Appendix: Solution of Burnashev's Problem and a Sharpening of the Erdős/Ko/Rado Theorem

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Motivated by a coding problem for Gaussian channels, Burnashev came to the following *Geometric Problem* (which he stated at the Information Theory Meeting in Oberwolfach, Germany, April 1982). For every $\delta > 0$, does there exist a constant $\lambda(\delta) > 0$ such that the following is true: "Every finite set $\{x_1, \dots, x_N\}$ in a Hilbert space H has a subset $\{x_{i_1}, \dots, x_{i_M}\}$, $M \geq \lambda(\delta)N$, without 'bad' triangles. (A triangle is *bad*, if one side is longer than $1 + \delta$ and the two others are shorter ($\leq$) than 1)"?

This is the case for Euclidean spaces. (A good exercise before the further reading!) We show that this is *not* so for infinite–dimensional Hilbert spaces. The proof is based on a sharpening of the famous Erdős–Ko–Rado Theorem and was given at the same meeting.

The publication of this note from 1982 was originally planned in a forthcoming book on Combinatorics by G. Katona. Since the completion of this book is still unclear and on the other hand the method of generated sets of [1] and the method of pushing and pulling of [2] are now available there is realistic hope that this direction of work with its open problems can now be continued. Therefore it should be made known and the late publication is justified.

The solution was found by a funny chance event: Burnashev pronounced the name "Hilbert" in the Russian way like "Gilbert", which gave us the inspiration to view the problem in a sequence space.

Let h be the Hamming distance. Define

$$G_k^n \triangleq \left(\left\{ (a_1, a_2, a_3, \dots) : a_t \in \left\{ 0, \frac{1}{\sqrt{2}} \right\}, 1 \leq t \leq n;\ a_t = 0, t > n;\ \sum_{t=1}^{n} a_t = \frac{k}{\sqrt{2}} \right\}, h \right),$$

Obviously, for $1 \leq k \leq n \quad G_k^n \subset H = \ell_2$ and for $a^n, b^n \in G_k^n$

$$h(a^n, b^n) \leq 2 \Leftrightarrow \|a^n - b^n\|_2 \leq 1. \tag{1}$$

We call $X \subset G_k^n$ **good**, if it contains no bad triangle. It suffices to show that for some k

$$g_k(n) \triangleq \max_{X \subset G_k^n,\ \text{good}} |X| = o\left(\binom{n}{k} \right). \tag{2}$$

Using the representation of subsets of an n–set as $(0-1)$–incidence vectors the determination of $q_2(n)$ leads to an extremal problem of independent interest, whose solution provides in all but one case an amazing sharpening of the well–known Erdős/Ko/Rado Theorem.

This says that for any family $\mathcal{B} \subset \mathcal{P}_\ell(\{1, \dots, n\})$ of all ℓ–element subsets of an n–set with the

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Intersection Property: $B \cap B' \neq \varnothing \qquad \forall B, B' \in \mathcal{B}$
necessarily

$$|\mathcal{B}| \leq \binom{n-1}{\ell-1}, \text{ if } n \geq 2\ell. \tag{3}$$

Our result is the

Theorem. *Let $n \geq 2\ell$, $\ell \geq 2$. For any $\mathcal{A} \subset \mathcal{P}_\ell\big((1,2,\dots,n)\big)$ with the*

Triangle Property: $\forall A, B, C \in \mathcal{A} : A \cap B \neq \varnothing,\ B \cap C \neq \varnothing \Rightarrow A \cap C \neq \varnothing$
we have

$$|\mathcal{A}| \leq \begin{cases} n & \text{if } \ell = 2 \text{ and } n \equiv 0 \mod 3 \\ \binom{n-1}{\ell-1} & \text{otherwise.} \end{cases} \tag{4}$$

Moreover, this bound is best possible.

Proof: The Triangle Property implies that $\mathcal{A}$ can be partitioned into families $\mathcal{A}(1), \dots, \mathcal{A}(T)$ such that

(a) The families $\mathcal{A}(t)$, $1 \leq t \leq T$, have the Intersection Property.
(b) The sets $A(t) \triangleq \cup\big\{A : A \in \mathcal{A}(t)\big\}$, $1 \leq t \leq T$, are disjoint.
(c) The numbers $\alpha_t \triangleq |A(t)|$ satisfy $\ell \leq \alpha_t \leq n$ for $1 \leq t \leq T$.

This and (3) imply

$$|\mathcal{A}| = \sum_{1 \leq t \leq T} |\mathcal{A}(t)| \leq \sum_{t:\alpha_t < 2\ell} \binom{\alpha_t}{\ell} + \sum_{t:\alpha_t \geq 2\ell} \binom{\alpha_t - 1}{\ell - 1}. \tag{5}$$

Case 1: The second sum equals 0
By Pascal's identity for $q \geq p \geq \ell$

$$\binom{q}{\ell} + \binom{p}{\ell} = \binom{q}{\ell} + \binom{p-1}{\ell} + \binom{p-1}{\ell-1} \leq \binom{q}{\ell} + \binom{p-1}{\ell} + \binom{q}{\ell-1} = \binom{q+1}{\ell} + \binom{p-1}{\ell}$$

and therefore with (b)

$$|\mathcal{A}| \leq \sum_{t:\alpha_t < 2\ell} \binom{\alpha_t}{\ell} \leq \binom{2\ell-1}{\ell} \frac{n}{\lfloor 2\ell - 1 \rfloor} + \binom{(2\ell-1)x}{\ell}, \text{ where } x = \frac{n}{2\ell-1} - \frac{n}{\lfloor 2\ell-1 \rfloor}. \tag{1}$$

Case 2: The second sum does not equal 0
We show first that for $2\ell > \gamma \geq \ell$, $\beta \geq \ell$, $\gamma + \beta \leq n$

$$\binom{\gamma}{\ell} + \binom{\beta-1}{\ell-1} \leq \binom{\gamma+\beta-1}{\ell-1}. \tag{7}$$

Clearly,

$$\ell(\gamma+\beta-1) \cdot \dots \cdot (\gamma+\beta-\ell+1) \geq \ell(\beta-1) \cdot \dots \cdot (\beta-\ell+1) + \ell\gamma^{\ell-1}$$
$$\geq \ell(\beta-1) \cdot \dots \cdot (\beta-\ell+1) + \gamma(\gamma-1) \cdot \dots \cdot (\gamma-\ell+2)(\gamma-\ell+1),$$

since $\gamma - \ell + 1 \leq \ell$, and thus (7) follows.

Using (7) we can shift terms from the first sum to the second sum in (5) and obtain finally an upper bound of the form

$$\sum_{i\in I} \binom{\rho_i - 1}{\ell - 1}; \qquad \sum \rho_i \leq n, \rho_i \geq 2\ell,$$

which is obviously smaller than $\binom{n-1}{\ell-1}$.

Thus we have

$$|\mathcal{A}| \leq \max\left(\binom{n-1}{\ell-1}, \binom{2\ell-1}{\ell}\frac{n}{\lfloor 2\ell-1\rfloor} + \binom{(2\ell-1)x}{\ell}\right) \tag{8}$$

where $x = \frac{n}{2\ell-1} - \frac{n}{\lfloor 2\ell-1\rfloor}$.

For $\ell = 2$ thus $|\mathcal{A}| \leq \begin{cases} n-1 & \text{if } n \not\equiv 0 \mod 3 \\ n & \text{if } n \equiv 0 \mod 3 \end{cases}$.

In all other cases it suffices to show

$$\binom{n-1}{\ell-1} \geq \binom{2\ell-1}{\ell-1}\frac{n}{\lfloor 2\ell-1\rfloor} + \binom{(2\ell-1)x}{\ell}.$$

In case $n = 2\ell$ this is true, because

$$\frac{2\ell}{\lfloor 2\ell-1\rfloor} = 1 \text{ and } \binom{(2\ell-1)x}{\ell} = \binom{1}{\ell} = 0.$$

In case $n \geq 2\ell + 1$ we have

$$\binom{n-1}{\ell-1} \geq \binom{2\ell-1}{\ell-1}\frac{n}{2\ell-1}, \text{ because } (n-1)(n-\ell+1) = n(n-\ell)+\ell-1 \geq n(\ell+1) \tag{9}$$

and thus $(2\ell-1)(n-1)(n-2)\cdot\dots\cdot(n-\ell+1) \geq (2\ell-1)(2\ell-2)\cdot\dots\cdot(\ell+1)n$.

Since for $0 \leq x \leq 1$ $\quad x\binom{2\ell-1}{\ell} \geq \binom{(2\ell-1)x}{\ell}$ (9) yields the result.

Translation of the result for G_2^n yields

$$g_2(n) = \begin{cases} n & \text{for } n \equiv 0 \mod 3 \\ n-1 & \text{otherwise} \end{cases}.$$

Thus we have as

Corollary. (Negative answer to Burnashev's Question for *every* $\delta > 0$)

$$\lim_{n\to\infty} g_2(n) \cdot |G_2^n|^{-1} = \lim_{n\to\infty} \frac{2}{n-1} = 0.$$

Problems

1. Let $M(N)$ be the guaranteed cardinality of a largest good subset of an N–set in H. We have just shown that $M(N) \leq 0(\sqrt{N})$.
   What is the exact asymptotic growth of $M(N)$?
2. What is the best choice of $\lambda(\delta)$ for the n–dimensional Euclidean space?
3. Generalize the Theorem to families of sets with the property:

$$|A \cap B| \geq d,\ |B \cap C| \geq d \Rightarrow |A \cap C| \geq d.$$

References

1. R. Ahlswede and L.H. Khachatrian, The complete intersection theorem for systems of finite sets, Preprint 95–066, SFB 343 "Diskrete Strukturen in der Mathematik", European J. Combinatorics, 18, 125–136, 1997.
2. R. Ahlswede and L.H. Khachatrian, A pushing–pulling method: new proofs of intersection theorems, Preprint 97–043, SFB 343 "Diskrete Strukturen in der Mathematik", Universität Bielefeld, Combinatorica 19(1), 1–15, 1999.