Appendix: On Edge–Isoperimetric Theorems for Uniform Hypergraphs

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1 Introduction

Denote by $\Omega = \{1, \ldots, n\}$ an n–element set. For all $A, B \in \binom{\Omega}{k}$, the k–element subsets of Ω, define the relation $\sim$ as follows:
$A \sim B$ iff A and B have a common shadow, i.e. there is a $C \in \binom{\Omega}{k-1}$ with $C \subset A$ and $C \subset B$. For fixed integer α, our goal is to find a family $\mathcal{A}$ of k–subsets with size α, having as many as possible $\sim$ –relations for all pairs of its elements. For $k = 2$ this was achieved by Ahlswede and Katona [2] many years ago. However, it is surprisingly difficult for $k \geq 3$, in particular there is no complete solution even for $k = 3$. Perhaps, the reason is the complicated behaviour for "bad α" so that the most natural and reasonable conjecture, which will be described in the last section and was mentioned already in [2], is false. Actually, our problem can also be viewed as a kind of isoperimetric problem in the sense of Bollobás and Leader ([4], see also [6]). They gave two versions. Partition the vertex set V of a graph $G = (V, E)$ into 2 parts A and A^c such that for fixed α $|A| = \alpha$ and

I. The subgraph induced by A has maximal number of edges
   or
II. The number of edges connecting vertices from A and A^c is as small as possible.

When G is regular, the two versions are equivalent. In our case we define $G = (V, E)$ by $V = \binom{\Omega}{k}$ and $E = \big\{\{A, B\} \subset V\colon A \neq B \text{ and } A \sim B\big\}$. Thus the original problem is an edge–isoperimetric problem for a certain regular graph. In order to solve our problem, in Section 2 we reduce it to another kind of problem, which we call "sum of ranks problem": For a lattice with a rank function find a downset of given size with maximal sum of the ranks of its elements. Similar questions were studied in [3], [6], and [8]. In Section 3, we go over to a continuous version of the problem and solve it for $k = 3$ and "good α". Some of the auxiliary results and ideas there extend also to general k. A related but much simpler result concerning a moment problem is presented in Section 4.

2 From Edge–Isoperimetric to Sum of Ranks Problem

In this section we reduce the edge–isoperimetric problem to the sum of ranks problem. Denote by $\mathcal{L}(n, k) = (S_{n,k}, \leq)$ the lattice defined by

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$$S_{n,k} = \{(x_1, \dots, x_k) : 1 \le x_1 < x_2 \dots < x_k \le n, x_i \in \mathbb{Z}^+\}$$

and $(x_1, \dots, x_k) \le (x'_1, \dots, x'_k) \Leftrightarrow x_i \le x'_i (1 \le i \le k)$. For $x^k \in S_{n,k}$, the rank of x^k is defined as $|x^k| = \sum_{i=1}^k x_i$ and for $W \subset S_{n,k}$, let $||W|| = \sum_{x^k \in W} |x^k|$. In addition we let $A = \{x_1, \dots, x_k\} \in \binom{\Omega}{k}$, with elements labelled in increasing order, correspond to $x^k = \Phi(A) \triangleq (x_1, \dots, x_k) \in S_{n,k}$, and, similarly, $\mathcal{A} \subset \binom{\Omega}{k}$ to $\Phi(\mathcal{A}) = \{\Phi(A) : A \in \mathcal{A}\}$. Moreover, for $\mathcal{A} \subset \binom{\Omega}{k}$ we introduce

$$\mathcal{P}(\mathcal{A}) = \{(A, B) \in \mathcal{A}^2 : A \sim B\}.$$

Using for $A \in \mathcal{A}$ and $1 \le i < j \le n$ the following "pushing to the left" or so–called switching operator $O_{i,j}$, which is frequently employed in combinatorial extremal theory:

$$O_{i,j}(A) = \begin{matrix} (A \setminus \{j\}) \cup \{i\} \text{ if } & (A \setminus \{j\}) \cup \{i\} \notin \mathcal{A},\ j \in A, \text{ and } i \notin A \\ A & \text{otherwise,} \end{matrix}$$

one can prove, by standard arguments, that for fixed α an $\mathcal{A} \subset \binom{\Omega}{k}$ with $|\mathcal{A}| = \alpha$, which maximizes $|\ \mathcal{P}(\mathcal{A})\ |$, can be assumed to be within a family of subsets, which are invariant under the pushing to left operator. It is also easy to see that such subsets correspond to a downset in $\mathcal{L}(n, k)$.

Lemma 1. *For $\alpha \in \mathbb{Z}^+$ $\max_{|\mathcal{A}|=\alpha} |\ \mathcal{P}(\mathcal{A})\ |$ is assumed by an $\mathcal{A} \subset \binom{\Omega}{k}$ s.t. $\Phi(\mathcal{A})$ is a downset in $\mathcal{L}(n, k)$.*

Now we are ready to show the first of our main results.

Theorem 1. *For fixed $\alpha \in \mathbb{Z}^+$, maximizing $|\ \mathcal{P}(\mathcal{A})\ |$ for $\mathcal{A} \subset \binom{\Omega}{k}$, $|\mathcal{A}| = \alpha$, is equivalent to finding a downset W in $\mathcal{L}(n, k)$ with $|W| = \alpha$ and maximal $||W||$.*

Proof. Assume that $\mathcal{A} \subset \binom{\Omega}{k}$, $W = \Phi(\mathcal{A})$ is a downset in $\mathcal{L}(n, k)$, and $|\mathcal{A}| = \alpha$.

For every $x^k \in W$ there are exactly

$$(x_{i+1} - x_i - 1)\binom{k-i}{k-1-i} = (x_{i+1} - x_i - 1)(k - i) \tag{1.1}$$

y^k's with $y^k \le x^k$, whose first i components coincide with those of x^k and the $(i+1)$-st components differ, and for which A and B have a common shadow if $x^k = \Phi(A)$ and $y^k = \Phi(B)$. (Here $x_0 \triangleq 0$.) By (1.1), for $x^k = \Phi(A)$ fixed, there is a total of

$$\sum_{i=0}^{k-1} (x_{i+1} - x_i - 1)(k - i) = \sum_{i=1}^{k} (k - i + 1)x_i - \sum_{i=0}^{k-1} (k - i)x_i - \sum_{i=0}^{k-1} (k - i)$$

$$= \sum_{i=1}^{k} x_i - \binom{k+1}{2} = |x^k| - \binom{k+1}{2} \tag{1.2}$$

B's with $\Phi(B) = y^k \le x^k$, $B \sim A$, and with $\Phi(B) \in \mathcal{A}$, because $\Phi(\mathcal{A})$ is a downset. Consequently

$$| \mathcal{P}(\mathcal{A}) |= 2 \sum_{x^k \in W} |x^k| - 2\binom{k+1}{2}|\mathcal{A}| = 2||W|| - 2\alpha\binom{k+1}{2}. \tag{1.3}$$

Thus our theorem follows from Lemma 1 and (1.3).

From now on we study our problem in the "sum–rank" version.

3 From the Discrete to a Continuous Model

A natural idea to solve a discrete problem for "good parameters" is to study the related continuous problem. Every $z^k \in \mathbb{Z}^k$ we let correspond to a cube $C(z^k) \triangleq \{x^k : \lceil x_i \rceil = z_i\}$ in $\mathbb{R}^k$. This mapping sends our $S_{U,k}$ for $U \in \mathbb{Z}^+$ to $\overset{\sim}{\rightarrow} S_{U,k} \triangleq \{x^k : 0 < x_1 < x_2 \cdots < x_k \le U,\ \lceil x_i \rceil \ne \lceil x_j \rceil, \text{ if } i \ne j\}$. Thus, keeping the partial order "$\le$", we can "embed" our $\mathcal{L}(U,k)$ into a "continuous lattice" $\overset{\sim}{\rightarrow} \mathcal{L}(U,k) = (\overset{\sim}{\rightarrow} S_{U,k}, \le)$. Moreover, the image $\overset{\sim}{\rightarrow} W \triangleq \Phi(W)$ of a downset W in $\mathcal{L}(U,k)$ is a downset in $\overset{\sim}{\rightarrow} \mathcal{L}(U,k)$, with (finite) integer–components for maximal points. Let μ be the Lebesgue measure on $\mathbb{R}^{k'}$, and let $k' \le k$ be specified by the context. For $W \subset \mathbb{R}^k$, define

$$||W|| = \int_W |x^k| d\mu, \quad \text{where} \quad |x^k| = \sum_j x_j. \tag{3.1}$$

Let $\mathcal{D}$ be the set of downsets in $\overset{\sim}{\rightarrow} \mathcal{L}(U,k)$ with finitely many maximal points. Since it is of no consequence if we add or substract a set of measure zero, we will frequently exchange "$<$" (or "$>$") and "$\le$" (or "$\ge$") in the sequel. It is enough in our problem for "good α" to consider $\max_{\mu(\overset{\sim}{\rightarrow} W)=\alpha, \overset{\sim}{\rightarrow} W \in \mathcal{D}} ||W||$ in $\overset{\sim}{\rightarrow} \mathcal{L}(U,k)$, and the following lemma is the desired bridge.

Lemma 2. *Suppose that $\overset{\sim}{\rightarrow} W \in \mathcal{D}$ has only maximal points with integer components, and so for a $W \subset \mathcal{L}(U,k)$ $\overset{\sim}{\rightarrow} W = \Phi(W)$.*

Then

$$|| \overset{\sim}{\rightarrow} W|| = ||W|| - \frac{k}{2}\alpha, \quad \text{where} \quad \alpha = \mu(\overset{\sim}{\rightarrow} W). \tag{3.2}$$

Proof.

$$\begin{aligned} || \overset{\sim}{\rightarrow} W|| = \textstyle\sum_{z^k \in W} ||C(z^k)|| &= \textstyle\sum_{z^k \in W} \int_{C(z^k)} |x^k| \mu(dx^k) \\ &= \textstyle\sum_{z^k \in W} \int_{z_k-1}^{z_k} dx_k \dots \int_{z_1-1}^{z_1} dx_1 \sum_{j=1}^k x_j \\ &= \textstyle\sum_{z^k \in W} \sum_{i=1}^k \int_{z_i-1}^{z_i} x_i dx_i = \sum_{z^k \in W} \sum_{i=1}^k \frac{1}{2}(2z_i - 1) \end{aligned} \tag{3.3}$$

and (3.2) follows, because $|W| = \mu(\overset{\sim}{\rightarrow} W)$. We say that $W \in \mathcal{D}$ can be reduced to $W' \in \mathcal{D}$, if $\mu(W') = \mu(W)$ and $||W'|| \ge ||W||$.

4 Cones and Trapezoids

Next we define cones and trapezoids, which will play important role in our problem. A cone in $\overset{\sim}{\rightarrow} S_{U,k}$ is a set

$$K_k(u) = \{x^k \in R^k : 0 < x_1 < \cdots < x_k \le u \text{ and } \lceil x_i \rceil \ne \lceil x_j \rceil \text{ for } i \ne j\}, \text{ with } u \le U. \tag{4.1}$$

Clearly, $\overset{\sim}{\rightarrow} S_{U,k}$ is a cone itself. It can be denoted by $K_k(U)$. A trapezoid $R_k(v,u)$ in $K_k(U)$ is a downset below $(v, u \dots u)$, where $0 < v \le u \le U$, i.e.

$$R_k(v,u) \triangleq \{x^k \in \overset{\sim}{\rightarrow} S_{U,k} : x_1 \le v, x_k \le u\} \tag{4.2}$$

and therefore $K_k(u) = R_k(u,u)$. Moreover, for $W \subset K_k(u)$ set

$$\overline{W}^{(u)} \triangleq K_k(u) \setminus W \tag{4.3}$$

and

$$\hat{W}^{(u)} \triangleq \{(\lfloor u \rfloor, \dots, \lfloor u \rfloor) - x^k : x^k \in \overline{W}^{(u)}\}. \tag{4.4}$$

For integral u one can easily verify that

$$W = \hat{V}^{(u)} \quad \text{for} \quad V = \hat{W}^{(u)} \tag{4.5}$$

and

$$R_k(v,u) = \hat{K}_k^{(u)}(u - v). \tag{4.6}$$

Lemma 3. *For $W \in \mathcal{D}$ and $W \subset K_k(u)$, $u \le U$,*

$$||W|| = ||K_k(u)|| - k\lfloor u \rfloor \mu(\hat{W}^{(u)}) + ||\hat{W}^{(u)}||. \tag{4.7}$$

Proof. According to the definitions of "$\wedge(u)$" and "|| ||",

$$\begin{aligned} ||W|| = \textstyle\int_W |x^k| \mu(dx^k) &= \textstyle\int_{K_k(u) \setminus \overline{W}^{(u)}} |x^k| \mu(dx^k) \\ &= ||K_k(u)|| - \textstyle\int_{\overline{W}^{(u)}} |x^k| \mu(dx^k) \\ &= ||K_k(u)|| - \textstyle\int_{\hat{W}^{(u)}} \sum_{j=1}^k \big(\lfloor u \rfloor - x_j\big) \mu(dx^k) \\ &= ||K_k(u)|| - k\lfloor u \rfloor \mu\big(\hat{W}^{(u)}\big) + ||\hat{W}^{(u)}||. \end{aligned}$$

Notice that for $u \notin \mathbb{Z}^+$ $\hat{W}^{(u)}$ is not in $\mathcal{L}(u,k)$.

Corollary 1. *For $u \in \mathbb{Z}^+$*

$$||K_k(u)|| = \frac{ku}{2} \mu(K_k(u)). \tag{4.8}$$

Proof. One can verify (4.8) by standard techniques in calculus for evaluating integrals, however, Lemma 3 provides a very elegant and simple way.

By (4.7) for $W \subset K_k(u)$

$$||W|| - ||\hat{W}^{(u)}|| = ||K_k(u)|| - ku\ \mu\big(\hat{W}^{(u)}\big) \tag{4.9}$$

and by (4.5) and (4.7) one can exchange the roles of W and $\hat{W}$. Therefore we have

$$||\hat{W}^{(u)}|| - ||W|| = ||K_k(u)|| - ku\ \mu(W). \tag{4.10}$$

"Adding (4.9) and (4.10)" and using the fact $\mu(K_k(u)) = \mu(W) + \mu\big(\hat{W}^{(u)}\big)$, we obtain (4.8). Next we establish a connection between $||K_k(u)||$ and $\mu(K_k(u))$ for not necessarily integral u. It can elegantly be expressed in terms of densities. We define the density of $W \subset \mathbb{R}^{k'}$ ($k' \leq k$ defined by context) as

$$d_{k'}(W) = \frac{||W||}{\mu(W)} \quad \text{and set} \quad d = d_k. \tag{4.11}$$

Then Corollary 1 takes the form

$$d\big(K_k(u)\big) = \frac{k}{2}u,\ \ u \in \mathbb{Z}^+. \tag{4.12}$$

We extend this formula to general u.

Lemma 4. *For $u \leq U$ not necessarily integers, denote by $\theta \triangleq \{u\} = u - \lfloor u \rfloor$ the fractional part of u. Then*

(i) $\quad \mu\big(K_k(u)\big) = \binom{\lfloor u \rfloor}{k} + \theta\binom{\lfloor u \rfloor}{k-1}$,

(ii) $\quad ||K_k(u)|| = \frac{ku}{2}\mu\big(K_k(u)\big) + \frac{k-1}{2}\theta(1-\theta)\binom{\lfloor u \rfloor}{k-1}$

and therefore

(iii) $\quad d\big(K_k(u)\big) = \frac{ku}{2} + \dfrac{\frac{k-1}{2}\theta(1-\theta)}{\frac{1}{k}\big(\lfloor u \rfloor + 1 - k\big) + (k-1)\theta}$.

Proof. By its definition

$$K_k(u) = K_k\big(\lfloor u \rfloor\big) \cup \big\{x^k : \lfloor u \rfloor < x_k \leq u \text{ and } (x_1, \ldots, x_{k-1}) \in K_{k-1}\big(\lfloor u \rfloor\big)\big\}$$
$$\triangleq K_k\big(\lfloor u \rfloor\big) \cup J \text{ (say)}. \tag{4.13}$$

On the other hand, according to the correspondence Φ between the discrete and the continuous models,

$$\mu\big(K_k(\lfloor u \rfloor)\big) = \binom{\lfloor u \rfloor}{k}, \mu\big(K_{k-1}(\lfloor u \rfloor)\big) = \binom{\lfloor u \rfloor}{k-1}. \tag{4.14}$$

Therefore $\mu(J) = \theta\binom{\lfloor u \rfloor}{k-1}$ and consequently (i) holds. Now

$$||K_k(u)|| = ||K_k\big(\lfloor u \rfloor\big)|| + ||J||. \tag{4.15}$$

By Corollary 1 and (4.14)

$$||K_k(\lfloor u\rfloor)|| = \frac{k\lfloor u\rfloor}{2}\binom{\lfloor u\rfloor}{k}. \tag{4.16}$$

Furthermore, by (4.8) for $k-1$ and by (4.14)

$$\begin{aligned} ||J|| &= \mu\big(K_{k-1}(\lfloor u\rfloor)\big)\int_{\lfloor u\rfloor}^{u} x_k\, dx_k + \int_{\lfloor u\rfloor}^{u} dx_k ||K_{k-1}(\lfloor u\rfloor)|| \\ &= \big(\lfloor u\rfloor + \tfrac{\theta}{2}\big)\theta\tbinom{\lfloor u\rfloor}{k-1} + \theta\tfrac{k-1}{2}\lfloor u\rfloor\tbinom{\lfloor u\rfloor}{k-1}. \end{aligned} \tag{4.17}$$

Combination of these three identities gives

$$||K_k(u)|| = \frac{k\lfloor u\rfloor}{2}\binom{\lfloor u\rfloor}{k} + \left(\lfloor u\rfloor + \frac{\theta}{2} + \frac{k-1}{2}\lfloor u\rfloor\right)\theta\binom{\lfloor u\rfloor}{k-1}$$

and thus

$$||K_k(u)|| = \frac{k\lfloor u\rfloor}{2}\binom{\lfloor u\rfloor}{k} + \left(\frac{k+1}{2}\lfloor u\rfloor + \frac{\theta}{2}\right)\theta\binom{\lfloor u\rfloor}{k-1}. \tag{4.18}$$

This and (i) imply

$$\begin{aligned} ||K_k(u)|| - \tfrac{ku}{2}\mu\big(K_k(u)\big) &= -\tfrac{k\theta}{2}\tbinom{\lfloor u\rfloor}{k} + \left(\tfrac{\lfloor u\rfloor}{2} - \tfrac{k-1}{2}\theta\right)\theta\tbinom{\lfloor u\rfloor}{k-1} \\ &= -\tfrac{k\theta}{2}\tbinom{\lfloor u\rfloor}{k} + \tfrac{\lfloor u\rfloor}{2}\theta\tbinom{\lfloor u\rfloor}{k-1} - \tfrac{k-1}{2}\theta^2\tbinom{\lfloor u\rfloor}{k-1} \\ &= -\tfrac{\theta\lfloor u\rfloor}{2}\tbinom{\lfloor u\rfloor-1}{k-1} + \tfrac{\lfloor u\rfloor}{2}\theta\tbinom{\lfloor u\rfloor}{k-1} - \tfrac{k-1}{2}\theta^2\tbinom{\lfloor u\rfloor}{k-1} \\ &= \tfrac{\lfloor u\rfloor}{2}\theta\tbinom{\lfloor u\rfloor-1}{k-2} - \tfrac{k-1}{2}\theta^2\tbinom{\lfloor u\rfloor}{k-1} = \tfrac{k-1}{2}\theta\tbinom{\lfloor u\rfloor}{k-1} - \tfrac{k-1}{2}\theta^2\tbinom{\lfloor u\rfloor}{k-1}, \end{aligned}$$

and therefore (ii).

Remark 1 (to Lemma 4).
Actually, we can derive a somewhat more general result along the same lines. *Let* $J_k(u,u') \triangleq \big\{(x_1,\dots,x_k) \mid u < x_1 < \cdots < x_k \le u'$ *and* $\lceil x_i\rceil \ne \lceil x_j\rceil$, *for* $i \ne j\big\}$, $u < u' \in \mathbb{R}$, $\theta \triangleq \lceil u\rceil - u$ *and* $\theta' = u' - \lfloor u'\rfloor \triangleq \{u'\}$, *then*

$$\mu\big(J_k(u,u')\big) = \binom{\lfloor u'\rfloor - \lceil u\rceil}{k} + \binom{\lfloor u'\rfloor - \lceil u\rceil}{k-1}(\theta+\theta') + \theta\theta'\binom{\lfloor u'\rfloor - \lceil u\rceil}{k-2} \tag{4.19}$$

and

$$||J_k(u,u')|| - k(u+u') = \frac{k-1}{2}\big[(\theta'-\theta)[1-(\theta+\theta')]\big]\binom{\lfloor u'\rfloor - \lceil u\rceil}{k-1} - \frac{\theta\theta'}{2}(\theta'-\theta)\binom{\lfloor u'\rfloor - \lceil u\rceil}{k-2}. \tag{4.20}$$

This can be seen as follows.
By shifting the origin, we can assume w.l.o.g., that $u = -\theta$, $\theta \in [0,1)$, i.e. $\lfloor u\rfloor = 0$. Then

$$\begin{aligned} J_k(u,u') = K_k\big(\lfloor u'\rfloor\big) &\cup \big(\{x_1 : -\theta < x_1 \le 0\} \times \{(x_2,\dots,x_k) : (x_2,\dots,x_k) \in K_{k-1}(\lfloor u'\rfloor)\big) \\ &\cup \big(\big\{(x_1,\dots,x_{k-1}) : (x_1,\dots,x_{k-1}) \in K_{k-1}(\lfloor u'\rfloor)\big\} \times \big\{x_k : \lfloor u'\rfloor < x_k \le u'\big\}\big) \\ &\cup \big(\{x_1 : -\theta < x_1 \le 0\} \times \big\{(x_2,\dots,x_{k-1}) \in K_{k-2}(\lfloor u'\rfloor)\big\} \times \{x_k : \lfloor u'\rfloor < x_k \le u'\}\big) \end{aligned}$$

and by the same argument as the one used in the proof of Lemma 4 we obtain (4.19) and (4.20).

5 The Cases $k = 2, 3$

Using the same idea as in the proof of Theorem 1 in [2] simple calculations lead to two alternatives.

Lemma 5. *For $k = 2$, $U \in \mathbb{Z}^+$ and $W \in \mathcal{D}$ consider*

$$m_1(W) \triangleq \max\{x : (x, y) \in W \text{ for some } y\}. \tag{5.1}$$

Then

(i) W can be reduced to a trapezoid, if $m_1(W) \leq \frac{U}{2}$

and

(ii) W can be reduced to a cone, if $m_1(W) \geq \frac{U}{2}$.

Now we turn our attention to $k = 3$ and drop all subscripts k (for example write $K(U)$ instead of $K_3(U)$ and so on).
For $W \subset K(U)$ we call the 2–dimensional set

$$S_u(W) \triangleq \{(x, y) : (x, y, u) \in W \text{ and } (x, y, u + \varepsilon) \notin W \text{ for all } \varepsilon > 0\} \tag{5.3}$$

a Z–surface of W at u.
We call this surface *regular*, when for some $(x, y) \in S_u(W)$ and some $\varepsilon > 0$ $(x, y, u + \varepsilon) \in K(U)$. Therefore $S_u(W)$ is irregular iff $u = U$. The Y– and X–surfaces are defined analogously. We present now the basic idea of "moving top layers from lower density to higher density".
Observe first that the condition $\mu(R(\nu, u)) = \alpha$ (for fixed α) forces v to depend continuously on u, say

$$v = V_\alpha(u). \tag{5.4}$$

There are again two alternatives.

Lemma 6. *For $k = 3$, $u \leq U$, and $U \in \mathbb{Z}^+$ any trapezoid $R(v, u)$ can be reduced to a cone or the trapezoid $R(V_\alpha(U), U)$.*

Proof. Fix α and $U \in \mathbb{Z}^+$. Then $||R(V_\alpha(u), u)||$ is a continuous function in u, which achieves a maximal value. So, if the lemma is not true, then there are a $U \in \mathbb{Z}^+$, an α, and a u_0 with $v_0 \triangleq V_\alpha(u_0) < u_0 < U$ and $R(v_0, u_0)$ achieves the maximal value. $R(v_0, u_0)$ has one regular Z–surface and one regular X-surface, namely

$$\begin{aligned} S_1 &\triangleq \{(x, y) : 0 < x < y \leq \lceil u_0 \rceil - 1, x \leq v_0 \text{ and } \lceil x \rceil \neq \lceil y \rceil\} \\ \text{and } S_2 &\triangleq \{(y, z) : \lceil v_0 \rceil < y < z \leq u_0 \text{ and } \lceil y \rceil \neq \lceil z \rceil\}. \end{aligned} \tag{5.6}$$

(c.f. Figure 1)

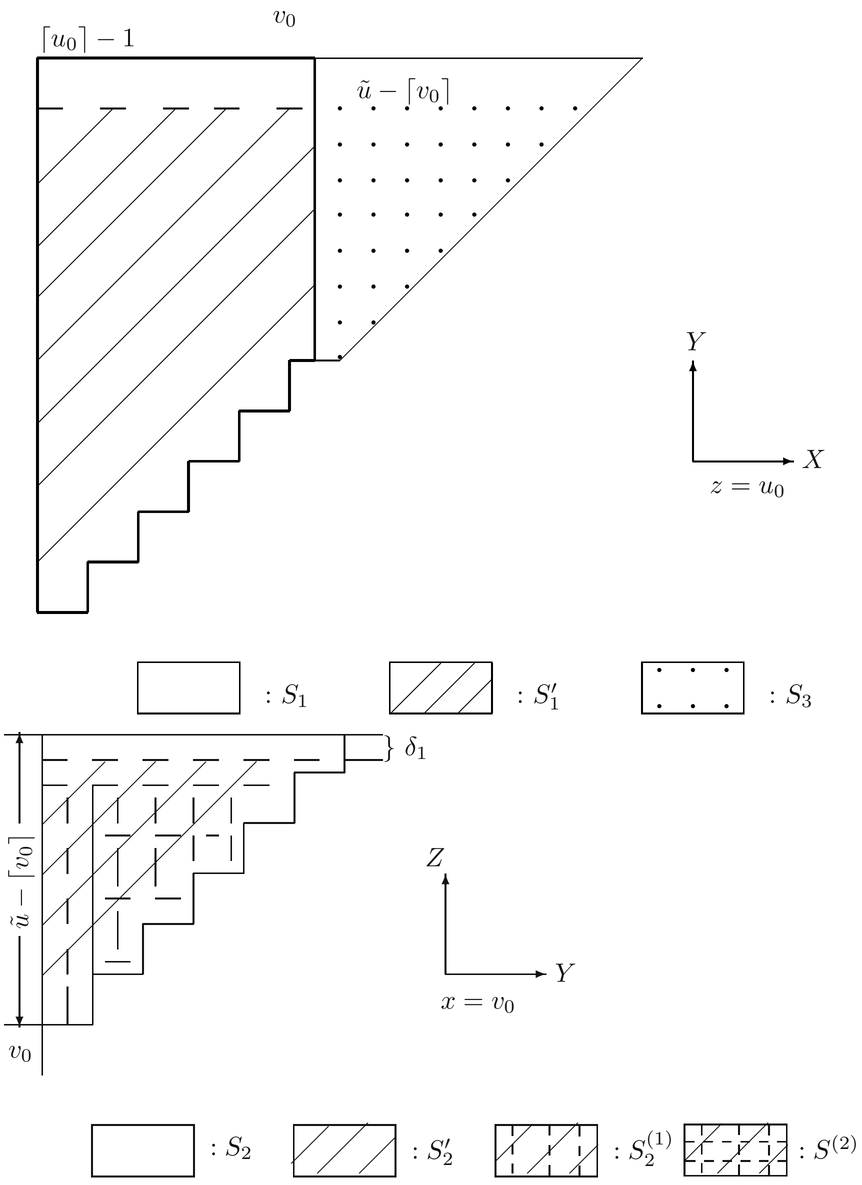


Fig. 1.

Case 1: $d(S_1) + u_0 < d(S_2) + v_0.$ (5.7)

Choose $\delta_1, \delta_2 > 0$ and define

$$\begin{aligned} D_1 &= S_1 \times \{z : u_0 - \delta_1 < z \le u_0\} \\ \text{and } D_2 &= \{x : v_0 < x \le v_0 + \delta_2\} \times S_2'. \end{aligned} \tag{5.9}$$

They satisfy

$$\mu(D_1) = \mu(D_2), \tag{5.10}$$

$$\delta_1 \le u_0 - \big(\lceil u_0 \rceil - 1\big), \delta_2 \le \big(\lfloor v_0 \rfloor + 1\big) - v_0, \tag{5.11}$$

and

$$d(S_1) + u_0 < d(S_2'') + v_0 \le d(S_2') + v_0, \tag{5.12}$$

where

$$S_2'' \triangleq S_2 \smallsetminus \big\{(y,z) : u_0 - \delta_1 < z \le u_0\big\} \tag{5.13}$$

and

$$S_2' \triangleq \begin{cases} S_2'' \smallsetminus \big\{(y,z) : v_0 < y \le v_0 + 1\big\} & \text{if } v_0 \in \mathbb{Z}^+ \\ S_2'' & \text{otherwise.} \end{cases} \tag{5.14}$$

The second inequality in (5.12) follows from Lemma 4 and our choice is possible by (5.7). Then

$$R' \triangleq \big(R(v_0,u_0) \smallsetminus D_1\big) \cup D_2 \in \mathcal{D} \tag{5.15}$$

is a trapezoid with measure α.
However by (5.9) - (5.14),

$$\begin{aligned} ||R'|| - ||R(v_0,u_0)|| &= ||D_2|| - ||D_1|| \\ &= \left[\mu(S_2') \int_{v_0}^{v_0+\delta_2} x dx + \delta_2 ||S_2'||\right] - \left[||S_1||\delta_1 + \mu(S_1) \int_{u_0-\delta_1}^{u_0} z dz\right] \\ &= \left[\big(\mu(S_2')\delta_2\big)\left(v_0 + \tfrac{\delta_2}{2}\right) + \big(\delta_2 \mu(S_2')\big) d(S_2')\right] - \left[\big(\mu(S_1)\delta_1\big) d(S_1) + \big(\mu(S_1)\delta_1\big)\left(u_0 - \tfrac{\delta_1}{2}\right)\right] \\ &= \mu(D_2)\left[v_0 + \tfrac{\delta_2}{2} + d(S_2')\right] - \mu(D_1)\left[d(S_1) + u_0 - \tfrac{\delta_1}{2}\right] \\ &= \mu(D_1)\left[\big(d(S_2') + v_0\big) - \big(d(S_1) + u_0\big) + \tfrac{\delta_1+\delta_2}{2}\right] > 0, \end{aligned}$$

a contradiction. Here the fourth equality follows from $\mu(S_2')\delta_2 = \mu(D_2)$ and $\mu(S_1)\delta_1 = \mu(D_2)$ (by (5.9)), the fifth equality follows from (5.10) and the inequality follows from (5.12).

Case 2: $d(S_1) + u_0 > d(S_2) + v_0$. One can come to a contradiction just like in case 1.
Case 3: $d(S_1) + u_0 = d(S_2) + v_0$. (5.16)

S_2 is a "shifted cone". One can calculate $d(S_2)$ and conclude with (5.16)

$$\lceil u_0 \rceil - 2 > v_0. \tag{5.17}$$

Consequently the following two surfaces are not empty:

$$\begin{aligned} S_1' &\triangleq \big\{(x,y) : 0 < x < y \le \lceil u_0 \rceil - 2, x \le v_0 \text{ and } \lceil x \rceil \ne \lceil y \rceil\big\} \\ \text{and } S_2^{(1)} &\triangleq \big\{(y,z) : \lceil v_0 \rceil < y < z \le u_0 - 1 \text{ and } \lceil y \rceil \ne \lceil z \rceil\big\} \\ &= S_2 \smallsetminus \big\{(y,z) : u_0 - 1 < z \le u_0\big\}. \end{aligned} \tag{5.19}$$

(See Figure 1) Assume first that

$$\mu(S_1') \geq \mu(S_2^{(1)}). \tag{5.20}$$

Let

$$\begin{aligned} D_1 &\triangleq \big\{(x,y,z) \in R(v_0,u_0) : u_0 - 1 < z \leq u_0\big\} \\ &= S_1 \times \big\{z : \lceil u_0 \rceil - 1 < z \leq u_0\big\} \cup S_1' \times \big\{z : u_0 - 1 < z \leq \lceil u_0 \rceil - 1\big\} \\ &\triangleq D_1' \cup D_1'', \\ D_2 &\triangleq \big\{(x,y,z) \in S_U : v_0 < x \leq x_0, z \leq u_0 - 1\big\} \\ &= \big\{x : v_0 < x \leq \lceil v_0 \rceil\big\} \times S_2^{(1)} \cup \Big[\textstyle\bigcup_{i \geq 2}\big(\{x : \lceil v_0 \rceil + i - 1 < x \leq v^{(i)}\} \times S_2^{(i)}\big)\Big], \end{aligned} \tag{5.22}$$

where

$$S_2^{(i)} = S_2^{(i-1)} \smallsetminus \big\{(y,z) : \lceil v_0 \rceil + 2 - i < x \leq \lceil v_0 \rceil + 3 - i\big\},$$

the last $v^{(i)}$ equals x_0, for the other i's $v^{(i)} = \lceil v_0 \rceil + i$, and finally x_0 is specified by

$$\mu(D_1) = \mu(D_2), \text{ if such an } x_0 \text{ exists.}$$

Otherwise continue with Case 4. Introduce now

$$R' = \big(R(v_0,u_0) \smallsetminus D_1\big) \cup D_2.$$

R' is a trapezoid with measure α. Now we have, with justifications given afterwards,

$$\begin{aligned} ||D_1|| &= \Big[\mu(S_1)\Big(u_0 - \tfrac{u_0 - \lceil u_0 \rceil + 1}{2}\Big)\big(u_0 - \lceil u_0 \rceil + 1\big) + ||S_1||\big(u_0 - \lceil u_0 \rceil + 1\big)\Big] \\ &\quad + \Big[\mu(S_1')\Big(\lceil u_0 \rceil - \tfrac{\lceil u_0 \rceil - u_0}{2} - 1\Big)\big(\lceil u_0 \rceil - u_0\big) + ||S_1'||\big(\lceil u_0 \rceil - u_0\big)\Big] \\ &= \mu(D_1')\Big(d(S_1) + u_0 - \tfrac{u_0 - \lceil u_0 \rceil + 1}{2}\Big) + \mu(D_1'')\Big[d(S_1') + \lceil u_0 \rceil - \tfrac{\lceil u_0 \rceil - u_0}{2} - 1\Big] \\ &= \big[\mu(D_1')d(S_1) + \mu(D_1'')d(S')\big] + (u_0 - 1)\big(\mu(D_1') + \mu(D_1'')\big) \\ &\quad + \tfrac{1}{2}\mu(D_1')\big(u_0 - \lceil u_0 \rceil + 1\big) + \tfrac{1}{2}\big(\lceil u_0 \rceil - u_0\big)\big(2\mu(D_1') + \mu(D_1'')\big) \\ &< \mu(D_1)\big(d(S_1) + u_0 - 1\big) + \tfrac{1}{2}\big(u_0 - \lceil u_0 \rceil + 1\big)\mu(D_1') + \tfrac{1}{2}\big(\lceil u_0 \rceil - u_0\big)\big(2\mu(D_1') + \mu(D_1'')\big) \\ &= \mu(D_1)\big(d(S_1) + u_0 - 1\big) + \tfrac{1}{2}\Big[\tfrac{\mu^2(D_1')}{\mu(S_1)} + 2\tfrac{\mu(D_1')\mu(D_1'')}{\mu(S_1')} + \tfrac{\mu(D_1'')^2}{\mu(S_1')}\Big] \\ &< \Big(d(S_1) + u_0 - 1 + \tfrac{\mu(D_1)}{2\mu(S_1')}\Big)\mu(D_1). \end{aligned} \tag{5.23}$$

Here the second and the fourth equality are obtained by

$$\mu(D_1') = \mu(S_1)\big(u_0 - \lceil u_0 \rceil + 1\big) \text{ and } \mu(D_1'') = \mu(S_1')\big(\lceil u_0 \rceil - u_0\big).$$

The first inequality follows from $d(S_1) > d(S_1')$ and $\mu(D_1) = \mu(D_1') + \mu(D_1'')$ and the second one follows from $\mu(S_1) > \mu(S_1')$. Similarly, since $d(S_2^{(1)}) < d(S_1^{(i)})$ and $\mu(S_2^{(1)}) > d(S_2^{(i)})$ for $i \geq 2$

$$||D_2|| > \left(d(S_2^{(1)}) + v_0 + \frac{\mu(D_2)}{2\mu(S_2^{(1)})}\right)\mu(D_2). \tag{5.24}$$

Finally, as S_2 and $S_2^{(1)}$ are shifted cones, by (iii) in Lemma 4, (5.6), (5.16), and (5.19)

$$d(S_2^{(1)}) + v_0 > d(S_2) - 1 + v_0 = d(S_1) + u_0 - 1. \tag{5.25}$$

So a contradiction $||R'|| - ||R(v_0, u_0)|| = ||D_2|| - ||D_1|| > 0$ follows from (5.19), (5.23), and (5.25). Therefore (5.20) must be false, i.e.

$$\mu(S_1') < \mu(S_2^{(1)}). \tag{5.26}$$

Let now $\overset{\sim}{\rightarrow} u \triangleq \lceil u_0 \rceil - 2$, $S_3 \triangleq K(\overset{\sim}{\rightarrow} u) \setminus S_1'$ (c.f. Figure 1), $\xi = 1 - \{v_0\}$, and $\eta = u_0 - \big(\lceil u_0 \rceil - 1\big)$, then by (5.26)

$$\mu(S_3) - \mu(S_1') > \mu(S_3) - \mu(S_2^{(1)}) = \big(\overset{\sim}{\rightarrow} u - \lceil v_0 \rceil\big)(\xi - \eta), \tag{5.27}$$

and by (i) in Lemma 4

$$\mu(S_3) = \frac{1}{2}\big[\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil\big)^2 - \big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil\big) + 2\xi\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil\big)\big] = \frac{\overset{\sim}{\rightarrow} u - \lceil v_0\rceil}{2}\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil \quad 1 + 2\xi \;. \tag{5.28}$$

However, by their definitions

$$\mu(S_1') + \mu(S_3) = \mu\big(K(\overset{\sim}{\rightarrow} u)\big) = \frac{1}{2}\big(\overset{\sim}{\rightarrow} u^2 - \overset{\sim}{\rightarrow} u\big). \tag{5.29}$$

Adding (5.27) to (5.29) we obtain

$$\mu(S_3) > \frac{1}{4}\big(\overset{\sim}{\rightarrow} u - 1\big) \overset{\sim}{\rightarrow} u + \frac{1}{2}\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil\big)(\xi - \eta). \tag{5.30}$$

(5.28) and (5.30) imply

$$\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil\big)\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil - 1 + \xi + \eta\big) > \frac{\overset{\sim}{\rightarrow} u}{2}\big(\overset{\sim}{\rightarrow} u - 1\big). \tag{5.31}$$

Simplifying (5.31), we obtain

$$\begin{aligned}
\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil\big)^2 &> \tfrac{\overset{\sim}{\rightarrow} u^2}{2} + \tfrac{\overset{\sim}{\rightarrow} u}{2} - \lceil v_0\rceil - (\xi + \eta)\big(\overset{\sim}{\rightarrow} u - \lceil v_0\rceil\big) > \tfrac{\overset{\sim}{\rightarrow} u^2}{2} - \tfrac{3}{2} \overset{\sim}{\rightarrow} u + \lceil v_0\rceil \\
&\quad (\text{as } \overset{\sim}{\rightarrow} u \geq \lceil v_0\rceil, \text{ see (5.17) and as } \xi + \eta \leq 2) \\
&= \tfrac{1}{2}\left(\overset{\sim}{\rightarrow} u - \tfrac{3}{2}\right)^2 - \tfrac{9}{8} + \lceil v_0\rceil, \text{ i.e.} \\
\overset{\sim}{\rightarrow} u - \lceil v_0\rceil &> \tfrac{\sqrt{2}}{2} \overset{\sim}{\rightarrow} u - \tfrac{3\sqrt{2}}{4}, \text{ or} \\
\lceil v_0\rceil &< \left(1 - \tfrac{\sqrt{2}}{2}\right) \overset{\sim}{\rightarrow} u + \tfrac{3\sqrt{2}}{4} = \left(1 - \tfrac{\sqrt{2}}{2}\right)\overline{u} - 1 + \tfrac{5\sqrt{2}}{4},
\end{aligned} \tag{5.32}$$

where $\overline{u} \triangleq \lceil u_0\rceil - 1 = \overset{\sim}{\rightarrow} u + 1$. On the other hand, by (iii) in Lemma 4 and (5.16) with $\eta' = \{u_0\}$

$$\begin{aligned}
d(S_1) = d(S_2) + v_0 - u_0 &\leq \left(u_0 + \lceil v_0\rceil + \tfrac{\eta'(1-\eta')}{\overline{u} - \lceil v_0\rceil - 1}\right) + v_0 - u_0 \\
&= v_0 + \lceil v_0\rceil + \tfrac{\eta'(1-\eta')}{\overline{u} - \lceil v_0\rceil - 1}.
\end{aligned} \tag{5.33}$$

Consider that S_1 is the union of a rectangle and a 2–dimensional cone (a triangle).

$$\begin{aligned}||S_1|| &= \tfrac{1}{2}\big(\lceil v_0\rceil^2 - \lceil v_0\rceil\big)\lceil v_0\rceil + v_0\big(\overline{u} - \lceil v_0\rceil\big)\left(\lceil v_0\rceil + \tfrac{v_0+\overline{u}-\lceil v_0\rceil}{2}\right) \\ &= \tfrac{1}{2}\big[\lceil v_0\rceil^2\big(\lceil v_0\rceil - 1\big) + v_0\big(\overline{u} - \lceil v_0\rceil\big)\big(v_0 + \lceil v_0\rceil + \overline{u}\big)\big],\end{aligned} \tag{5.34}$$

and

$$\mu(S_1) = \frac{1}{2}\big(\lceil v_0\rceil^2 - \lceil v_0\rceil\big) + v_0\big(\overline{u} - \lceil v_0\rceil\big). \tag{5.35}$$

(5.33) - (5.35) imply

$$\begin{aligned}&\left(v_0 + \lceil v_0\rceil + \tfrac{\eta'(1-\eta')}{\overline{u}-\lceil v_0\rceil-1}\right)\left(\tfrac{1}{2}\big(\lceil v_0\rceil^2 - \lceil v_0\rceil\big) + v_0\big(\overline{u} - \lceil v_0\rceil\big)\right) \\ &\geq \tfrac{1}{2}\big[\lceil v_0\rceil^2\big(\lceil v_0\rceil - 1\big) + v_0\big(\overline{u} - \lceil v_0\rceil\big)\big(v_0 + \lceil v_0\rceil + \overline{u}\big)\big], \text{ i.e.}\end{aligned}$$

$$\begin{aligned}&\lceil v_0\rceil\big(\lceil v_0\rceil - 1\big)\tfrac{\eta'(1-\eta')}{\overline{u}-\lceil v_0\rceil-1} \geq v_0\big(\overline{u} - \lceil v_0\rceil\big)\left(\overline{u} - v_0 - \lceil v_0\rceil - \tfrac{2\eta'(1-\eta')}{\overline{u}-\lceil v_0\rceil-1}\right) - v_0\big(\lceil v_0\rceil^2 - \lceil v_0\rceil\big) \\ &= v_0\big(\overline{u}^2 - 3\lceil v_0\rceil\overline{u} + \lceil v_0\rceil^2\big) + v_0\lceil v_0\rceil + v_0\big(\overline{u} - \lceil v_0\rceil\big)\left[\big(\lceil v_0\rceil - v_0\big) - \tfrac{2\eta'(1-\eta')}{\overline{u}-\lceil v_0\rceil-1}\right] \\ &\geq \big(\lceil v_0\rceil - 1\big)\left[\big(\overline{u}^2 - 3\overline{u}\lceil v_0\rceil + \lceil v_0\rceil^2\big) + \lceil v_0\rceil - \big(\overline{u} - \lceil v_0\rceil\big)\tfrac{2\eta'(1-\eta')}{\overline{u}-\lceil v_0\rceil-1}\right],\end{aligned}$$

i.e.

$$\begin{aligned}\overline{u}^2 - 3\overline{u}\lceil v_0\rceil + \lceil v_0\rceil^2 &\leq \big(2\overline{u} - \lceil v_0\rceil\big)\tfrac{\eta'(1-\eta')}{\overline{u}-\lceil v_0\rceil-1} - \lceil v_0\rceil \\ &\leq \tfrac{1}{4}\tfrac{2\overline{u}-\lceil v_0\rceil}{\overline{u}-\lceil v_0\rceil-1} - \lceil v_0\rceil.\end{aligned} \tag{5.36}$$

Comparing (5.32) and (5.36), one can conclude

$$\begin{aligned}&\left[\left(1 - \tfrac{\sqrt{2}}{2}\right) + \tfrac{5\sqrt{2}-4}{4\overline{u}}\right]^2 - 3\left[\left(1 - \tfrac{\sqrt{2}}{2}\right) + \tfrac{5\sqrt{2}-4}{4\overline{u}}\right] + 1 \\ &< \tfrac{1}{\overline{u}} \cdot \tfrac{1}{2\sqrt{2}\overline{u}-5\sqrt{2}} - \tfrac{1}{\overline{u}}\left[\left(1 - \tfrac{\sqrt{2}}{2}\right) + \tfrac{\sqrt{2}-4}{4\overline{u}}\right] \\ &= \tfrac{1}{\overline{u}}\left(\tfrac{1}{2\sqrt{2}\overline{u}-5\sqrt{2}} - \tfrac{5\sqrt{2}-4}{4\overline{u}}\right) - \tfrac{1}{\overline{u}}\left(1 - \tfrac{\sqrt{2}}{2}\right), \quad \text{or}\end{aligned}$$

$$\begin{aligned}&\left(1 - \tfrac{\sqrt{2}}{2}\right)^2 - 3\left(1 - \tfrac{\sqrt{2}}{2}\right) + 1 < \\ &\tfrac{1}{4\overline{u}}(3\sqrt{2} + 2) + \tfrac{1}{\overline{u}}\left(\tfrac{1}{2\sqrt{2}\overline{u}-5\sqrt{2}} - \tfrac{4(5\sqrt{2}-4)+(5\sqrt{2}-4)}{16\overline{u}}\right).\end{aligned} \tag{5.37}$$

One can check that (5.37) does not hold unless $\overline{u} < 8$, or $\lceil u_0\rceil \leq 8$. However, it is not difficult to check that (5.16) and (5.26) cannot hold simultaneously for $4 < u \leq 8$. Finally using the condition $U \notin \mathbb{Z}^+$ it follows that $U \geq 4$. One can also check the lemma for $3 < u \leq 4$.

Case 4

If an x_0 with $\mu(D_1) = \mu(D_2)$ does not exist, i.e. D_1 is too big to find a D_2 with the same measure, we choose a proper h, $0 < h < 1$, such that for

$$\begin{aligned}D_1 &\triangleq \big\{(x, y, z) \in R(v_0, u_0) : u_0 - h < z \leq u_0\big\} \text{ and} \\ D_2 &\triangleq \big\{(x, y, z) \in S_U : v_0 < x < y \leq u_0 - h\big\}, \; \mu(D_1) = \mu(D_2).\end{aligned}$$

D_2 is a shifted cone. By the arguments leading to Lemma 4, (c.f. (4.18), (4.19) in Remark to Lemma 4) we get for its density

$$d(D_2) \geq 3\lceil v_0 \rceil + \tfrac{3}{2}\big[u_0 - h - \lceil v_0 \rceil - \big(1 - \{v_0\}\big)\big] - \frac{\{v_0\}\big(1-\{v_0\}\big)}{\big|u_0-h-\lceil v_0 \rceil\big|^+ + 2\big(1-\{v_0\}\big)}$$
$$= \tfrac{3}{2}(u_0 + v_0 - h) - \frac{\{v_0\}\big(1-\{v_0\}\big)}{\big|u_0-h-\lceil v_0 \rceil\big|^+ + 2\big(1-\{v_0\}\big)} \;.$$

However, by (5.16) and Lemma 4

$$d(D_1) = d(S_1) + u_0 - \frac{h}{2} = d(S_2) + v_0 - \frac{h}{2} \leq v_0 + \lceil v_0 \rceil + u_0 - \frac{h}{2} + \frac{1}{4}.$$

Then

$$d(D_2) - d(D_1) \geq \tfrac{u_0}{2} + \tfrac{v_0}{2} - \lceil v_0 \rceil - h - \tfrac{1}{4} - \frac{\{v_0\}\big(1-\{v_0\}\big)}{\big|u_0-h-\lceil v_0 \rceil\big|^+ + 2\big(1-\{v_0\}\big)}$$
$$> \tfrac{1}{2}\big(u_0 - \lceil v_0 \rceil\big) - h - \tfrac{3}{4}.$$

Thus by (5.16), for $u_0 > 8$

$$d(D_2) > d(D_1).$$

For $\lceil u_o \rceil \leq 8$ we check it directly.

Remark 2. *For $m \in \mathbb{Z}^+$ denote by $\mathcal{D}_m$ the set of downsets of $\overset{\sim}{\rightarrow} \mathcal{L}(U)\big(\triangleq \overset{\sim}{\rightarrow} \mathcal{L}(U,3)\big)$ with m maximal points. We can show that $\max_{\mu(W)=\alpha, W\in\mathcal{D}_m} ||W||$ can be achieved, as well.*

More precisely, define a metric on the set $\big\{(x^i, y^i, z^i)_{i=1}^k : (x^i, y^i, z^i) \in \mathbb{R}^3\big\}$ as the sum of Euclidean (or L_1-) metrics of the k components points. Then for fixed $\mu(W) = \alpha$, $W \in \mathcal{D}_m$, $||W||$ is a continuous function of its maximal points.

6 On Regular Surfaces

Lemma 7. *Every $W \in \mathcal{D}$ can be reduced to a $W' \in \mathcal{D}$, which has of each of the regular $X-, Y-$ and $Z-$ surfaces at most one (for $U \in Z^+$).*

Proof. Suppose there exists a W that canot be reduced to such kind of W'. W.l.o.g. by Remark 1 we assume W achieves $\underset{m'\leq m}{\rightarrow} \max_{\mu(W)=\alpha, W\in\mathcal{D}_{m'}} ||W||$, (recalling $\mathcal{D} = \bigcup_{m=1}^{\infty} \mathcal{D}_m$ by its definition).

Case 1: Suppose W has at least 2 regular z–surfaces, say S_i at i, for $i = 1, 2$, and

$$d(S_1) + u_1 \leq d(S_2) + u_2. \tag{6.1}$$

Using the same method as in the proof of Lemma 6, Case 1, one can obtain a contradiction. Furthermore, we can see that W has 2 regular X–surfaces iff $\hat{W}^{(u)}$ has 2 regular Z–surfaces. Since W and $\hat{W}^{(u)}$ must achieve the maximal value simultaneously, we are left with **Case 2:** W has at least 2 regular Y–surfaces S_1 at v_1 and S_2 at v_2 with

$$d(S_1) + v_1 \leq d(S_2) + v_2 \tag{6.2}$$

and of each of the regular $Z-$ and $X-$ surfaces at most one. Let $S_2' = S_2$, if $v_2 \notin Z$, and otherwise let $S_2' = S_2 \smallsetminus \big\{(x,z) \mid v_1 < z \leq v_1 + 1\big\}$. Since W has no 2 regular Z–surfaces nor X–surfaces, S_2 is rectangular, consequently $d(S_2') > d(S_2)$. Thus we can use S_2' to replace S_2 and play the same game as before to arrive at a contradiction.

7 Main Result in Continuous Model, $k = 3$

Theorem 2. *For $U \in \mathbb{Z}^+$ and fixed α every $W \in \mathcal{D}$ with $\mu(W) = \alpha$ can be reduced to a cone or the trapezoid $R\big(V_\alpha(U), U\big)$.*

Proof. Assume the theorem is not true. Then by Remark 1 and Lemma 6 there exists a $W \in \mathcal{D}$ with m maximal points achieving maximal value of $||W||$ over $\bigcup_{m' \leq m} \mathcal{D}_m$, which is neither a cone nor a trapezoid. Moreover, by Lemma 7 we can assume that W has at most one regular $X-$, at most one regular $Y-$, and at most one regular $Z-$ surface.

Case 1: W has only one (regular or irregular) Z–surface at $u \leq U$. Then W has one or two maximal points, whose third components must be u. **Subcase 1.1:** W has one maximal point, say $P = (w, v, u)$. Because $v = \lceil u \rceil - 1$ implies W is a trapezoid, we assume $w < v \leq \lceil u \rceil - 1$. Thus, W has one Z–surface S_1 and one Y–surface, which are shown in Figure 2 (a).
We are going to use the same idea as before. However, it is not enough to exchange the layers. Instead of it we will exchange cylinders. (a) Suppose $w \geq u - \lceil v \rceil$.
We choose $0 < h_1 < u - \lceil v \rceil$ and define $S_2 \triangleq \big\{(y,z) : v < y < z \leq u - h_1$ and $\lceil y \rceil \neq \lceil z \rceil\big\}$, $D_1 = S_1 \times \big\{z : u - h_1 < z \leq u\big\}$, $D_2 \triangleq \big\{x : 0 < x \leq w\big\} \times S_2$, and $W' = (W \smallsetminus D_1) \cup D_2$ such that

$$\mu(D_1) = \mu(D_2). \tag{7.1}$$

Then $W' \in \mathcal{D}$ and furthermore, if we denote $\{v\}$ by θ and use the arguments of the proof of Lemma 4 (see Remark to Lemma 4), then we obtain

$$d(S_2) - (v + u - h_1) = \frac{(\theta' - \overline{\theta})\big[1 - (\theta' + \overline{\theta})\big] - \overline{\theta}\theta'(\theta' - \overline{\theta})\big(\lfloor u - h_1 \rfloor - \lceil v \rceil\big)^{-1}}{\big(\lfloor u - h_1 \rfloor - \lceil v \rceil - 1\big) + 2(\theta' + \overline{\theta}) + 2\overline{\theta}\theta'\big(\lfloor u - h_1 \rfloor - \lceil v \rceil\big)^{-1}} \triangleq \eta_1, \tag{7.2}$$

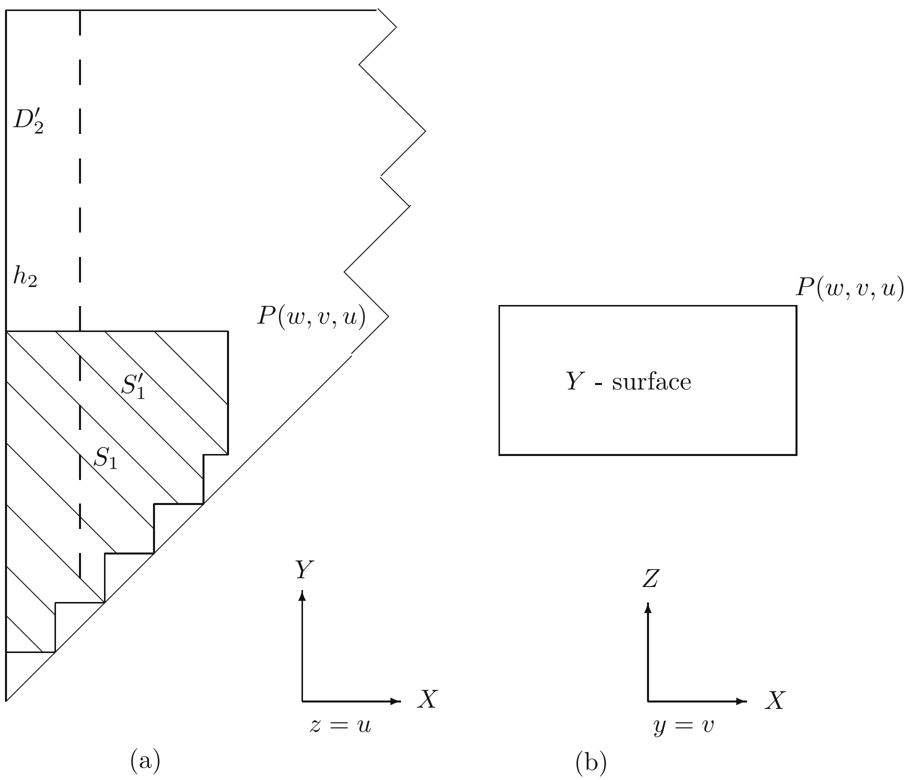


Fig. 2 (a).

where $\theta' \triangleq \{u-h_1\}$ and $\overline{\theta} = 1-\theta = \lceil v \rceil - v$, if $u-h_1-\lceil v \rceil > 1$. By Lemma 4 and Corollary 2,

$$d(S_1) - v \le \frac{\theta(1-\theta)}{\lceil v \rceil - 1 + 2\theta} \triangleq \eta_2. \tag{7.3}$$

Consequently

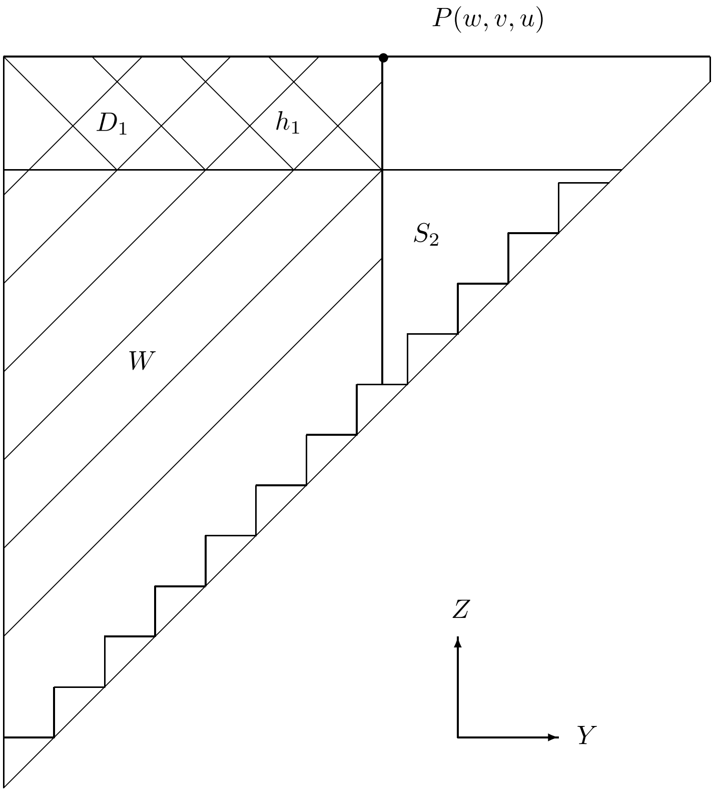
$$d(S_2) - \big(d(S_1)+u\big) \ge -h_1 + \eta_1 - \eta_2. \tag{7.4}$$

Therefore, by simple calculation

$$\begin{aligned} &||W'|| - ||W|| = ||D_2|| - ||D_1|| \\ &= \mu(D_2)\left(d(S_2)+\tfrac{w}{2}\right) - \mu(D_1)\left(d(S_1)+u-\tfrac{h_1}{2}\right) \\ &= \mu(D_2)\left[d(S_2) - \big(d(S_1)+u\big) + \tfrac{w}{2} + \tfrac{h_1}{2}\right] \ge \mu(D_2)\left[\tfrac{w}{2} - \tfrac{h_1}{2} + \eta_1 - \eta_2\right]. \end{aligned} \tag{7.5}$$

By (7.2),

$$\eta_1 \ge -\,\frac{\overline{\theta}(1-\overline{\theta})}{\lfloor u-h_1 \rfloor - \lceil v \rceil - 1 + 2\overline{\theta}} = \frac{-\theta(1-\theta)}{\lfloor u-h_1 \rfloor - \lceil v \rceil - 1 + 2(1-\theta)}. \tag{7.6}$$



(c)

Fig. 2 (b).

Thus, (7.3) and (7.6) imply

$$\eta_1 - \eta_2 \geq -\frac{1}{2}. \tag{7.7}$$

However, when $h_1 \leq u - \lceil v \rceil - 1$, (7.5) and (7.2) imply the contradiction

$$||W'|| > ||W||. \tag{7.8}$$

When $u - \lceil v \rceil - 1 \leq h_1 < u - \lceil v \rceil$, S_2 becomes a rectangle (c.f. Figure 3) and $d(S_2) = v + u - h_1 + \frac{\overline{\theta}}{2} - \frac{u - \lceil v \rceil - h_1}{2}$. Then use

$$\eta_1 = \frac{1-\theta}{2} - \frac{u - \lceil v \rceil - h_1}{2}, \tag{7.9}$$

and (7.8) holds again. (b) If $w < u - \lceil v \rceil$, then we choose $0 < h_2 < w$ and let $S_1' = S_1 \smallsetminus \big\{(x,y) : 0 < x \leq h_2\big\}$, $S_2' = \big\{(y,z) : v \leq y < z < u, \lceil y \rceil \neq \lceil z \rceil\big\}$, $D_1' \triangleq S_1' \times \big\{z : \lceil v \rceil < z \leq u\big\}$, and $D_2' = S_2 \times \{x : 0 < x \leq h_2\}$ with

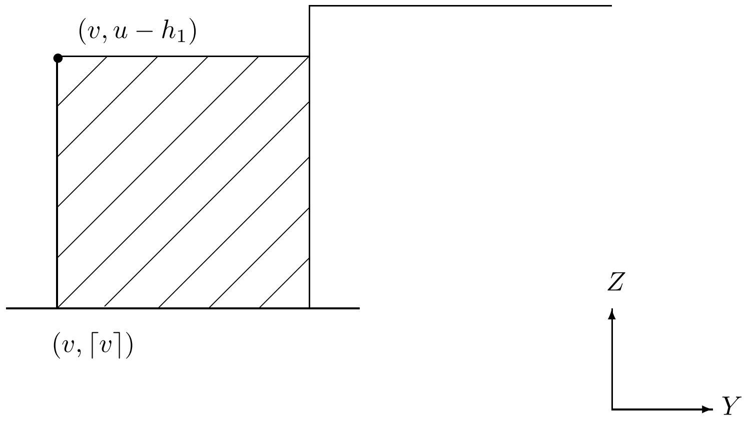


Fig. 3.

$\mu(D_1') = \mu(D_2')$. Considering $(W \smallsetminus D_1') \cup D_2'$ in a similar way we arrive at a contradiction. (c.f. Figure 2 (a)) **Subcase 1.2:** W has 2 maximal points.

According to our assumption on regular surfaces the Z–surface S_1 of W must be as in Figure 4.

Then we follow the same reasoning as in the previous subcase in the shadow part (i.e. exchange cylinders in the shadow part $\{(x, y, z) \in S_U \mid x \leq v_o\}$, where v_0 is the smaller first component in the 2 maximal points) and obtain a contradiction.

Case 2: W has 2 Z–surfaces. Since W and $\hat{W}$ always simultaneously achieve their maximum, we can assume $\hat{W}$ has 2 $\mathcal{Z}$–surfaces too, because otherwise we can use $\hat{W}$, which has been studied in Case 1 already, instead of W. However, $\hat{W}$ has 2 Z–surfaces iff W has one regular X–surface, and

$$\{(0, y, z) \in S_U\} \smallsetminus W \neq \varnothing. \tag{7.10}$$

Thus we can assume W has one regular X–surface and (7.10) holds.

Then by our assumption W has 2 maximal points, say $P_1 = (w_1, v_1, U)$ and $P_2 = (w_2, v_2, u)$ and $v_1 < \lceil U \rceil - 1$. **Subcase 2.1:** $\lceil v_1 \rceil \geq \lfloor u \rfloor$. Then $w_1 < w_2$,

because P_2 is maximal. Recalling that in our proof under subcase 1.1 we only exchange the points (x, y, z) with $x \leq w$, and $y \geq \lceil v \rceil$, in the present case we can use the plane $x = w_1$ to cut S_U into 2 parts and repeat the same reasoning as in subcase 1.1 to obtain a contradiction in the part $x \geq w_1$.

Moreover, for this kind of W's, $\hat{W}^{(U)}$ has 2 maximal points, $\hat{P}_1 = (\hat{w}_1, \hat{v}_1, U)$ and $\hat{P}_2 = (\hat{w}_2, \hat{v}_2, \hat{u})$ with $\hat{w}_1 = U - \lceil v_1 \rceil$, $\hat{v}_1 = U - v_1$, $\hat{w}_2 = U - u$, $\hat{v}_2 = U - \lceil w_1 \rceil$, $\hat{u} = U - w_1$, i.e. $\hat{w}_1 = \lceil \hat{v}_1 \rceil - 1$, $\hat{v}_2 = \lceil \hat{u} \rceil - 1$ and $\hat{w}_2 \geq \hat{w}_1$. Therefore, the following subcase 2.2 can be cancelled from our list. **Subcase 2.2:** $w_1 = \lceil v_1 \rceil - 1$,

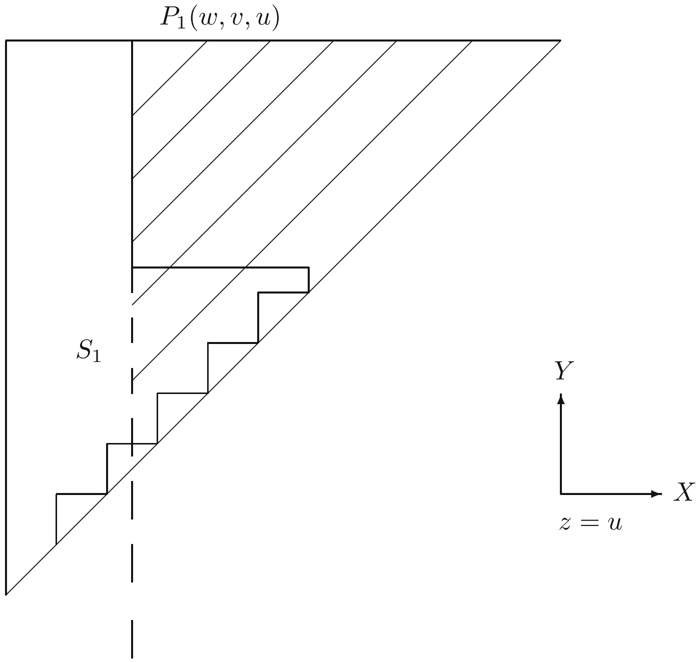


Fig. 4.

$v_2 = \lceil u \rceil - 1$, and $w_2 \geq w_1$. **Subcase 2.3:** $w_1 = \lceil v_1 \rceil - 1$, $v_2 = \lceil u \rceil - 1$, $w_2 < w_1$, and $\lceil v_1 \rceil < u$. In this subcase, there are one regular Z–surface and one regular Y–surface passing P_1.

Denote by $S_1 = \big\{(x,y) : y \leq v_1 \ \lceil x \rceil \neq \lceil y \rceil \big\}$ the irregular Z–surface, by S_2 the regular X–surface at w_2, a shifted cone, and by $S_3 = \big\{(y,z) : \lceil y \rceil \neq \lceil z \rceil, (0,y,z) \in S_U \setminus W\big\}$ as in Figure 5.
Then $\overset{\sim}{\rightarrow} W \triangleq W \cap \big\{(x,y,z) : y > v_1\big\}$ is a cylinder with base S_2. Therefore we can assume

$$v_2 - v_1 = \lceil u \rceil - 1 - v_1 > U - u, \tag{7.11}$$

because otherwise, by Lemma 5, we can replace $\overset{\sim}{\rightarrow} W$ by a cylinder with the same size 2–dimensional trapezoid base and the same height, and then reduce W to a downset with 2 regular Y–surfaces. If $d(S_1) + U < d(S_3)$, then we can repeat our reasoning as before and arrive at a contradiction. So we only need to consider

$$d(S_1) + U \geq d(S_3), \tag{7.12}$$

which, in fact, is also impossible. By Lemma 4

$$d(S_1) = v_1 + \frac{\theta(1-\theta)}{\mid \lfloor v_1 \rfloor - 1 \mid^+ + 2\theta} \triangleq v_1 + \eta. \tag{7.13}$$

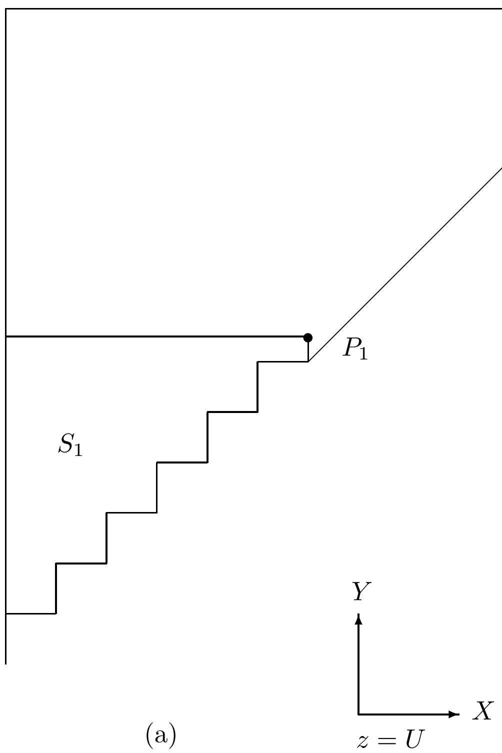


Fig. 5 (a).

Partitioning S_3 into a rectangle S_3' and a (2–dimensional) cone S_3', we obtain

$$||S_3|| = \frac{1}{2}\big(\lceil u \rceil - 1 + v_1 + U + u\big)\mu(S_3') + \big(U + \lceil u \rceil - 1\big)\mu(S_3''), \tag{7.14}$$

$$\mu(S_3') = \big(\lceil u \rceil - 1 - v_1\big)(U - u), \mu(S_3'') = \binom{U - (\lceil u \rceil - 1)}{2}, \tag{7.15}$$

and

$$\mu(S_3) = \mu(S_3') + \mu(S_3''). \tag{7.16}$$

(see Figure 5 (c).) Thus, it follows from (7.12) – (7.16) that

$$\frac{1}{2}\big[U - u - (\lceil u \rceil - 1) + v_1\big]\big(\lceil u \rceil - 1 - v_1\big)(U - u) - \big(\lceil u \rceil - 1 - v_1\big)\binom{U - (\lceil u \rceil - 1)}{2} + \eta\,\mu(S_3) \geq 0. \tag{7.17}$$

(7.11) and (7.17) imply

$$\eta\ \mu(S_3) > \big(\lceil u \rceil - 1 - v_1\big)\binom{U - (\lceil u \rceil - 1)}{2}. \tag{7.18}$$

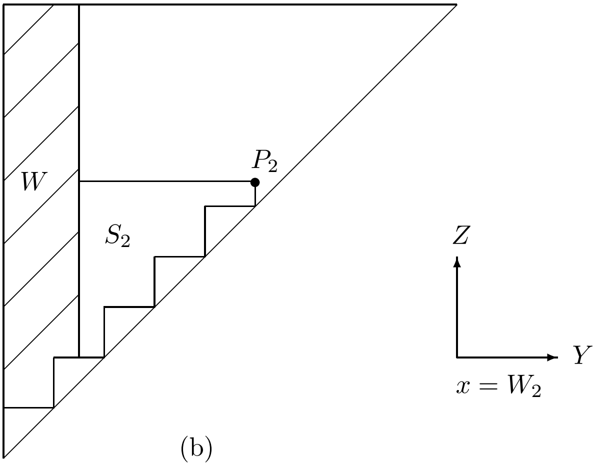


Fig. 5 (b).

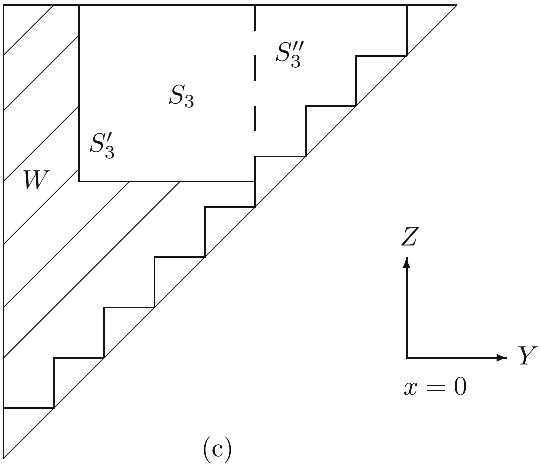


Fig. 5 (c).

However, by (7.15) and (7.16)

$$\frac{\mu(S_3)}{\left(\lceil u\rceil - 1 - v_1\right)\binom{U-(\lceil u\rceil-1)}{2}} = \frac{U-u}{\binom{U-(\lceil u\rceil-1)}{2}} + \frac{1}{\lceil u\rceil - 1 - v_1} \leq 4, \text{ if } U - \left(\lceil u\rceil - 1\right) \geq 2. \tag{7.19}$$

On the other hand, by the definition of η, $\eta \leq \frac{1}{4}$, which contradicts (7.18) and (7.19). When $U - \lceil u\rceil - 1 \leq 1$, we can directly derive a contradiction.

Thus we are left with the case $w_1 < \lceil v_1\rceil - 1$ (and $\lceil v_1\rceil < u$), i.e. both of the regular $X-$ and $Y-$surfaces pass through P_1, or in other words neither of the surfaces passes through P_2 unless P_2 shares one of them with P_1. In fact, all of the following 3 subcases are not new to us.

Subcase 2.4: There is no regular surface passing through P_2, i.e. $P_2 = \big(\lceil u\rceil - 2, \lceil u\rceil - 1, u\big)$. Then the top part of W, namely, $W_t \triangleq W \cap \big\{(x,y,z) : z > u\big\}$ is a cylinder with a 2 dimensional trapezoid $R_2(w_1, v_1)$ (its irregular Z–surface) as base. By similar reasoning with Lemma 5 as after (7.11) we can assume $v_1 = \lfloor u\rfloor$, which has been treated in the subcase 2.1.

Subcase 2.5: P_1 and P_2 share a regular X–surface, i.e. $w_1 = w_2$ and $v_2 = \lceil u\rceil - 1$. Then $\hat{W}^{(U)}$ falls into subcase 2.4.

Subcase 2.6: P_1 and P_2 share a regular Z–surface, i.e. $v_1 = v_2$, and $w_2 = \lceil v_2\rceil - 1$. Then $\hat{W}^{(U)}$ falls into subcase 2.3.

8 A Last Auxiliary Result

Lemma 8. *For $U \in \mathbb{Z}^+, U \geq 6$, $\alpha = \binom{U}{3} - \binom{m}{3} < \frac{1}{2}\binom{U}{3}$ and $m \in \mathbb{Z}^+$*

$$||R\big(V_\alpha(U), U\big)|| > ||K(u)||, \quad \text{if} \;\; \mu\big(K(u)\big) = \alpha = \mu\big(R_\alpha(U), U\big). \tag{8.1}$$

Proof. At first let us restrict ourselves to $U \geq 12$. We know from (i) in Lemma 4 that

$$6\mu\big(K(u)\big) = 6\binom{\lfloor u\rfloor}{3} + 6\theta\binom{\lfloor u\rfloor}{2} = (u-1)^3 - \left\{\left[3\left(\theta - \frac{1}{2}\right)^2 + \frac{1}{4}\right]\lfloor u\rfloor - (1-\theta)^3\right\}. \tag{8.2}$$

Therefore,

$$d\big(K(u)\big) \geq \frac{3}{2}u > \frac{3}{2}\left[6\mu\big(K(u)\big)^{\frac{1}{3}} + 1\right] = \frac{3}{2}\left[(6\alpha)^{\frac{1}{3}} + 1\right]. \tag{8.3}$$

On the other hand for $\eta > 0$, by (8.2)

$$\begin{aligned}
&\big[u - (1+\eta)\big]^3 = (u-1)^3 - 3\eta(u-1)^2 + 3\eta^2 - \eta^3 \\
&= 6\mu\big(K(u)\big) - \left[3\eta(u-1)^2 - 3\eta^2(u-1) + \eta^3 - \left[3\left(\theta - \tfrac{1}{2}\right)^2 + \tfrac{1}{4}\right]\lfloor u\rfloor + (1-\theta)^3 + \eta^3\right] \\
&= 6\mu\big(K(u)\big) - \left[3\eta\lfloor u\rfloor^2 - 3\left(2\eta\overline{\theta} + \eta^2 - \overline{\theta}(1-\overline{\theta}) + \tfrac{1}{3}\right)\lfloor u\rfloor + (\overline{\theta} + \eta)^3\right] \\
&\leq 6\mu\big(K(u)\big) - 3\lfloor u\rfloor\left[\lfloor u\rfloor\eta - \left(2\eta + \eta^2 + \tfrac{1}{3}\right)\right], \\
&\text{where} \quad \overline{\theta} \triangleq 1 - \theta.
\end{aligned} \tag{8.4}$$

Let

$$\eta = \xi - \frac{2\theta\overline{\theta}}{(\lfloor u\rfloor - 2) + 6\theta} > 0 \tag{8.5}$$

and η, ξ will be defined later. Then by (8.4) and (8.5),

$$d\big(K(u)\big) \le \frac{3}{2}\left[6\mu\big(K(u)\big)^{\frac{1}{3}} + (1+\xi)\right]. \tag{8.6}$$

when

$$\lfloor u \rfloor \ge \frac{2\eta + \eta^2 + \frac{1}{3}}{\eta}, \tag{8.7}$$

Choose $\xi_1 = 0.12$ and $\xi_2 = 0.035$, to estimate $d\big(K(u)\big)$ and $d\big(K(U)\big)$, resp. By our assumption $u \ge 7\frac{20}{29}$, if $U = 12$, and $u > 8$, if $U \ge 13$. Then one can verify (8.6) with (8.5) for u, ξ_1 (or U, ξ_2). So, by (8.7)

$$d\big(K(u)\big) \le \tfrac{3}{2}\left[\big(6\mu(u)\big)^{\frac{1}{3}} + 1 + \xi_1\right] = \tfrac{3}{2}\left((6\alpha)^{\frac{1}{3}} + 1 + \xi_1\right) \tag{8.8}$$

$$d\big(K(U)\big) \le \tfrac{3}{2}\left[\big(6\mu\big(K(U)\big)\big)^{\frac{1}{3}} + 1 + \xi_2\right]. \tag{8.9}$$

Setting $\alpha = \lambda\mu\big(K(U)\big)$, by Lemmas 3 and 4, (8.3), and (8.8), we obtain

$$\begin{aligned}
&||R\big(V_\alpha(U),U\big)|| - ||K(u)|| = \tfrac{3}{2}U\mu\big(K(u)\big) - 3\big(\mu\big(K(U)\big) - \alpha\big) + ||\hat{R}^{(U)}\big(V_\alpha(U),U\big)|| - ||K(u)|| \\
&\ge \tfrac{3}{2}\Big\{\left[\big(6\mu\big(K(U)\big)\big)^{\frac{1}{3}} + 1 + \xi_2\right]\big(2\alpha - \mu\big(K(U)\big)\big) + \left[\big(6\big[\mu\big(K(U)\big) - \alpha\big]\big)^{\frac{1}{3}} + 1\right] \\
&\cdot\big(\mu\big(K(U)\big) - \alpha\big) - \left[(6\alpha)^{\frac{1}{3}} + 1 + \xi_1\right]\alpha\Big\} = \tfrac{3}{2}\sqrt[3]{6}\mu\big(K(u)\big)f(\lambda), \quad \text{where} \\
&f(\lambda) = 2\lambda - 1 + (1-\lambda)^{\frac{1}{3}} - \lambda^{\frac{1}{3}} - \frac{\xi_2 + (\xi_1 - 2\xi_2)\lambda}{\big(6\mu\big(K(U)\big)\big)^{\frac{1}{3}}},
\end{aligned} \tag{8.10}$$

is concave in λ. Let $\varepsilon_1 = \frac{2.7}{\big(6\mu\big(K(U)\big)\big)^{\frac{1}{3}}}$, $\varepsilon_2 = \frac{2.68/2^{\frac{5}{3}}}{\big(6\mu\big(K(U)\big)\big)^{\frac{1}{3}}}$ and $M \in \mathbb{Z}^+$ be specified by

$$\binom{M}{3} \le \frac{1}{2}\binom{U}{3} < \binom{M+1}{3}. \tag{8.11}$$

Then

$$\varepsilon_1 < \frac{3}{U} = \frac{\binom{U-1}{2}}{\binom{U}{3}} = \frac{\binom{U}{3} - \binom{U-1}{3}}{\mu\big(J(U)\big)}, \tag{8.12}$$

and as $\frac{\left[2\binom{M+1}{2}\right]^3}{\left[6\binom{M}{3}\right]^2} = \frac{M(M-1)}{(M+1)}$, by (8.11) and $M > 9$ (when $U > 12$),

$$\begin{aligned}
\frac{1}{2}\frac{\binom{M}{2}}{\binom{U}{3}} &= \frac{1}{4}\left[\frac{M(M-1)}{(M+1)^2}\right]^{\frac{1}{3}} \frac{\big(6\binom{M+1}{3}\big)^{\frac{1}{3}}}{\mu\big(K(U)\big)} > \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{2}{3}}\left[\frac{M(M-1)}{(M+1)}\right]^{\frac{1}{3}} \frac{1}{\left[6\mu\big(K(U)\big)\right]^{\frac{1}{3}}} \\
&\ge \frac{3}{2^{\frac{5}{3}}}(0.72)^{\frac{1}{3}}\frac{1}{\left[6\mu\big(K(U)\big)\right]^{\frac{1}{3}}} = \frac{2.68884\ldots}{2^{\frac{5}{3}}}\frac{1}{\left[6\mu\big(K(U)\big)\right]^{\frac{1}{3}}} > \varepsilon_2.
\end{aligned} \tag{8.13}$$

However, with Taylor's expansion,

$$\begin{aligned}
f(\varepsilon_1) &\ge 2\varepsilon_1 - \tfrac{4}{3}\varepsilon_1 + \tfrac{4}{9}\varepsilon_1^2 - \varepsilon_1^{\frac{4}{3}} - \frac{\xi_2 + (\xi_1 - 2\xi_2)\varepsilon_1}{\big(6\mu\big(K(U)\big)\big)^{\frac{1}{3}}} \\
&= \frac{1}{\left[6\mu\big(K(U)\big)\right]^{\frac{1}{3}}}\left(\tfrac{2}{3} \times 2.7 - 2.7 \times \varepsilon_1^{\frac{1}{3}} - \xi_2\right) \\
&+ \frac{\varepsilon_1}{\left[6\mu\big(K(U)\big)\right]^{\frac{1}{3}}}\left[\tfrac{4}{9} \times 2.7 - (\xi_1 - 2\xi_2)\right] > 0.
\end{aligned} \tag{8.14}$$

Moreover, set $g(x) = (1+x)^{\frac{4}{3}} - (1-x)^{\frac{4}{3}}$. Then

$$g(0) = g''(0) = 0, g'(0) = \frac{8}{3} \text{ and } g''(x) > -0.6254,$$

when $0 \leq x \leq 2\varepsilon_2 < 0.1551$. Thus, by the definition of ε_2 and Taylor's expansion again

$$\begin{aligned} f\left(\tfrac{1}{2} - \varepsilon_2\right) &= -2\varepsilon_2 + \left(\tfrac{1}{2}\right)^{\frac{4}{3}} g(2\varepsilon_2) - \frac{\xi_1\left(\frac{1}{2}-\varepsilon_2\right)+2\xi_2\varepsilon_2}{\left[6\mu\left(K(U)\right)\right]^{\frac{1}{3}}} \\ &\geq -(2\varepsilon_2) + \left(\tfrac{1}{2}\right)^{\frac{4}{3}} \tfrac{8}{3}(2\varepsilon_2) - 0.6254(2\varepsilon_2)^3 - \frac{\xi_1\left(\frac{1}{2}-\varepsilon_2\right)+2\xi_2\varepsilon_2}{\left[6\mu\left(K(U)\right)\right]^{\frac{1}{3}}} \qquad (8.15) \\ &= 2\varepsilon_2 \left[-1 + \tfrac{2^{\frac{5}{3}}}{3} - 0.6254(2\varepsilon_2)^2 - \tfrac{2^{\frac{2}{3}}}{2.68}\left[\tfrac{1}{2}\xi_1(1-2\varepsilon_2) + \xi_2(2\varepsilon_2)\right]\right] \\ &\geq 2\varepsilon_2[-1 + 1.05826\dots - 0.0150\dots - 0.0332\dots] > 0. \end{aligned}$$

(8.14), (8.15) and the convexity of f imply $f(\lambda) > 0$, when $\lambda \in \left[\varepsilon_1, \frac{1}{2} - \varepsilon_2\right]$, or, in other words, if $U \geq 12$ and $\varepsilon_1\mu\big(K(U)\big) \leq \alpha \leq \left(\frac{1}{2} - \varepsilon_2\right)\mu\big(K(U)\big)$, then $||R\big(V_\alpha(U), U\big)|| > ||K(U)||$. On the other hand (8.12) and the assumption on α together imply $\alpha > \varepsilon_1\mu\big(K(U)\big)$. Moreover it follows from the assumption on α, (8.11) and (8.13), that $\alpha \leq \left(\frac{1}{2} - \varepsilon_2\right)\mu\big(K(U)\big)$, unless

$$\alpha = \binom{U}{3} - \binom{M+1}{3} \text{ and } \binom{M}{3} \leq \alpha \leq \binom{M+1}{3}, \qquad (8.16)$$

where M is defined by (8.11).

However (8.16) implies $\hat{R}^{(U)}\big(V_\alpha(U), U\big) = K(M+1)$ and $u \in [M, M+1]$. Therefore

$$\hat{R}^{(U)}\big(V_\alpha(U), U\big) \setminus K(u) = \big\{(x,y,z) : u < z \leq M+1, 0 < x < y < M, \lceil x \rceil \neq \lceil y \rceil\big\} \triangleq \Delta, \text{ say.} \qquad (8.17)$$

This and Lemma 4 imply

$$d(\Delta) = M + \frac{M+1+u}{2} \geq 2M + \frac{1}{2}. \qquad (8.18)$$

Moreover, one can easily check in our case (i.e. $U \geq 12$) that $M \geq \frac{3}{4}U$, which together with (8.18) means that

$$d(\Delta) > \frac{3}{2}U. \qquad (8.19)$$

This and Lemmas 3, 4 imply

$$\begin{aligned} &||R\big(V_\alpha(U), U\big)|| - ||K(U)|| = \tfrac{3}{2}U\mu\big(K(U)\big) - \tfrac{3}{2}U\left(\mu\big(K(U)\big) - \alpha\right) \\ &+ \Big(||\hat{R}^{(U)}\big(V_\alpha(U), U\big)|| - ||K(u)||\Big) = \tfrac{3}{2}U\left[\alpha - \left(\mu\big(K(U)\big) - \alpha\right)\right] + ||\Delta|| \qquad (1) \\ &= \tfrac{3}{2}U\left(\mu\big(K(u) - \hat{R}^{(U)}\big(V_\alpha(U), U\big)\right) + ||\Delta|| = \left(d(\Delta) - \tfrac{3}{2}U\right)\mu(\Delta) > 0. \end{aligned}$$

i.e. so far, we have shown (8.1) for $U \geq 12$. Finally, we check (8.1) directly for $U = 6, 7, \ldots, 11$.

Remark 3. *For $U < 6$, there is no room for $\alpha = \binom{U}{3} - \binom{M}{2} < \frac{1}{2}\binom{U}{3}$.*

9 Main Result for $k = 3$ and Good α

Now let us return to our main problem in the discrete model. Denote by $R^*(v, u)$ the downset of $(v, u-1, u)$ $(v, u \in \mathbb{Z}^+)$ in $\mathcal{L}(U, 3)$ and by $K^*(u)$ the downset of $(u-2, u-1, u)$ $(u \in \mathbb{Z}^+)$ in $\mathcal{L}(U, 3)$. Then Lemmas 2,3, and 8 and Theorems 1 and 2 together imply immediately this solution.

Theorem 3. *Let $U \in \mathbb{Z}^+$, $U \geq 6$, then*

(i) For $\alpha = \binom{U}{3} - \binom{m}{3} \leq \frac{\binom{U}{3}}{2}$ for some $m \in \mathbb{Z}^+$, $\max_{|\mathcal{A}|=\alpha} \mathcal{P}(\mathcal{A})$ is achieved by $\mathcal{R}^*(U-m, U)$.

(ii) For $\alpha = \binom{m}{3} \geq \frac{\binom{U}{3}}{2}$ for some $m \notin \mathbb{Z}^+$ $\max_{|\mathcal{A}|=\alpha} \mathcal{P}(\mathcal{A})$ is achieved by $K^*(m)$.

10 A False Natural Conjecture for $k = 3$ and General α; There Is "Almost" No "Order" at All

We conclude our paper by taking a look at general α. Both, the result for $k = 2$ in [2] and our result for $k = 3$ and good α suggest that the following conjecture is reasonable, namely, that for $k = 3$ and α with

$$\binom{U}{3} - \binom{a+1}{3} < \alpha < \binom{U}{3} - \binom{a}{3} \leq N(\alpha) < \frac{\binom{U}{3}}{2}, \tag{10.1}$$

where $a \in \mathbb{Z}^+$ and $N(\alpha)$ is a function depending only on α, if U is big enough, the following configuration W is optimal for maximizing $\mathcal{P}(\mathcal{A})$:

(i) take the $\binom{U}{3} - \binom{a+1}{3}$ points (x, y, z) with $x \leq U - (a+1)$ in $S_{U,3}$

(ii) add the $\alpha - \left[\binom{U}{3} - \binom{a+1}{3}\right]$ points $(U-a, y, z)$ where (y, z) are points of a quasi–star or a quasi–complete graph in the sense of [2] according to the value of $\alpha - \left[\binom{U}{3} - \binom{a+1}{3}\right]$.

However, this conjecture, which has been made by several authors, is false.

Example 1: For $\alpha_0 \triangleq \left[\binom{U}{3} - \binom{U-2}{3}\right] - (U-2) - (U-3) = \binom{U}{3} - \binom{U-2}{3} - 2U + 5$ (when U is big enough), the$W$ described above is $S_1 \setminus (S_2 \cup S_3)$ where $S_1 \triangleq \big\{(x, y, z) \in S_{U,3} : x = 1, 2\big\}$.

$$S_2 \triangleq \big\{(2, 3, U), (2, 4, U), \ldots, (2, U-2, U), (2, U-1, U)\big\},$$

and S_3 is listed in (10.2) below.

Now let us consider the configuration W' with $W' \triangleq S_1 \setminus (S_2 \cup S_3')$, where S_3' is also listed in (10.2).

$$S_3: \ (2,3,U-1),(2,4,U-1),\dots,(2,U-2,U-1),(2,U-3,U-2),(2,U-4,U-2)$$
$$S_3': \ (1,2,U),(1,3,U),(1,4,U),\dots,(1,U-2,U),(1,U-1,U). \tag{10.2}$$

Thus, $||S_3|| > ||S_3'||$ when $U > 10$ and therefore $||W|| < ||W'||$. This example tells us that a solution for general α, even when $k = 3$, is much more challenging. Actually, if we pay a little bit more attention to it, we will find a deeper result just at our hands. People working on these kinds of problems usually wish to find "an order", more precisely a nested optimal sequence such as

$$W_1 \subset W_2 \subset W_3 \subset \dots$$

where W_i is optimal for size i. It is not surprising that in many cases, obviously including our problem, there is no order at all. In these cases, and in particular for our case, we define M_k as the maximal integer s.t. the optimal nested chain with length M_k i.e. the optimal nested chain

$$W_1 \subset W_2 \subset W_3 \subset \dots \subset W_{M_k} \tag{10.3}$$

exists. Considering our problem we only need to study the α-s with $\alpha \le \frac{1}{2}\binom{U}{3}$, because we can take "complements". Therefore we wish M_k to be close to $\frac{1}{2}\binom{U}{3}$. In fact in [2], it was shown that $M_2 \ge \frac{1}{2}\binom{U}{2} - \frac{U}{2}$, and that therefore M_2 is asymptotically equal to $\frac{1}{2}\binom{U}{3}$ (i.e. $\frac{\frac{1}{2}\binom{U}{2}-M_2}{\binom{U}{2}} \to 0$).

However, it is surprising that there is a jump between M_2 and M_3, because M_3 is asymptotically close to zero as can be seen from the following result.

Theorem 4.

$$M_3 < \binom{U}{3} - \binom{U-2}{3} \triangleq \alpha_2 \quad \textit{for} \quad U > U_0. \tag{10.4}$$

Proof. Assume the result is false. Then there is a nested optimal chain $W_1 \subset W_2 \subset \dots \subset W_{\alpha_2}$.

Let α_0, W and W' be defined as in Example 1 and set $\alpha_1 \triangleq \binom{U}{3} - \binom{U-1}{3}$. Then (when U is big enough) $\alpha_1 < \alpha_0 < \alpha_2$ and therefore $W_{\alpha_1} \subset W_{\alpha_0} \subset W_{\alpha_2}$. First of all, we draw attention to the fact that in the proofs in Section 3, we actually have already proved that the optimal configurations in Theorem 3 are unique (except if $\alpha = \frac{1}{2}\binom{U}{3}$.) Therefore, $W_{\alpha_1} = R^*(1,U)$ and $W_{\alpha_2} = R^*(2,U)$ or

$$(1,U-1,U) \in W_{\alpha_1} \quad \text{and} \quad (2,U-1,U) \in W_{\alpha_2} \tag{10.5}$$

and so

$$(1, U-1, U) \in W_{\alpha_0}. \tag{10.6}$$

Consequently,

$$W_{\alpha_0} \neq W'. \tag{10.7}$$

Moreover, there exists an $(x_0, y_0, z_0) \in W_{\alpha_0}$ with $x_0 \geq 3$, because otherwise by Theorems 2 and 3 in [2] $||W_{\alpha_0}|| = ||W||$, which would contradict Example 1 (here W and W' are defined as in Example 1). However, $(x_0, y_0, z_0) \notin R^*(2, U) = W_{\alpha_2} \supset W_{\alpha_0}$, a contradiction.

11 A Related Topic: The Maximal Moments for the Family of Measurable Symmetric Downsets

Next let us drop the condition $\lceil x \rceil \neq \lceil y \rceil$, $\lceil y \rceil \neq \lceil z \rceil$ used in the definition of $S_{U,3}$ in previous sections, i.e. consider the lattice $\alpha'(U,3) \triangleq (S'_{U,3}, \leq)$, $S'_{U,3} \triangleq \big\{(x,y,z) \in R^3 : 0 \leq x \leq y \leq z\big\}$. The problem becomes more smooth and therefore much simpler. To see this, we mention here two observations.

(a) To guarantee the formula analogous to (4.8), we don't have to require $u \in \mathbb{Z}^+$.

(b) One can simply derive a lemma analogous to Lemma 6, by standard methods in calculus (such as to take right derivatives and so on).

In fact, in a similar but much simpler way we can prove the following result.

Theorem 5. *For $U \in R$ let $I_U = [0,U]^3 \subset \mathbb{R}^3$ and let $\mathcal{F}_\alpha$ be the family of the Lebesgue measurable subsets S of I_U, satisfying*

(i) For every $S \in \mathcal{F}_\alpha$ $\mu(S) = \alpha$.
(ii) For every permutation π on $\{1,2,3\}$ and every $S \in \mathcal{F}_\alpha$ $(x_1, x_2, x_3) \in S$ implies $x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)} \in S$.
(iii) For every $S \in \mathcal{F}_\alpha$, $(x,y,z) \in S$ and $(x',y',z') \leq (x,y,z)$. Also $(x',y',z') \in S$.

Then $\max_{S \in \mathcal{F}_\alpha} ||S||$, where $||S|| = \int_S (x+y+z)dx\,dy\,dz$, is achieved by a set $S^* \in \mathcal{F}_\alpha$ of the form

$$S^* = \begin{cases} \big\{(x,y,z) : \min\{x,y,z\} \leq v\big\} \text{ for some } v = v(\alpha), \text{ if } \alpha \leq \frac{U^3}{2} \\ \big\{(x,y,z) : 0 \leq x,y,z \leq u\big\} \text{ for some } u = u(\alpha), \text{ if } \alpha \geq \frac{U^3}{2}. \end{cases}$$

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