence by (4.11) and (4.12)

$$|X| \leq \binom{4k+1}{2k+1} + \binom{4k+1}{2k+2} \leq \binom{4k+2}{2k}.$$

To complete the proof of the theorem, it remains to treat the case $n = 7$, $\ell = 3$. This can be easily done using a similar approach. ■

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