# About the number of step functions with restrictions

Rudolf Ahlswede and Vladimir Blinovsky

*We obtain the asymptotic formula for the number of scaled step functions with the restrictions on the length and height of steps (shapes of Young diagrams) of given area in the neighborhood of a given curve. This allows us to find the asymptotics of the whole number of such functions and find the limit shape-the curve of concentration of the step functios.* [1]

## 1 Introduction

Consider the $x \geq 0,\ y \geq 0$ quarter of the plain and step functions with integer nodes which start from some point on the $Y$ axis and end in some point on the $X$ axis. In this work we deal with the number of such step functions, for which the area under these functions is asymptotically equal to $n$ and $n \to \infty$.

The main problem we solved in this work is to obtain the asymptotics of the number of scaled step functions in the neighborhood of a given curve when the length and height of steps are taking values from given sets $\mathcal{A}$ and $\mathcal{B}$ correspondingly. The case when there are no such restrictions, i.e. $\mathcal{A} = \mathcal{B} = \{1, 2, \ldots\}$ was, considered in [3] and independently in [5] by using an essentially different method. In [4] the case was considered, where there are restrictions only on the height of steps i.e. $\mathcal{A} = \{1, 2, \ldots\}$ and $\mathcal{B}$ is an arbitrary set of positive integers. Here we complete these investigations by the case where $\mathcal{A}$ and $\mathcal{B}$ are given sets of positive integers. This case also differs from the previous ones by some new difficulties in the proof. In particular we don't know another method of obtaining the asymptotics of the whole number of step functions with the restrictions on the height and length of steps. At the same time the asymptotics in the case without restrictions is known due to a result of Hardy- Ramanujan [2] because the number of step functions without restrictions is the number of partitions of $n$ which in turn has the logarithmic asymptotics $\sim \pi\sqrt{n\frac{2}{3}}$. The case when $\mathcal{A} = \{1, 2, \ldots\}$ can be done explicitly also.

The problem considered here can be viewed as the large deviations problem. The difference $2C - L(f)$ (see next section for notations) is the rough logarithmic asymptotics (asymptotics under several parameters) of the probability that the step function belong to the

[1]This work is partially supported by RFFI grant No 03-01-00592 and 03-01-00098 and INTAS grant No 00-738.

neighborhood of the curve $f$. At the same time the obtaining of the explicit expression for the functional $L(\cdot)$ uses the tools from the large deviation theory in functional spaces. In particular in the proof of the Theorem 1 we use the following scheme (which we describe roughly) : we consider instead of given curve the spline with the nodes on the curve and for consecutive nodes $(c_i, c_{i+1})$ we find the asymptotics of the number of the step functions which start in the neighborhood of $c_i$ and end in the neighborhood of $c_{i+1}$. The it is shown that the whole number of step functions in the neighborhood of $f$ is the product of these numbers when $i$ runs over all nodes. Each this number has the rough logarithmic asymptotics given by Lemma 2 and depends only on differences between corresponding coordinates of $c_i$ and $c_{i+1}$. When finding the logarithmic asymptotics of the whole number of step functions in the neighborhood of $f$, the logarithm of the product of evaluations of pairs of consecutive nodes gives the integral sum whose limit gives the integral appeared in the expression for $L(f)$. Besides this procedure we have to take into account some technical and routine details concerning the singular properties of the function $f$.

In the Theorem 2 using variational method we find the maximum of the functional $L(\cdot)$ over sufficiently smooth functions and prove that the same maximum achieved on the set of ' 'all' functions from $\mathcal{C}$. Here we also have to take into account some technical details to make the calculus of variations to be correct. Also we find limit shape, on which achieved the maximum of the functional $L(\cdot)$. Taking into account Lemma 2 we see that this shape attract the most of the step functions. The range of the problems of finding the limit shapes of random step functions with given area which can be viewed as random Young diagram was considered in the paper [1]. There was offered the number of limit shapes under different distributions.

At last in the Lemma 2 we find the logarithmic asymptotics of the whole number of step functions with restrictions.

Note the one of the differences between the usual large deviations scheme: we first consider the restrictions of the functions on the interval $[1/r, r]$ and last limit which we take is the limit when $r \to \infty$. This is done, because the considered ensemble of step functions is not exponentially compact (see [5] for definitions) under $L^1 - -$metrics.

Let's start with precise formulations. Here we consider only the case when sets $\mathcal{A},\ \mathcal{B} \subset \{1, 2, \ldots\}$ are infinite: $|\mathcal{A}|,\ |\mathcal{B}| = \infty$. The case of finite sets we shall consider somewhere else. By step function we mean a piecewise constant non increasing function with nodes in integer points (later we consider the scaling of the points with integer coordinates and consider the scaled step functions). We consider the case when the lengths of the step function are taken from the set $\mathcal{A}$ and the difference between consecutive different values of these step functions (height of the steps) are taken from the set $\mathcal{B}$.

First we formulate the lemmas we need and next formulate the main results in two theorems. Everywhere in the text we assume that expressions like $c\sqrt{n}$ are integers. We will omit the label 'restricted' and speak simply about step functions.

## 2 Formulation of results

Let $\ell_{\min}^{\mathcal{A}} = \min\{\ell \in \mathcal{A}\}$, ($\ell_{\min}^{\mathcal{B}} = \min\{\ell \in \mathcal{B}\}$) and $\#_{\mathcal{A}}^{k,\epsilon}(a)$ ($\#_{\mathcal{B}}^{k,\epsilon}(a)$) is the number of sequences $(x_i)_1^k,\ x_i \in \mathcal{A},\ (x_i \in \mathcal{B})$ such that $\frac{1}{k}\sum_{i=1}^k x_i \in U(a,\epsilon)$, where $U(a,\epsilon) = \{x \in R:\ |x-a| < \epsilon\}$.

**Lemma 1** *The following relation is valid*

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\frac{\ln \#_{\mathcal{A}}^{k,\epsilon}}{k} = J_{\mathcal{A}}(a) \triangleq \inf_{\lambda\le 0}\left(\ln\sum_{\ell\in\mathcal{A}} e^{\lambda\ell} - \lambda a\right),\ \ a \ge 0. \tag{1}$$

Now we formulate the result about the asymptotics of the number $\#_{\mathcal{A},\mathcal{B}}^{n,\epsilon_1,\epsilon_2}(a,b)$ of step functions starting from some point $(c\sqrt{n}, d\sqrt{n})$ and ending in the rectangle $x = (c+a\pm\epsilon_1)\sqrt{n},\ y = (d-b\pm\epsilon_2)\sqrt{n}$. Note that $\#_{\mathcal{A},\mathcal{B}}^{n,\epsilon_1,\epsilon_2}(a,b)$ depends only on $a, b$ but not on $c, d$.

**Lemma 2** *The following equalities are valid:*

$$\lim_{\epsilon_1,\epsilon_2\to 0}\limsup_{n\to\infty}\frac{\ln \#_{\mathcal{A},\mathcal{B}}^{n,\epsilon_1,\epsilon_2}(a,b)}{\sqrt{n}} = \lim_{\epsilon_1,\epsilon_2\to 0}\liminf_{n\to\infty}\frac{\ln \#_{\mathcal{A},\mathcal{B}}^{n,\epsilon_1,\epsilon_2}(a,b)}{\sqrt{n}} = N\left(\frac{b}{a}\right),$$

*where*

$$N(\xi) = \max_{\kappa\in[0,\min\{1,\xi\}]}\kappa\left(J_{\mathcal{A}}(1/\kappa) + J_{\mathcal{B}}(\xi/\kappa)\right).$$

Now we define the set of functions $\mathcal{C}$. It contains the functions such that for each $f \in \mathcal{C}$ there exists an $\hat{f},\ f = \hat{f}$ *a.s.* and $\hat{f}$ is non-negative, non-increasing, continuous from the right on $[0,\infty)$ and

$$\int_0^\infty f(x)dx \le 1.$$

For given $n, \delta$ let $S_{n,\delta}$ be the set of step functions $\varphi_{n,\delta}$ for which was applied the scaling: all linear sizes of step functions are divided by $\sqrt{n}$ i.e. we consider the step functions with the nodes from the lattice $\frac{1}{\sqrt{n}}Z \times \frac{1}{\sqrt{n}}Z$ which are non increasing functions in the positive quarter $x, y \ge 0$ such that

$$\int_0^\infty \varphi_{n,\delta}(x)dx = 1 \pm \delta.$$

Let $S_{n,\delta,r}$ be the set of restrictions of $S_{n,\delta}$ on the interval $[1/r, r]$. Let also

$$B(f,\epsilon,r) = \left\{y \in L^1[1/r,r]:\ \int_{1/r}^r |y(x) - f(x)|dx < \epsilon\right\}$$

be the $L^1-$ ball of radius $\epsilon > 0$ on the interval $[1/r, r]$.

Next we formulate our main result.

**Theorem 1** *Let*

$$\#^{n,\epsilon,\delta,r}(f) = \#\{S_{n,\delta,r} \bigcup B(f,\epsilon,r)\},$$

*then the following equalities are valid*

$$\begin{aligned} & \lim_{r\to\infty} \limsup_{\delta\to 0} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\ln \#^{n,\epsilon,\delta,r}(f)}{\sqrt{n}} \\ = \; & \lim_{r\to\infty} \liminf_{\delta\to 0} \liminf_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\ln \#^{n,\epsilon,\delta,r}(f)}{\sqrt{n}} = L(f), \end{aligned}$$

*where*

$$L(f) = \begin{cases} \int_0^\infty N(-\hat{f}'(x))dx, & f \in \mathcal{C}, \\ -\infty, & f \notin \mathcal{C}. \end{cases}$$

Denote by $\mathcal{C}_1$ the class of functions from $\mathcal{C}$ which have continuous first derivative and by $\mathcal{C}_2$ the class of functions from $\mathcal{C}$ with continuous second derivative.

**Theorem 2** *The following relations are valid*

$$\max_{f\in\mathcal{C}} L(f) = \max_{f\in\mathcal{C}_2} L(f) = 2C, \tag{2}$$

*and if*

$$\arg\max_{f\in\mathcal{C}_2} L(f) = f_{\max}, \tag{3}$$

*then $f_{\max}$ is determined by the equation*

$$\sum_{\ell\in\mathcal{A}} e^{-Cx\ell} \sum_{\ell\in\mathcal{B}} e^{-Cf_{\max}(x)\ell} = 1 \tag{4}$$

*and $C$ is determined by the equation*

$$\int_0^\infty f_{\max}(x)dx = 1. \tag{5}$$

Note, that from (4) and (5) follows that if $\psi(x)$ is the solution of the equation

$$\sum_{\ell\in\mathcal{A}} e^{-x\ell} \sum_{\ell\in\mathcal{B}} e^{-\psi(x)\ell} = 1,$$

then $C = \sqrt{\int_0^\infty \psi(x)dx}$. The integrability of $\psi(x)$ we will prove at the end of the paper.

Let $\#^{n,\delta}_{\mathcal{A},\mathcal{B}} = \#\{S_{n,\delta}\}$ be the whole number of step functions in $S_{n,\delta}$.

**Theorem 3** *The following equalities are valid*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{\ln \#^{n,\delta}_{\mathcal{A},\mathcal{B}}}{\sqrt{n}} = \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{\ln \#^{n,\delta}_{\mathcal{A},\mathcal{B}}}{\sqrt{n}} = 2C.$$

# 3 Proofs

The proof the Lemma 1 is similar to the one for Cramer's theorem for large deviations of the sum of i.i.d. random variables. On the set $\mathcal{A}$ we define the counting measure $\mu:\ \mu(\ell)=1,\ \ell\in\mathcal{A}$ and $\mu(z)=0$ otherwise. Analogically to the probabilistic case we define the sequence $(X_i)_1^k$ of 'pseudo' random variables ('pseudo' means that we consider not normed measures) taking values in $\mathcal{A}$ such that $\mu(X_i\in A)=\mu(A)$. Also define the product measure

$$\mu^k\left(\bigotimes_{i=1}^k A_i\right)=\mu^k(X_i\in A_i,\ i=1,\dots,k)=\prod_{i=1}^k\mu(A_i).$$

Since $U(a,\epsilon)$ is an open convex set, by the standard subadditivity argument (see for ex. [6]) it follows that the limit

$$\lim_{k\to\infty}\frac{\ln\mu^k\left(\frac{1}{k}\sum_{i=1}^k X_i\in U(a,\epsilon)\right)}{k}\tag{6}$$

exists. Note, that $\mu^k\left(\frac{1}{k}\sum_{i=1}^k X_i\in U(a,\epsilon)\right)=\#_{\mathcal{A}}^{k,\epsilon}$. We omit the details of the proof of the Lemma which are literally the same as the proof of the Cramer's theorem [5].

Note also that $J_{\mathcal{A}}$ is a $\bigcap-$convex, differentiable, monotone increasing function and $J_{\mathcal{A}}(\ell_{\min}^{\mathcal{A}})=0,\ J_{\mathcal{A}}(a)=-\infty,\ a<\ell_{\min}^{\mathcal{A}}$.

**Proof of Lemma 2.** By simple use of the Chebyshev's inequality it is easy to obtain the inequality

$$\gamma_{n,k,\epsilon}(a)\triangleq\frac{\ln\#\{(x_i)_1^k:\ \sum_{i=1}^k x_i\in\sqrt{n}U(a,\epsilon)\}}{\sqrt{n}}\le(a+\epsilon)H\left(\frac{k}{\sqrt{n}(a+\epsilon)}\right),\tag{7}$$

where $H(z)=-z\ln z-(1-z)\ln(1-z)$. Since $H(z)\overset{z\to0}{\to}0$ and $H$ is monotone increasing on $[0,1/2]$ we conclude that if $k<\sqrt{n}\delta_1$, then $\gamma_{n,k,\epsilon}(a)\overset{\delta\to0}{\to}0$. The same is valid for the set $\mathcal{B}$ instead of $\mathcal{A}$. Let $a>0,\ b\ge0$ be some reals. If a step function starts in $(c\sqrt{n},d\sqrt{n})$ and reaches rectangle $x=(c+a\pm\epsilon_1)\sqrt{n},\ y=(d-b\pm\epsilon_2)\sqrt{n}$ in some point $(x_p,y_p)$ and on this interval it has $k$ horizontal segments, it is possible for it to have $k-1,k$ or $k+1$ vertical segments. Denote

$$A_{k_1,j}^{n,\epsilon_1,\epsilon_2}(a,b)=\sum_{k=k_1}^{\min\{(a+\epsilon_1)\sqrt{n},(b+\epsilon_2)\sqrt{n}+j\}}\#_{\mathcal{A}}^{k,\epsilon_1}\left(\frac{a\sqrt{n}}{k}\right)\#_{\mathcal{B}}^{k-j,\epsilon_2}\left(\frac{b\sqrt{n}}{k-j}\right),\tag{8}$$

$\epsilon_1,\epsilon_2>0$, then

$$\#_{\mathcal{A},\mathcal{B}}^{n,\epsilon_1,\epsilon_2}(a,b)=\left(A_{1,0}^{n,\epsilon_1,\epsilon_2}(a,b)+A_{1,1}^{n,\epsilon_1,\epsilon_2}(a,b)+A_{1,-1}^{n,\epsilon_1,\epsilon_2}(a,b)\right)2^{\sqrt{n}o_{\epsilon_1,\epsilon_2}(1)}.\tag{9}$$

Here term $2^{\sqrt{n}o_{\epsilon_1,\epsilon_2}(1)}$ arises from the fact that we count the step functions according to their end points in the rectangle and one step function can have several nodes in the same rectangle.

We underline that we write the same symbol $o(1)$ for different values which have the property that they tend to zero as parameters tend to infinity or zero according to their meaning and the first limit is taken for the left parameter in their order. In particular $o_{\epsilon_1,\epsilon_2}(1)$ in the left hand side and right hand side of (9) are different and first $\epsilon_1 \to 0$ and then $\epsilon_2 \to 0$.

Next we omit index $j$ in $A^{n,\epsilon_1,\epsilon_2}_{k_1,j}(a,b)$ because the value $j$ does not change the asymptotics of the expressions. Next using Lemma 1 and (8) we have

$$\ln A^{n,\epsilon_1,\epsilon_2}_{\delta_1\sqrt{n}}(a,b) = \max_{k=\delta_1\sqrt{n},\ldots,\max\{(a+\epsilon_1)\sqrt{n},(b+\epsilon_2)\sqrt{n}\}} \left[k\left(J_{\mathcal{A}}\left(\frac{a\sqrt{n}}{k}\right)+J_{\mathcal{B}}\left(\frac{b\sqrt{n}}{k}\right)\right)\right]+o_{k,\epsilon_1,\epsilon_2}(1)\sqrt{n}. \tag{10}$$

Denote

$$A(\kappa,\xi) = \kappa(J_{\mathcal{A}}(1/\kappa)+J_{\mathcal{B}}(\xi/\kappa)), \kappa,\xi \geq 0.$$

Then the expression in the square brackets in the right hand side of (8) can be written as $a\sqrt{n}A(k/(a\sqrt{n}),\, b/a)$. Function $A(\kappa,\xi)$ for given $\xi \in [0,\infty)$ is continuous and non increasing when

$$0 \leq \kappa \leq \kappa_1 \stackrel{\Delta}{=} \min\{1/\ell^{\mathcal{A}}_{\min}, \xi/\ell^{\mathcal{B}}_{\min}\}$$

and $= -\infty$ when $\kappa > \kappa_1$. Note also, that for sufficiently small $\kappa > 0$, $A(\kappa, b/a) > -\infty$ for the arbitrary $a, b > 0$ and hence making $\delta_1 > 0$ sufficiently small we achieve the situation where

$$\max_{\kappa\in[\delta_1/a,\min\{1,b/a\}]} A(\kappa, b/a) > -\infty$$

and the interval $[\delta_1/a, \min\{1, b/a\}]$ is nonempty. Thus we have $\frac{\ln A^{n,\epsilon_1,\epsilon_2}_{\delta_1\sqrt{n}}(a,b)}{\sqrt{n}} > -\infty$.

Combining this inequality, relations (9) (10) we have $(a, b > 0)$

$$\begin{aligned} a \max_{\kappa\in[\delta_1/a,\min\{1,b/a\}]} A(\kappa,b/a) + o_{\delta_1,\epsilon_1}(1) + o_{\delta_1,\epsilon_2}(1) &\leq \liminf_{n\to\infty} \frac{\ln \#^{n,\epsilon_1,\epsilon_2}_{\mathcal{A},\mathcal{B}}(a,b)}{\sqrt{n}} \qquad (11)\\ \leq \limsup_{n\to\infty} \frac{\ln \#^{n,\epsilon_1,\epsilon_2}_{\mathcal{A},\mathcal{B}}(a,b)}{\sqrt{n}} &\leq a \max_{\kappa\in[\delta_1/a,\min\{1,b/a\}]} A(\kappa,b/a) + o_{\delta_1,\epsilon_1}(1) + o_{\delta_1,\epsilon_2}(1). \end{aligned}$$

Because the estimated value in (11) does not depends on $\delta_1$ we can choose $\delta_1 \to 0$ and obtain the relations

$$\begin{aligned} & a \sup_{\kappa\in(0,\min\{1,b/a\}]} A(\kappa,b/a) + o_{\epsilon_1}(1) + o_{\epsilon_2}(1) \qquad (12)\\ \leq\; & \liminf_{n\to\infty} \frac{\ln \#^{n,\epsilon_1,\epsilon_2}_{\mathcal{A},\mathcal{B}}(a,b)}{\sqrt{n}} \leq \limsup_{n\to\infty} \frac{\ln \#^{n,\epsilon_1,\epsilon_2}_{\mathcal{A},\mathcal{B}}(a,b)}{\sqrt{n}} \\ \leq\; & a \sup_{\kappa\in(0,\min\{1,b/a\}]} A(\kappa,b/a) + o_{\epsilon_1}(1) + o_{\epsilon_2}(1). \end{aligned}$$

Note, that $A(\kappa, b/a)$ is continuous at $\kappa = 0$, $A(0, b/a) = 0$ and thus we can change the range of $\kappa$ in the sup in (12) to $[0, \min\{1, b/a\}]$ and change sup to max.

Now if $b = 0, a > 0$, then for sufficiently small $\epsilon_1, \epsilon_2$

$$\frac{\ln \#^{n,\epsilon_1,\epsilon_2}_{\mathcal{A},\mathcal{B}}}{\sqrt{n}} = o_{\epsilon_1}(1) + o_{\epsilon_2}(1). \tag{13}$$

This is because the value $k$ cannot exceed $O(\epsilon_2\sqrt{n}/\ell^{\mathcal{B}}_{\min})$, otherwise $\#^{k,\epsilon_2}_{\mathcal{B}}(b) = 0$ and the estimation (7) shows that $\frac{\ln \#^{n,\epsilon_1,\epsilon_2}_{\mathcal{A},\mathcal{B}}}{\sqrt{n}}$ is as in (13). Also $\max_{\kappa\in[0,\min\{1,b/a\}]} A(\kappa,0) = 0$ and hence the formula from Lemma 2 is also valid in the case $b = 0$. Lemma 2 is proved.

**Proof of the Theorem 1.** First of all we investigate the properties of the function

$$N(\xi) = \max_{\kappa\in[0,\min\{1,\xi\}]} \kappa\left(J_{\mathcal{A}}(1/\kappa) + J_{\mathcal{B}}(\xi/\kappa)\right). \tag{14}$$

Function $N(\xi)$ is $\bigcap-$convex. Indeed $\lambda_1(\lambda_2)$, at which the inf in the definition of $J_{\mathcal{A}}(J_{\mathcal{B}})$ from (14) is achieved should satisfy the equalities

$$\kappa\frac{\sum_{\ell\in\mathcal{A}} \ell e^{\lambda_1\ell}}{\sum_{\ell\in\mathcal{A}} e^{\lambda_1\ell}} = 1,\ \kappa\frac{\sum_{\ell\in\mathcal{B}} \ell e^{\lambda_1\ell}}{\sum_{\ell\in\mathcal{B}} e^{\lambda_1\ell}} = \xi. \tag{15}$$

Here $\kappa$ is the value at which the max in the definition of $N(\xi)$ is achieved. Then $\lambda_1, \lambda_2$ should also satisfy the equality

$$\sum_{\ell\in\mathcal{A}} e^{\lambda_1\ell}\sum_{\ell\in\mathcal{B}} e^{\lambda_2\ell} = 1 \tag{16}$$

(here we assume $0\cdot\infty = 1$). Equations (15) are obtained by setting the derivatives in $\lambda_1(\lambda_2)$ under the inf in the expressions for $J_{\mathcal{A}}(1/\kappa)(J_{\mathcal{B}}(\xi/\kappa))$ to zero and expression (16) is obtained by setting the derivative of $A(\kappa,\xi)$ by $\kappa$ to zero and using equations (15). It can be easily seen that, when $\xi > 0$ and $\kappa$ vary in the interval $[0,\min\{1/\ell^{\mathcal{A}}_{\min}, \xi/\ell^{\mathcal{B}}_{\min}]$ and equations (15) are valid, $A'(\kappa,\xi)$ varies in the range from $-\infty$ to $\infty$. Because $A'_\kappa(\kappa,\xi)$ is continuous when (15) are valid, we obtain that there exists a $\kappa$ such that (15) are valid and $A'_\kappa(\kappa,\xi) = 0$.

When $\xi = 0$ we have $\kappa = 0, \lambda_1 = 0$. In this case we set $\lambda_2 = -\infty$. Then (15), (16) are also valid in the case $\xi = 0$.

Since for $\xi > 0$ $A(\kappa,\xi)$ is finite iff $\kappa\in[0,\min\{1/\ell^{\mathcal{A}}_{\min}, \xi/\ell^{\mathcal{B}}_{\min}\}]$, $A''_{\kappa\kappa}(\kappa,\xi) < 0$ and $A(\kappa,\xi) = -\infty$ otherwise, we obtain that $A(\kappa,\xi)$ is $\bigcap-$convex. If $\xi = 0, A(0,0) = 0$ and $A(\kappa,0) = -\infty,\ \kappa > 0$, hence $A(\kappa,0)$ is $\bigcap-$convex also.

Now we have the system of equations (15), (16) which determines $\lambda_1(\xi), \lambda_2(\xi), \kappa(\xi)$ in all cases where $N(\xi) > -\infty$. From the previous considerations follows that $N(\xi) > -\infty$ when $\xi \geq 0$.

Function $N(\xi)$ is $\bigcap-$convex. Indeed, from the relations (15), (16) follows that

$$N(\xi) = -\lambda_1(\xi) - \xi\lambda_2(\xi) \tag{17}$$

and

$$N''_{\xi\xi} = -\lambda_2'(\xi) \leq 0. \tag{18}$$

Also we need the following estimate

$$N(z+\xi) - N(z) \leq N(\xi),\ z,\xi \geq 0. \tag{19}$$

Indeed

$$N'(z) = -\lambda_2(z)$$

and

$$N'(z+\xi) - N'(z) = \lambda_2(z) - \lambda_2(z+\xi) \leq 0.$$

Now to prove Theorem 1 we prove two statements from which the theorem follows.

**Statement 1** *The following bound is valid*

$$\lim_{r\to\infty} \limsup_{\delta\to 0} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\ln \#^{n,\epsilon,\delta,r}(f)}{\sqrt{n}} = K_1(f) \leq L(f). \tag{20}$$

**Statement 2** *The following bound is valid*

$$\lim_{r\to\infty} \liminf_{\delta\to 0} \liminf_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\ln \#^{n,\epsilon,\delta,r}(f)}{\sqrt{n}} = K_2(f) \geq L(f) \tag{21}$$

**Proof of Statement 1.** First note that if $f \notin \mathcal{C}$, then $K_1(f) = -\infty$. Indeed if $\int_0^\infty f dx > 1$ or $\nu(f < 0) > 0$ or $\nu(\hat{f}' > 0) > 0$ ($\nu$ is Lebesgue measure), then for sufficiently large $r$ and $n$ and small $\epsilon, \delta > 0 \quad B(f,\epsilon,r) \bigcap S_{n,\delta,r} = \emptyset$.

We fix some $r > 1$ and $f \in \mathcal{C}$ and consider the decomposition

$$\hat{f} = \hat{f}^1 + \hat{f}^2,$$

where $\hat{f}^1$ is absolutely continuous and $\hat{f}^2$ singular monotone components of $\hat{f}$. Note that $\hat{f}^2$ is continuous from the right, because $\hat{f}$ is.

**Now we consider three cases. First case:** $\hat{f}(x) > 0,\ x \in [0,\infty)$, **second case:** $\hat{f}(x) = 0$ for some $x \in (0,\infty)$ and **last case:** $\hat{f}(x) \equiv 0$.

Let's make a note. Step functions $\varphi_{n,\delta}$ have no horizontal and (certainly) no vertical segments after it achieves the $X$-axis (when $\varphi_{n,\delta}(x) = 0$ for the first time). Otherwise we would have an infinite number of step functions, because we can continue the step function after $x_0$, where $\varphi_{n,\delta}(x_0) = 0$ for the first time by an infinite number of horizontal segments from $\mathcal{A}$.

Now we partition the interval $[1/r, r]$ into consecutive intervals $[x_i, x_{i+1}]$ and to make our considerations proper we must assume that every step function $\varphi_{n,\delta}$ has a node in each rectangle $R_i = (x_i \pm o_\epsilon(1), \hat{f}(x_i) \pm o_\epsilon(1))$. We will prove that every $\varphi_{n,\delta}$ has the property

$$|\varphi_{n,\delta}(x_i) - \hat{f}(x_i)| < o_\epsilon(1).$$

Thus to have a $\varphi_{n,\delta}$, which has a node in $R_i$, we should consider only $\varphi_{n,\delta}$ which have a node with coordinate $x \in (x_i \pm o_\epsilon(1))$.

At the same time it is possible to have the situation where $\varphi_{n,\delta}$ has very long horizontal segments, so long, that they pass throw the intervals $x_i \pm o_\epsilon(1)$ without nodes. The smaller

$\hat{f}(x_i)$ is, the longer segments can occur. It will become clear soon that the logarithmic asymptotics of the number $\#\{S_{n,\delta,r}\bigcap B(f,\epsilon,r)\}$ and the number of $\varphi_{n,\delta}\in B(f,\epsilon,r)$ without long segments (length exceeds $o_\epsilon(1)$) is the same, if we consider the functions $f$ such that $\hat{f}(f) > C_0 > 0$. If $\hat{f}(r)=0$ and $r_1$ is the smallest number with this property, our next reduction in that form does not work and we will consider some $r_0$ such that $\hat{f}(r_0) > C_0 > 0$ for some $C_0$ and make the considerations similar to the first case of positive $\hat{f}$ not on the whole interval $[1/r,r]$, but on the interval $[1/r,r_0]$ (surely we should have $1/r < r_0$) and on the interval $[r_0,r]$) estimate the number of restrictions of $\varphi_{n,\delta}$ such that

$$\left|\int_{r_0}^{r}\varphi_{n,\delta}(x)dx-\int_{r_0}^{r}f(x)dx\right| < o_\epsilon(1)$$

and $\int_{r_0}^{r}f(x)dx\to 0$ as $r_0\to r_1$. It means that

$$\int_{r_0}^{r}\varphi_{n,\delta}(x)dx=o_{\epsilon,r_0}(1). \tag{22}$$

The whole number of restrictions on the interval $[r_0,r]$ of the step functions $\varphi_{n,\delta}$, which satisfy (22), is small, less than the whole number (without restrictions) of step functions with area $o_\epsilon(1)$. By the Hardy-Ramanujan formula this number is less than $o_{\epsilon,r_0}(1)\sqrt{n}$.

Let's proceed first with the case $\hat{f}>0,\ x\geq 0$. Let $x_0\in[1/r,r]$ be such that $|\hat{f}'(x_0)|<C_1$ for some $C_1$. If $\varphi_{n,\delta}\in B(f,\epsilon,r)$, then

$$|\varphi_{n,\delta}(x_0)-\hat{f}(x_0)| < o_\epsilon(1). \tag{23}$$

Indeed it is easy to check, considering the graphs of the functions $\varphi_{n,\delta},\hat{f}$, that otherwise, because functions are monotone, the area of the gap between the graphs of these functions in the neighborhood of $x_0$ would be positive and non vanished as $\epsilon\to 0$ which is the contradiction with the condition $\varphi_{n,\delta}\in B(f,\epsilon,r)$.

Now consider the whole set $B_0=S_{n,\delta,r}\bigcap B(f,\epsilon,r)$. Let $B_1\subset B_0$ be the set of step functions which contains exactly one horizontal segment of length exceeding $o_\epsilon(1)$ when $x\in[0,r]$. This segment can start at any point $0,1/\sqrt{n},\dots,r$ on $X$ and can have the length $\rho$ such that $\rho\leq\rho_{\max}$, where

$$(\hat{f}(r)-o_\epsilon(1))\rho_{\max}=1+\delta. \tag{24}$$

This follows from the fact that the area under this segment should be less than $1+\delta$ and the length can be maximal, when the $Y$ coordinate of the segment is minimal (this is why we take care about positivity of $\hat{f}$).

Let $B_{11}$ be the subset of $B_1$ consisting of the step functions with the property that $B_{11}$ is the maximal subset of $B_1$ over the choice of long horizontal segment $\sigma$. Then

$$\#\{B_1\}\leq rn\rho_{\max}\#\{B_{11}\}\leq\frac{rn(1+\delta)}{\hat{f}(r)-o_\epsilon(1)}\#\{B_{11}\}. \tag{25}$$

The coefficient before $\#\{B_{11}\}$ is the number of possible choices of segment $\sigma$.

Now we make the following procedure with the elements from $B_{11}$. We take the segment $\sigma$ and consider the partition of it into $\lfloor \frac{\rho\sqrt{n}}{\ell} \rfloor$ parts for some $\ell \in \mathcal{A}$. If $\ell \nmid \rho\sqrt{n}$ then we simply omit the rest after the division and have $\lfloor \frac{\rho\sqrt{n}}{\ell} \rfloor$ parts of length $\ell$. Then between each consecutive parts we insert the vertical segment of length $\ell_{\min}^{\mathcal{B}}$. It will be $\lfloor \frac{\rho\sqrt{n}}{\ell} \rfloor$ insertions. Then we lift up our construction simultaneously with the whole left to $\sigma$ piece step function in such a way that a proper connection with the right to $\sigma$ part of $\varphi_{n,\delta}$ (the new step function should be also monotone) can be made. We receive new step function $\varphi_n^1$, such that $\varphi_n^1(x) \geq \varphi_{n,\delta}(x)$,

$$\max_x(\varphi_n^1(x) - \varphi_{n,\delta}(x)) < \frac{\rho}{\ell}\ell_{\min}^{\mathcal{B}} \tag{26}$$

and

$$\int_0^\infty (\varphi_n^1(x) - \varphi_{n,\delta}(x))dx < \frac{\rho^2}{2\ell}\ell_{\min}^{\mathcal{B}}. \tag{27}$$

Here (26) is obvious and (27) follows from the fact that the area under $\varphi_{n,\delta}$ after lifting increases on the area of the triangle with edges $\rho$, $\frac{\rho}{\ell}\ell_{\min}^{\mathcal{B}}$.

From (26) and (27) follows that choosing small $\epsilon > 0$ and sufficiently large $\ell \in \mathcal{A}$ we can make the expression in the right hand side of (26) arbitrary small and if

$$\varphi_{n,\delta,r} \in B(f,\epsilon,r)$$

then

$$\varphi_{n,r}^1 \in B(f,2\epsilon,r).$$

Also from (27) we have

$$\int_0^\infty \varphi_{n,r}^1(x)dx < 1 + 2\delta$$

and hence $\varphi_{n,r}^1 = \varphi_{n,2\delta,r}$. Note, that we can do this procedure for each $\varphi_{n,\delta} \in B_{11}$ and obtain the new set $\tilde{B}_{11}$. Also this procedure establishes the one-to-one correspondence between $B_{11}$ and $\tilde{B}_{11}$. Since, as it is shown, $\tilde{B}_{11}$ is the subset of $B(f,2\epsilon,r)$, elements of $\tilde{B}_{11}$ have nodes in the arbitrary interval $x_0 \pm o_\epsilon(1)$, $x_0 \in [0,r]$, and $\tilde{B}_{11}$ consists of step functions $\varphi_{n,2\delta}$. We denote this set by $S'_{n,2\delta}(f,2\epsilon,r)$. We have

$$\#\{B_{11}\} \leq \#\{S'_{n,2\delta}(f,2\epsilon,r)\} \tag{28}$$

and from (25) follows

$$\#\{B_1\} \leq \frac{r(1+\delta)}{\hat{f}(r) - o_\epsilon(1)} n\#\{S'_{n,2\delta}(f,2\epsilon,r)\}. \tag{29}$$

Now the reader easily reconstruct the similar considerations for the subsets $B_i$ when $i$ long segments appear. We omit these details and only demonstrate the final relation

$$\#\{S_{n,\delta}\bigcap B(f,\epsilon,r)\} \leq \#\left\{\bigcup_{i=0}^{\lfloor r/o_\epsilon(1)\rfloor}\left(B_i\bigcap S'_{n,\delta}(f,\epsilon,r)\right)\right\} \leq C_2 n^{\lfloor r/o_\epsilon(1)\rfloor + 1/2}\#\{S'_{n,2\delta}(f,2\epsilon,r)\}. \tag{30}$$

Hence, if we estimate the value $\#\{S'_{n,2\delta}(f,2\epsilon,r)\}$, then we find the estimate for $\#\{S_{n,\delta}\bigcap B(f,\epsilon,r)\}$ and the coefficient in the right hand side of (30) does not influence the logarithmic asymptotics of the estimates.

Now we will show how to construct the upper estimate for the number $\#\{S'_{n,2\delta}(f,2\epsilon,r)\}$. In order to do it we preliminary make the choice of the intervals, on which we count the number of restrictions of $\varphi_{n,2\delta}$ with prescribed properties. Then we will use the multiplicative property: the number of restrictions of step functions on the whole interval $[1/r,r]$ is equal to the product of the number of restrictions of step functions on the subintervals which are the partition of $[1/r,r]$. Now we come to the precise formulations.

We define the measure $\beta((a,b])=\hat{f}^2(a)-\hat{f}^2(b)$. From the regularity of the Lebesgue measure follows that there exists an open set $B\in[1/r,r]$, $\beta(B)=\beta([1/r,r])$ such that for the arbitrary given $\delta_1>0$, $\nu(B)<\delta_1$. The set $B$ is the union of not more than a countable number of intervals $B_i$ and for some $m$ we have $\beta\left(\bigcup_{i>m}B_i\right)<\delta_1$. Next we add to every interval $B_i$, $i\le m$ its boundary points and obtain the closed intervals $\bar{B}_i$ and the union $\bigcup_{i=1}^m\bar{B}_i$ is the union of a finite number $m_1$ of nonintersecting closed intervals $[a_i,b_i]$, $i\le m_1$, $\bigcup_{i=1}^{m_1}[a_i,b_i]=\bigcup_{i=1}^m\bar{B}_i$. Then

$$\nu\left(\bigcup_{i=1}^{m_1}[a_i,b_i]\right)<\delta_1$$

and the set $[1/r,r]\setminus\bigcup_{i=1}^{m_1}(a_i,b_i)=\bigcup_{i=1}^{s}[c_i,d_i]$ is the union of a finite number $s$ of nonintersecting closed intervals and $\{c_i,d_i\}\subset\{1/r,r,a_i,b_i\}$. For every $i=1,\dots,s$ consider the decomposition

$$[c_i,d_i]=\bigcup_{j=1}^{s_i}[c_i^j,d_i^j]$$

of the interval $[c_i,d_i]$ into $s_i$ consecutive subintervals of 'almost' equal length $d_i^j-c_i^j\approx(d_i-c_i)/s_i$. 'Almost' means the following: we assume that $|\hat{f}'(x)|<C_1$, $x\in\{c_i^j,d_i^j\}$ for some constant $C_1$, otherwise, we slightly move points $c_i^j$, $d_i^j$ in order to satisfy this condition and previous conditions, connected with choices of these intervals.

The set $S'_{n,2\delta}(f,2\epsilon,r)$ has the property that each step function $\varphi_{n,2\delta}\in S'_{n,2\delta}(f,2\epsilon,r)$ has a node in each rectangle $R_i^j=\{x=c_i^j\pm o_\epsilon(1),y=\hat{f}(c_i^j)\pm o_\epsilon(1)\},\tilde{R}_i^j=\{x=d_i^j\pm o_\epsilon(1),y=\hat{f}(d_i^j)\pm o_\epsilon(1)\}$. The total number $\Phi_{i,j,n}$ of the restrictions of the step functions from $S'_{n,2\delta}(f,2\epsilon,r)$ starting in the rectangle $\Delta=(x=c_i^j\pm o_\epsilon(1),\hat{f}(c_i^j)\pm o_\epsilon(1))$ can be estimated as follows

$$\frac{\Phi_{i,j,n}}{\sqrt{n}}\le\Delta x_i^jN\left(-\frac{\Delta\hat{f}_i^j}{\Delta x_i^j}\right)+o_{n,\epsilon}(1),$$

where $\Delta x_i^j=d_i^j-c_i^j$, $\Delta\hat{f}_i^j=\hat{f}(d_i^j)-\hat{f}(c_i^j)$. This follows from the Lemma 2 and the fact that the number of nodes in $\Delta$ is $O(n)$.

Next we will use the multiplicative property for $\Phi_{i,j,n}$, which tells that the number $\Phi_n=\#\{S'_{n,2\delta}(f,2\epsilon,r)\}$ is upper bounded by the product of the restrictions of these functions on subintervals, which form the partition of the large interval (with intersections of the subintervals only on boundaries). Since $\{[c_i^j, d_i^j], [a_\ell, b_\ell]\}$ is the partition of the interval $[1/r, r]$ we have

$$
\begin{aligned}
\frac{\ln \Phi_n}{\sqrt{n}} &\leq \frac{\ln\left(\prod_{i,j}\Phi_{i,j,n} \times \prod_\ell \Phi_{\ell,n}\right)}{\sqrt{n}} \\
&\leq \sum_{i,j}\frac{\ln \Phi_{i,j,n}}{\sqrt{n}} + \sum_\ell \frac{\Phi_{\ell,n}}{\sqrt{n}} + s' o_{n,\epsilon}(1) \\
&\leq \sum_{i,j}\Delta x_i^j N\left(-\frac{\Delta \hat{f}_i^j}{\Delta x_i^j}\right) + \sum_\ell \Delta x_\ell N\left(-\frac{\Delta \hat{f}_\ell}{\Delta x_\ell}\right) + s' o_{n,\epsilon}(1). \qquad (31)
\end{aligned}
$$

Here $\Phi_{\ell,n}$ is the number of restrictions of the step functions from $S'_{n,2\delta}(f, 2\epsilon, r)$ on the interval $[a_\ell, b_\ell]$, $s' = \sum_{i=1}^s s_i + m_1$, $\Delta x_\ell = b_\ell - a_\ell$, $\hat{f}_\ell = \hat{f}(b_\ell) - \hat{f}(a_\ell)$. Here we once more use the Lemma 2.

Then the whole number $\#^{n,\delta,\epsilon,r}(f)$ is estimated by the product of $\Phi_n$ and the numbers $\alpha_{n,r}$ and $\beta_{n,r}$ of restrictions of the step functions on the interval $[0, 1/r]$ or $[r, \infty)$ correspondingly. Each of these numbers can be estimated by the number of step functions $\tilde{\varphi}_n$ without restrictions on steps such that $\max_{x\in[0,\infty)} \tilde{\varphi}_n(x) \leq (1+2\delta)/r$. Indeed on the left interval $[0, 1/r]$, if we exchange axis $X$ and $Y$ we obtain from $\varphi_n(x)$, $x \in [0, 1/r]$ step function $\tilde{\varphi}_n$. Also, for $x \geq r$, $\varphi_{n,2\delta}(x) \leq (1+2\delta)/r$, because $\int_0^\infty \varphi_{n,2\delta}(x)dx \leq 1 + 2\delta$.

Later we will prove that

$$
\frac{\ln \alpha_{n,r}}{\sqrt{n}},\ \frac{\ln \beta_{n,r}}{\sqrt{n}} = o_{n,\epsilon,r}(1). \qquad (32)
$$

Next we show that the contribution of $\sum_\ell$ to the estimate (31) can be made arbitrary small. Indeed $N(\xi)$ is $\bigcap-$convex and from Jensen inequality follows that

$$
\sum_\ell \Delta x_\ell N\left(-\frac{\Delta \hat{f}_\ell}{\Delta x_\ell}\right) \leq N\left(-\frac{\sum_\ell \Delta \hat{f}_\ell}{\sum_\ell \Delta x_\ell}\right)\sum_\ell \Delta x_\ell.
$$

Then we have

$$
\begin{aligned}
N(z) &= \max_{\kappa\in[0,\min\{1,z\}]} \kappa(J_{\mathcal{A}}(1/\kappa) + J_{\mathcal{B}}(z/\kappa)) \\
&\leq \max_{\kappa\in[0,\min\{1,z\}]}\left(\inf_{\lambda\leq 0}(\kappa \ln \sum_{\ell\in\mathcal{A}} e^{\lambda_1\ell} - \lambda_1) + \inf_{\lambda_2\leq 0}(\kappa\ln\sum_{\ell\in\mathcal{B}} e^{\lambda_2\ell} - z\lambda_2)\right) = (1+z)H\left(\frac{z}{1+z}\right).
\end{aligned}
$$

We have, if $\beta((1/r, r]) = \hat{f}^2(1/r) - \hat{f}^2(r) > 0$, that for some $C_4, C_5 > 0$, $\infty > C_5 > \hat{f}(1/r) - \hat{f}(r) > |\sum_\ell \Delta \hat{f}_\ell| > C_4$. This follows from the choice of intervals $[a_\ell, b_\ell]$, where measure $\beta$ concentrates. Also we have $\sum_\ell \Delta x_\ell < \nu(B) < \delta_1$ and hence if we define $z = -\sum_\ell \Delta \hat{f}_\ell / \sum_\ell \Delta x_\ell$, then

$$
\frac{N(z)}{z}\left(-\sum_\ell \Delta \hat{f}_\ell\right) < C_5 \frac{N(z)}{z} \to 0 \text{ as } \delta_1 \to 0.
$$

If $\hat{f}^2(1/r) - \hat{f}^2(r) = 0$, then we do not consider intervals $[a_\ell, b_\ell]$ at all.

Thus the sum on $\ell$ in the right hand side of (31) can be made arbitrary small as $\delta_1 \to 0$.

Next we deal with the term $\sum_{i,j}$ in (31). From (19) we have ($\Delta \hat{f}_i^j = \Delta \hat{f}_i^{1j} + \Delta \hat{f}_i^{2j}$)

$$N\left(-\frac{\Delta \hat{f}_i^j}{\Delta x_i^j}\right) \le N\left(-\frac{\Delta \hat{f}_i^{1j}}{\Delta x_i^j}\right) + N\left(-\frac{\Delta \hat{f}_i^{2j}}{\Delta x_i^j}\right).$$

Using Jensen inequality we obtain

$$\sum_{i,j} \Delta x_i^j N\left(-\frac{\Delta \hat{f}_i^{2j}}{\Delta x_i^j}\right) \le \gamma \triangleq \left(\sum_{i,j} \Delta x_i^j\right) N\left(-\frac{\sum_{i,j} \Delta \hat{f}_i^{2j}}{\sum_{i,j} \Delta x_i^j}\right). \tag{33}$$

Since $\sum_{i,j} \Delta \hat{f}_i^{2j} = \beta\left(\bigcup_{i>m} B_i\right) < \delta_1$ and $\sum_{i,j} \Delta x_i^j > r - 1/r - \delta_1$ we have

$$\gamma \to 0 \text{ as } \delta_1 \to 0. \tag{34}$$

Now we estimate the term $\alpha_{n,r}$ ($\beta_{n,r}$ will have the same estimation). As we mentioned before $\alpha_{n,r}$ has as upper bound the logarithm of the number of partitions of $n(1 \pm 2\delta)$ with maximal element of each partition less than $\sqrt{n}/r$. We must estimate from above the number of solutions of the equation

$$\sum_{i=1}^{\sqrt{n}/r} i x_i \le n(1+2\delta), \;\; x_i = 0, 1, 2, \ldots.$$

This can be done by simple using of Chebyshev's inequality, we omit the details and it follows that

$$\limsup_{r\to 0} \limsup_{n\to\infty} \frac{\ln \# \left\{ (x_i)_1^{\sqrt{n}/r} \; \sum_{i=1}^{\sqrt{n}/r} i x_i \le n(1+2\delta) \right\}}{\sqrt{n}} = 0. \tag{35}$$

Taking together all established facts (31), (32) (33), (34), (35) we can write the following chain of inequalities

$$\begin{aligned} \frac{\ln \#^{n,\delta,\epsilon,r}(f)}{\sqrt{n}} &\le \frac{\ln \Phi_n}{\sqrt{n}} + s' o_{n,\epsilon,\delta}(1) + o_{\delta_1}(1) + o_r(1) \\ &\le \sum_{i,j} \Delta x_i^j N\left(-\frac{\Delta \hat{f}_i^{1j}}{\Delta x_i^j}\right) + s' o_{n,\epsilon,\delta}(1) + o_{\delta_1}(1) + o_r(1) \end{aligned}$$

or

$$\limsup_{n\to\infty} \frac{\ln \#^{n,\delta,\epsilon,r}(f)}{\sqrt{n}} \le \sum_{i,j} \Delta x_i^j N\left(-\frac{\Delta \hat{f}_i^{1j}}{\Delta x_i^j}\right) + s' o_{\epsilon,\delta}(1) + o_{\delta_1}(1) + o_r(1). \tag{36}$$

We can rewrite the $\sum$ in the right hand side of (36) as follows

$$\sum_{i,j} \Delta x_i^j N\left(-\frac{1}{\Delta x_i^j} \int_{c_i^j}^{d_i^j} \hat{f}^{1\prime}(x) dx\right) = \int_{\bigcup_i [c_i, d_i]} N(-f_c(x)) dx,$$

where $f_c$ is a piecewise constant function such that for a given partition $\{[c_i^j, d_i^j]\}$ of the set $\bigcup_i [c_i, d_i]$

$$f_c(x) = \frac{1}{\Delta x_i^j} \int_{c_i^j}^{d_i^j} \hat{f}^{1\prime}(x)dx, \ x \in [x_i^j, d_i^j).$$

Taking $\epsilon \to 0, \delta \to 0$ we obtain from (36)

$$\begin{aligned} F &= \limsup_{\delta\to 0} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\ln \#^{n,\delta,\epsilon,r}(f)}{\sqrt{n}} \leq \sum_{i,j} \Delta x_i^j N\left(-\frac{\Delta \hat{f}_i^{1j}}{\Delta x_i^j}\right) + o_{\delta_1,r}(1) \\ &= \int_{\bigcup_i [c_i,d_i]} N(-f_c(x))dx + o_{\delta_1,r}(1). \end{aligned}$$

Next we take $\omega_i = d_i^j - c_i^j$ such that $\omega = \max_i \omega_i \to 0$ and we have $f_c(x) \to \hat{f}^{1\prime}(x)$ a.s. on $\bigcup_i [c_i, d_i]$ and

$$\begin{aligned} F &\leq \limsup_{\omega\to 0} \int_{[c_i,d_i]} N(-f_c(x))dx + o_{\delta_1,r}(1) \qquad (37) \\ &\leq \int_{\bigcup_i [c_i,d_i]} \limsup_{\omega\to 0} N(-f_c(x))dx = \int_{\bigcup_i [c_i,d_i]} N(\limsup_{\omega\to 0}(-f_c(x))dx \\ &= \int_{\bigcup_i [c_i,d_i]} N(-\hat{f}^{1\prime}(x))dx = \int_{\bigcup_i [c_i,d_i]} N(-\hat{f}'(x))dx \\ &\leq \int_{1/r}^{r} N(-\hat{f}'(x))dx. \end{aligned}$$

Here for simplicity we omit $o_{\delta_1}(1) + o_r(1)$ in the last relations. The first equality follows from the continuity of $N(\xi)$. In the second inequality we use Fatou Lemma which is possible to use, because $N(-\hat{f}'(x))$ is integrable on $[0, \infty)$ as follows from Theorem 2 (we will prove it later). The second equality in (37) follows from the fact that, if $y \in L^1([a, b], dx)$, then

$$\lim_{q\to\infty} \frac{1}{|D_q|} \int_{D_q} y(x)dx = y(x_0) \ a.s.,$$

where $(D_q)_1^\infty$ is an arbitrary sequence of closed intervals with nonempty interior such that $\bigcap_q D_q = \{x_0\}$. The third inequality in (37) follows from the fact that $\hat{f}^{1\prime} = \hat{f}'$ *a.s.*

Now we take $\delta \to 0, \delta_1 \to 0, r \to \infty$ at both sides of (37) and obtain the inequality

$$F \leq \int_0^\infty N(-\hat{f}'(x))dx.$$

Thus (20) is proved for strictly positive functions.

Now we describe how to deal with functions $f \in \mathcal{C}$ such that $\hat{f}(x_0) = 0$ for some $0 < x_0 < \infty$ and $x_0$ is minimal with this property. We will not show the whole proof in this case, because in many steps it is similar to the first case, but we will underline the differences in the proof. Consider once more the interval $[1/r, r]$, $r > x_0$ and another interval $[1/r, r_0]$, $r_0 < x_0$. On the interval $[1/r, r_0]$ we make the same considerations and estimates as in the first case on the interval $[1/r, r]$. Thus we estimate the number of restrictions $S'_{n,2\delta}(f, 2\epsilon, \{r, r_0\})$, it has the same meaning as $S'_{n,2\delta}(f, 2\epsilon, r)$ but the restrictions of step functions are on the interval $[1/r, r_0]$. Then the number of restrictions of the step functions on intervals $[0, 1/r], [r, \infty)$ are estimated in the same way as in the first case and their asymptotics is $o_{n,r}(1)$. The number of restrictions of step functions on the interval $[r_0, r]$ is $o_{\epsilon,r_0}(1)$ as $\epsilon \to 0$ and $r_0 \to r_1$. This is due to the same argument as in estimating $\alpha_{n,r}$ or $\beta_{n,r}$ : the number of these restrictions is less than the number of step functions $\varphi_{n,\delta}$ with $\max_x \varphi_{n,\delta} \leq o_\epsilon(1)+\hat{f}(r_0) \to 0$ as $\epsilon \to 0,\ r_0 \to x_0$. Actually we can construct the upper bound in this case only on interval $[1/r, r_0]$ instead of $[1/r, r]$ and then $r_0 \to r$, but we choose the last interval to make the formulations of the Theorem 1 uniform in all cases. As before we obtain the estimate

$$F \leq \int_{1/r}^{r_0} N(-\hat{f}'(x))dx + o_{r_0,r}(1).$$

Taking $r_0 \to x_0, r \to \infty$ we obtain

$$F \leq \int_0^{x_0} N(-\hat{f}'(x))dx = \int_0^{\infty} N(-\hat{f}'(x))dx$$

and Statement 1 is proved in the second case.

The last case, when $\hat{f} \equiv 0$ can be done in a simple way. On the intervals $[0, 1/r],\ [r, \infty)$ we have as before the number of restrictions of $\varphi_{n,\delta}$ is $o_{n,r}(1)$ and $\int_{1/r}^{r} \varphi_{n,\delta}(x)dx < \epsilon$. Thus the number of restrictions of $S_{n,\delta}$ on $[1/r, r]$ is less than the whole number of step functions with the area $\leq \epsilon n$, which due to Hardy-Ramanujan result is $\sqrt{n} o_\epsilon(1)$. The product of these numbers of restrictions on the different parts of $[0, \infty)$ as before gives the upper bound on $\#^{n,\delta,\epsilon,r}(f)$ and

$$F \leq 0 = \int_0^{\infty} N(-\hat{f}'(x))dx.$$

Statement 1 is completely proved.

**Proof of Statement 2.** This proof is simpler than the proof of the upper bound (20), because now we do not care about the long horizontal segments. Choose the partition of the interval $[1/r, r]$ into $s$ consecutive intervals $[a_i, b_i]$ of equal length $\Delta = b_i - a_i = (r - 1/r)/s$. Note, that in the proof of (20) we consider also the contributions to $\#^{n,\delta,\epsilon,r}(f)$ of step functions whose restrictions does not belong to $B(f, \epsilon, r)$. To prove (21) we should restrict our attention only to the step functions whose restrictions belong to $B(f, \epsilon, r)$. As before we consider the subset of step functions $\varphi_{n,\delta}$ such that they have the node in each rectangle $(x = a_i \pm o_\epsilon(1), y = \hat{f}(a_i) \pm o_\epsilon(1)),\ i = 1, \ldots, s$ and in rectangle $(x = b_s \pm o_\epsilon(1), y = \hat{f}(b_s) \pm o_\epsilon(1))$. We choose $\varphi_{n,\delta},\ x \in [0, 1/r]$ or $x \in [r, \infty)$ in an arbitrary way such that $\varphi_{n,\delta} \in S_{n,\delta}$. Because we have restrictions on steps, sometimes it can happen that it is not possible to continue the step function with given restriction on $[1/r, r]$ to the intervals $[0.1/r)$ or $[r, \infty)$ without violation of the restrictions on steps. In such cases we shift the step function in vertical direction by not more than $\ell^{\mathcal{B}}_{\min}/\sqrt{n}$ units of the scaled integer lattice and in horizontal direction by not more than $\ell^{\mathcal{A}}_{\min}/\sqrt{n}$ units to obtain the step function which starts at $(0, p)$

and ends in $(q, 0)$ for some $p, q$. Because the number of shifts is finite it does not change the logarithmic asymptotics of the number of step functions.

Now we estimate the $L^1([1/r, r], dx)-$distance between the restrictions $S_{n,\delta,r}$ and $\hat{f}$. It can be easily seen that if the pair of monotone non-increasing functions $y, \varphi_{n,\delta}$ is such that

$$|y(x) - \varphi_{n,\delta}(x)| < \epsilon_1 \tag{38}$$

when $x = a, b,\ a < b$, then

$$\int_a^b |y_1(x) - y_2(x)| dx \leq (b - a)(y_1(a) - y_1(b) + 2\epsilon_1). \tag{39}$$

This is because the area restricted by the curves $y, \varphi_{n,\delta}$ and lines $x = a, x = b$ is covered by the rectangle with edges $y = y_1(a) + \epsilon_1,\ y = y_1(b) - \epsilon_1,\ x = a, x = b$. Let (38) be true for $y(x) = \hat{f}(x)$ and all $x = a_i$ and $x = b_i$. Then by (39) we have for every given $r$, sufficiently small $\Delta = \max_i(b_i - a_i)$ and $\epsilon_1$

$$\int_{1/r}^r |\varphi_{n,\delta}(x) - f(x)| dx < \epsilon. \tag{40}$$

Next as in the proof of the upper bound (20) the logarithm of the number of restrictions $S_{n,\delta}$ on interval $[a_i, b_i]$ is estimated from below by the value

$$\sqrt{n}\left[(b_i - a_i)N\left(\frac{\hat{f}(a_i) - \hat{f}(b_i)}{b_i - a_i}\right) + o_{n,\epsilon,\delta}(1)\right]. \tag{41}$$

Actually the step function can have a node in any point from the scaled lattice in the rectangle $(x = a_i \pm o_\epsilon(1), y = \hat{f}(a_i) \pm o_\epsilon(1))$ and end in rectangle $(x = b_i \pm o_\epsilon(1), y = \hat{f}(b_i) \pm o_\epsilon(1))$, but the number of points in these rectangles is $O(n)$ and this does not influence the logarithmic asymptotics in (41).

As before the contribution of all intervals $[a_i, b_i]$ in the lower estimation of $\#\{S_{n,\delta,r} \bigcap B(f, \epsilon, r)\}$ is bounded by the sum of values (41):

$$\sqrt{n}\left[\sum_{i=1}^s \Delta x_i N\left(-\frac{\hat{f}(b_i) - \hat{f}(a_i)}{b_i - a_i}\right) + s o_{n,\epsilon,\delta}(1)\right]$$

and taking into account the choice of $\varphi_{n,\delta}$ on the intervals $[0, 1/r), [r, \infty)$ we obtain the logarithmic asymptotics of the lower bound of the number $\#^{n,\epsilon,\delta,r}$ :

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\ln \#^{n,\epsilon,\delta,r}}{\sqrt{n}} &\geq \sum_{i=1}^s \Delta x_i N\left(-\frac{\Delta \hat{f}_i^1}{\Delta x_i} - \frac{\Delta \hat{f}_i^2}{\Delta x_i}\right) + s o_{\epsilon,\delta}(1) \\
\geq\ & \sum_{i=1}^s \Delta x_i N\left(-\frac{\Delta \hat{f}_i^1}{\Delta x_i}\right) + s o_{\epsilon,\delta}(1) \\
\geq\ & \sum_{i=1}^s \Delta x_i N\left(\frac{\int_{b_i}^{a_i} \hat{f}^{1\prime}(x) dx}{\Delta x_i}\right) + s o_{\epsilon,\delta}(1) \\
\geq\ & \sum_{i=1}^s \int_{a_i}^{b_i} N(-\hat{f}^{1\prime}(x)) dx + s o_{\epsilon,\delta}(1) = \int_{1/r}^r N(-\hat{f}^{1\prime}(x)) dx + s o_{\epsilon,\delta}(1).
\end{aligned}$$

Here the second inequality follows from the fact that $N(\xi)$ is a monotone function.

Taking limits from both sides of the last chain of inequalities we obtain the inequality

$$\liminf_{r\to\infty}\liminf_{\delta\to 0}\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\ln \#^{n,\epsilon,\delta,r}}{\sqrt{n}} \geq L(f).$$

This proves (21) and the Statement 2.

Now we turn to the proof of Theorem 2. First we prove, that

$$\sup_{f\in\mathcal{C}} L(f) = \sup_{f\in\mathcal{C}_1} L(f). \tag{42}$$

We will prove more, namely that the sup in the right hand side of (42) is achieved on the functions $y \in \mathcal{C}_1$ such that

$$\int_0^\infty y(x)dx < 1. \tag{43}$$

To prove this it is enough to show that for $\epsilon > 0$ and each $f \in \mathcal{C}$ such that

$$\int_0^\infty f(x)dx \leq 1$$

there exists a $y \in \mathcal{C}_1$ such that

$$\int_0^\infty |N(-y'(x)) - N(-\hat{f}'(x))|dx < \epsilon_1 \tag{44}$$

and $\int_0^\infty y(x)dx < 1$. Choose $x_0 > 0$ such that

$$\int_0^{1/x_0} N(-\hat{f}'(x))dx,\ \int_{x_0}^\infty N(-\hat{f}'(x))dx < \epsilon_1. \tag{45}$$

This is always possible, because

$$\int_0^\infty N(-\hat{f}'(x))dx < \infty.$$

Now denote $a = ess\inf_{x\notin A_0} \hat{f}'(x)$, where $A_0 = \{x :\ \hat{f}'(x) = 0\}$ and denote for $\beta > \alpha > \gamma > 0$ the restricted function

$$\hat{f}'(x,\alpha,\beta,\gamma) = \begin{cases} \hat{f}'(x), & \alpha - a \leq -\hat{f}'(x) \leq \beta,\ x \in [1/x_0, x_0], \\ a-\gamma, & -\hat{f}'(x) < \alpha - a, x \in [1/x_0, x_0], \\ -\beta, & -\hat{f}'(x) > \beta,\ x \in [1/x_0.x_0], \\ 0, & x \notin [1/x_0, x_0] \end{cases}$$

and $\hat{f}(x,\alpha,\beta,\gamma) \to 0$ as $x \to \infty$. Next we choose $\beta \to \infty$ and $\alpha \to 0$. Then we can choose $\beta > \alpha > \gamma > 0$ such that

$$\int_{1/x_0}^{x_0} |N(-\hat{f}'(x,\alpha,\beta,\gamma)) - N(-\hat{f}'(x))|dx < \epsilon_1 \tag{46}$$

and

$$\int_{1/x_0}^{x_0} |\hat{f}'(x,\alpha,\beta,\gamma) - \hat{f}'(x)|dx < \delta_1, \tag{47}$$

$$\int_0^\infty \hat{f}(x,\alpha,\beta,\gamma)dx < \int_0^\infty f(x)dx - \delta. \tag{48}$$

Inequalities (46), (47) follow from standard arguments about the restricted functions. When necessity of the restriction of the function $\hat{f}'(x,\alpha,\beta,\gamma)$ from below dictates by the validity of the inequality (48), and the fact that $N'_\xi(z) \overset{z\to 0}{\to} \infty$ (see estimations (55)).

Now we approximate uniformly the function $-\hat{f}'(x,\alpha,\beta,\gamma)$ on $[1/x_0, x_0]$ by the simple function $\chi(x)$ with a finite number of values, such that

$$\gamma - a \le \chi(x) \le -\hat{f}'(x,\alpha,\beta,\gamma) \tag{49}$$

and on $[0, 1/x_0)$ and $(x_0, \infty)$ function $\chi(x) = 0$. At last approximate $\chi(x)$ by a continuous function $\bar{\chi}(x)$ :

$$\int_0^\infty |\chi(x) - \bar{\chi}(x)|dx < \delta_2. \tag{50}$$

This approximation can be done by using the standard arguments (see for ex.[8], p.86). It is important and follows from the proof of possibility of such approximation, that the Lebesgue measure

$$\nu^0 = \nu(\{x : |\chi(x) - \bar{\chi}(x)| \neq 0\}) \tag{51}$$

can be made arbitrary small and

$$\hat{s} \max_x \chi(x) \ge \bar{\chi}(x) \ge \min_x \chi(x), \tag{52}$$

where $\hat{s}$ is the number of different values of $\chi(x)$. For the arbitrary $\delta_3 > 0$ we can choose $\bar{\chi}$ such that it satisfies the additional condition

$$\bar{\chi}(x) = 0,\ x > x_0 + \delta_3. \tag{53}$$

We set $y'(x) = -\bar{\chi}(x)$ and require $y(x) \to 0$ as $x \to \infty$.

Next we have the final chain of relations

$$\begin{aligned} \int_0^\infty y(x)dx &= \int_0^\infty \left( \int_x^\infty \bar{\chi}(z)dz \right) dx \le \int_0^{x_0} \left( \int_0^{x_0} \chi(z)dz \right) dx + \delta_2(x_0 + \delta_3) \\ &\le \int_0^\infty f(x)dx - \delta + \delta_2(x_0 + \delta_3). \end{aligned} \tag{54}$$

Here in the first inequality we use (50) and in the last inequality (48) and (49).

Next estimation is for $L(\cdot)$ :

$$
\begin{aligned}
&|L(\bar{\chi}(x)) - L(-\hat{f}'(x))|dx = \int_0^{1/x_0} |N(\bar{\chi}(x)) - N(-\hat{f}'(x))|dx \qquad (55)\\
+ &\int_{1/x_0}^{x_0} |N(\bar{\chi}(x)) - N(-\hat{f}'(x))|dx + \int_{x_0}^{\infty} |N(\bar{\chi}(x)) - N(-\hat{f}'(x))|dx \\
\leq &\ 2\epsilon_1 + 2\nu^0 N(\hat{s}\beta) + \int_{1/x_0}^{x_0} |N(-\hat{f}'(x,\alpha,\beta,\gamma)) - N(-\hat{f}'(x))|dx \\
+ &\int_{1/x_0}^{x_0} |N(-\hat{f}'(x,\alpha,\beta,\gamma)) - N(\chi(x))|dx + \int_{1/x_0}^{x_0} |N(\chi(x)) - N(\bar{\chi}(x))|dx \\
\leq &\ 3\epsilon_1 + 2\nu^0 N(\hat{s}\beta) + N'_\xi(\gamma - a)(\delta_1 + \delta_2).
\end{aligned}
$$

We should explain some of the estimates above. In the second inequality we use the fact that $\nu(\{x: \ \bar{\chi}(x) > 0\} \bigcap A_0) \leq \nu^0$ and $\max N(\chi(x)) \leq N(\hat{s}\beta)$. In the last inequality we use the fact that $N'(z)$ is monotonically decreasing. Now we choose $\epsilon_1 < \epsilon/3$, then $\alpha, \beta > 0$ such that (46), (48) and (47) are valid. Then we choose $\nu^0, \delta_1, \delta_2, \delta_3$ such that $\delta_2(x_0 + \delta_3) < \delta/2$ and

$$N'_\xi(\gamma - a)(\delta_1 + \delta_2) + 2\nu^0 N(\hat{s}\beta) < \epsilon - 3\epsilon_1.$$

Then the right hand side of (54) and (55) is less than $\int_0^\infty f(x)dx - \delta/2$ and $\epsilon$ correspondingly. Then we choose corresponding to the previous choice of the parameters functions $\hat{f}'(x,\alpha,\beta,\gamma), \chi(x), \bar{\chi}(x)$. This proves (42) with the assertion connected with (43).

Now we are ready to prove that

$$\sup_{f \in \mathcal{C}_1} L(f) = \sup_{f \in \mathcal{C}_2} L(f). \qquad (56)$$

Taking into account previous considerations we should prove that for arbitrary $\epsilon > 0, f \in \mathcal{C}_1$ such that

$$\int_0^\infty f(x)dx < 1 - \delta$$

there exists $y \in \mathcal{C}_2$ such that

$$|L(y) - L(f)| < \epsilon.$$

The proof of this fact is quite similar to the proof of (42) and we describe the simple idea omitting routine details. It is necessary to repeat considerations of the previous proof of (42) with one exception: now we consider not an arbitrary simple function $\chi(x)$ but a step function- a function which is constant on intervals. This is possible to do when function $f'$ is continuous, if instead of function $\hat{f}'(x,\alpha,\beta,\gamma)$ we consider the function

$$
f_1'(x,\alpha,\beta) = \begin{cases} f'(x), & \alpha \leq -f'(x) \leq \beta, x \in [1/x_0, x_0], \\ \alpha, & -f'(x) < \alpha, \ x \in [1/x_0, x_0], \ or \ x \notin [1/x_0, x_0], \\ -\beta, & -f'(x) > \beta, \ x \in [1/x_0, x_0] \end{cases}
$$

and

$$\int_0^\infty f_1(x,\alpha,\beta) < 1 - \delta.$$

Then all previous considerations are valid with 1 instead of $\int_0^\infty f(x)dx$.

When constructing the function $\bar\chi(x) \in \mathcal{C}_2$ we connect the consecutive steps of function $\chi(x)$ by a smooth curve and obtain the smooth curve $\chi(x)$. It is easy to see that all conditions and inequalities can be satisfied. This proves (56).

To prove the still remaining statement of Theorem 2 we first prove Theorem 3. From Theorem 1 and the inequality ($f \in \mathcal{C}$)

$$\#^{n,\delta,\epsilon,r}(f_{\max}) \leq \#^{n,\delta}_{\mathcal{A},\mathcal{B}}$$

follows that

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{\ln \#^{n,\delta}_{\mathcal{A},\mathcal{B}}}{\sqrt{n}} \geq \lim_{r\to\infty}\liminf_{\delta\to 0}\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\ln \#^{n,\delta,\epsilon,r}(f_{\max})}{\sqrt{n}} = \int_0^\infty N(-f'_{\max}))dx = 2C, \tag{57}$$

where $C$ is determined by the relations (4), (5). The last equality follows from the following relations:

$$N(-f'_{\max}) = -\lambda_1(-f'_{\max}) + f'_{\max}(x)\lambda_2(-f'_{\max}(x)). \tag{58}$$

Comparing (15), (16) and (4) we see that

$$\lambda_1(-f'_{\max}(x)) = -Cf_{\max}(x),\ \ \lambda_2(-f'_{\max}(x)) = -Cx.$$

substituting these values into (58) we have

$$\int_0^\infty N(-f'_{\max}(x))dx = 2C.$$

Here we use integration by parts.

After (57) all what we need to prove is the inequality

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{\ln \#^{n,\delta}_{\mathcal{A},\mathcal{B}}}{\sqrt{n}} \leq \int_0^\infty N(-f'_{\max}(x))dx.$$

Once more we consider the interval $[1/r, r]$ and the set of restrictions $S_{n,\delta}$ on this interval and on intervals $[0, 1/r)$, $(r, \infty)$. As before for sufficiently large $r$ the number of restrictions of $S_{n,\delta}$ on $[0, 1/r)$ and $(r, \infty)$ is small:

$$\frac{\ln \alpha_{n,r}}{\sqrt{n}},\ \frac{\ln \beta_{n,r}}{\sqrt{n}} = o_{n,r}(1). \tag{59}$$

Now we must estimate the number of restrictions $S_{n,\delta,r}$, the whole number $\#^{n,\delta}_{\mathcal{A},\mathcal{B}}$ is upper bounded by the product of the numbers of restrictions on these three intervals.

Note that, if $\varphi_{n,\delta,r}$ is the restriction of the step function $\varphi_{n,\delta}$, then

$$\int_{1/r}^r \varphi_{n,\delta,r}(x)dx \leq 1 + \delta - \frac{1}{r}\varphi_{n,\delta}(1/r). \tag{60}$$

Thus

$$\varphi_{n,\delta}(x) \le \varphi_{n,\delta}(1/r) \le r(1+\delta),\ x \in [1/r, r]. \tag{61}$$

Now by using standard arguments it is easy to see that the set of monotone non-increasing functions $\varphi(x)$ on $[1/r, r]$ with the restrictions

$$\int_{1/r}^{r} \varphi(x)dx \le 1 + \delta - \frac{1}{r}\varphi(1/r) \tag{62}$$

and (61) is compact in $L^1([1/r, r], dx)$ topology. We denote this compact by $K_r$.

From the proof of the Theorem 1 follows that for each $f \in \mathcal{C}$ we can write

$$K(f, \epsilon, r) = \limsup_{n\to\infty} \frac{\ln \#\{S_{n,\delta,r} \bigcap B(f, \epsilon, r)\}}{\sqrt{n}} = \int_{1/r}^{r} N(-\hat{f}'(x))dx + o_{\epsilon}(1).$$

Next for every $f \in \mathcal{C}$ we choose $\epsilon_f$ such that

$$\left| K(f, \epsilon, r) - \int_{1/r}^{r} N(-\hat{f}'(x))dx \right| < \epsilon_2 = o_{\epsilon_f}(1).$$

Then for every function $f \in K_r$ we consider the ball $B(f, \epsilon_f, r)$ and from this set of balls we can choose a finite number of balls $B(f_i, \epsilon_{f_i}, r)$ such that $K_r \subset \bigcup_i B(f_i, \epsilon_{f_i}, r)$.

Next we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\ln \#\{S_{n,\delta,r}\}}{\sqrt{n}} \le \max_i \frac{\ln \#\{S_{n,\delta,r} \bigcap B(f_i, \epsilon_{f_i}, r)\}}{\sqrt{n}} \\ = \ &\limsup_{n\to\infty} \max_i \int_{1/r}^{r} N(-\hat{f}_i'(x))dx + \epsilon_2 \le \sup_{f\in\mathcal{C}^r} \int_{1/r}^{r} N(-\hat{f}'(x))dx + \epsilon_2, \end{aligned} \tag{63}$$

where $\mathcal{C}^r$ is the set of restrictions of functions from $\mathcal{C}$ on $[1/r, r]$. As in Theorem 2 we can take in (63) the set $\mathcal{C}_2^r$ instead of $\mathcal{C}$, where $\mathcal{C}_2^r \subset \mathcal{C}^r$ is the set of functions on $[1/r, r]$ with continuous second derivative. The proof of this fact is the same as in the proof of Theorem 2 and even simpler, because we consider here the finite interval $[1/r, r]$.

Because the functional

$$L^r(f) = \int_{1/r}^{r} N(-\hat{f}'(x))dx$$

is convex, to find the extremal of this functional it is enough to find the local extremal of this functional. We have the following problem: find

$$f_{\max} = \arg\max_{f\in\mathcal{C}_2^r} L^r(f). \tag{64}$$

We will try to find the local extremal among the functions

$$f \in \mathcal{C}_2^r \ \textit{such that}\ |f'(x)| > y_0 \tag{65}$$

for some $y_0 > 0$. If we find such local unconditional extremal for sufficiently small $y_0$ it will be the solution of the problem (64).

If (65) is valid, then

$$y'(x) = f'(x) + th'(x) < -y_0, \ x \in [1/r, r] \tag{66}$$

for sufficiently small $t$ and we restrict our attention on such $h(x) \in \mathcal{C}_2^r$ that

$$\int_{1/r}^{r} h(x)dx = 0, \ h(1/r) = h(r) = 0, \tag{67}$$

then $y \in \mathcal{C}_2^r$. Since $y'(x) < -y_0, \ y \in \mathcal{C}_2^r$ and $N(\xi), \lambda_2(\xi)$ are increasing functions. we have

$$0 < N'_\xi(-y'(x)) = -\lambda_2(-y'(x)) < -\lambda(y_0).$$

Hence we can move the derivative before the integral and obtain

$$\frac{d}{dt}L^r(f + th)|_{t=0} = \int_{1/r}^{r} N'_\xi(-f'(x))h'(x)dx \triangleq M.$$

If $f$ is extremal, then $M = 0$ and integrating by parts we obtain

$$N'_\xi(-f'(x))h(x)|_{1/r}^{r} - \int_{1/r}^{r} h\frac{d}{dx}N'_\xi(-f'(x))dx = 0. \tag{68}$$

Since $h \in \mathcal{C}_2^r$ is arbitrary such that (65), (67) are valid, standard arguments from calculus of variations (see for ex.[7]) show, that for the integral in (68) to vanish it is necessary that

$$\frac{d}{dx}N'_\xi(-f'(x)) = C = const. \tag{69}$$

The first term in the left hand side of (68) vanishes, because of the condition (67).

Equation (69) has the solution

$$N'_\xi(-f'(x)) = C(x + \alpha), \ C, \alpha = const$$

and hence

$$\lambda_2(-f'(x)) = -C(x + \alpha). \tag{70}$$

At the same time from (15), (16) follows that

$$\frac{d\lambda_1}{d\lambda_2} = f'(x)$$

i.e

$$\lambda_1(x) = -C\int f'(x)dx + const = -C(f(x) + \beta), \ \beta = const. \tag{71}$$

Using (16), (70), (71) we obtain the equality

$$\sum_{\ell \in \mathcal{A}} e^{-C(x+\alpha)\ell} \sum_{\ell \in \mathcal{B}} e^{-C(y+\beta)\ell} = 1.$$

This equation determines (up to constants $C, \alpha, \beta$) $y$ as a function of $x$. Also we have the condition

$$\int_{1/r}^{r} y(x)dx \leq 1 - \frac{1}{r}y(1/r). \tag{72}$$

Values $\alpha$ and $\beta$ are simply the shifts along $X$ and $Y$ axis of the graph of function $y(x)$, when $\alpha = \beta = 0$. Shifts along the $Y$ axis do not influence explicitly the functional

$$L^r(f) \triangleq \int_{1/r}^{r} N(-\hat{f}'(x))dx$$

and the best choice of $\beta$ which gives the max of the functional $L^r(f)$ is such that condition (72) is the least restrictive, i.e. when $y(r) = 0$. Since $y'(x)$ is a continuous function of $\alpha, \beta, C$, functional $L^r(y)$ is also a continuous function of these variables. Also it is easy to see for some $y_0$, $|y'(x)| > y_0$, $x \in [1/r, r]$ for every given $\alpha, C, r$. But we must consider the possibility that sup of $L^r(f)$ is achieved when $\alpha \to \infty$. In such case

$$y'(x) \to C_1 = const, \; x \in [1/r, r]. \tag{73}$$

If this is the case when $L^r(\cdot)$ achieves its sup, then

$$\sup_{f \in \mathcal{C}} L^r(f) = \left(r - \frac{1}{r}\right) N(-C_1).$$

We will show that it is not true (this is not the sup of $L^r(f)$). Indeed, if $y'(x) = C_1$ and $y(r) = 0$, then $-C_1 = \frac{y(1/r)}{r-1/r}$ and from (72) follows that

$$y(1/r) \leq \frac{2}{r + 1/r}$$

or

$$-C_1 \leq \frac{2}{r^2 - 1/r^2}.$$

Since

$$\int_{1/r}^{r} N(-y'(x))dx \leq (1 - C_1)rH\left(\frac{-C_1}{1 - C_1}\right) \to 0 \text{ as } r \to \infty,$$

for large $r$ functional $L^r(-y'(x))$ vanishes and for large $r$, $y(x)$ cannot be sup of $L^r(f)$, because it is easy to introduce a function $f$ which satisfies condition (72) and for which $L^r(f)$ increases with $r$ (for example when $f(x) = \psi(x)$, where $\psi(x)$ is defined in (75)).

Hence the extremum of $L^r(f)$ is achieved for some finite $\alpha$. For given $C$, $|y'(x)|$ increases, when $\alpha$ decreases and $\alpha \geq -1/r$. We give a rough upper estimate of $\max L^r(y)$ and assume that $\alpha = -1/r$ and instead of the condition (72) we consider the more rough condition

$$\int_{1/r}^{r} y(x)dx \leq 1. \tag{74}$$

Then, if $\psi(x)$ is the solution of the equation

$$\sum_{\ell \in \mathcal{A}} e^{-x\ell} \sum_{\ell \in \mathcal{B}} e^{-\psi(x)\ell} = 1, \tag{75}$$

we have

$$1 = \sum_{\ell \in \mathcal{A}} e^{-x\ell} \sum_{\ell \in \mathcal{B}} e^{-\psi(x)\ell} \leq \sum_{\ell=1}^{\infty} e^{-x\ell} \sum_{\ell=1}^{\infty} e^{-\psi(x)\ell} \frac{1}{(e^x - 1)(e^{\psi(x)} - 1)}$$

and thus $\psi(x)$ is an integrable function on $[0, \infty)$ :

$$\begin{aligned} \psi(x) &\leq -\ln(1 - e^{-x}), \\ \int_0^\infty \psi(x)dx &\leq -\int_0^\infty \ln(1 - e^{-x})dx = \int_0^\infty \frac{x}{e^x - 1}dx = \frac{\pi^2}{6}. \end{aligned}$$

We have

$$y(x) = \frac{1}{C}\psi\left(C\left(x - \frac{1}{r}\right)\right) - \beta \tag{76}$$

and

$$y(r) = \frac{1}{C}\psi\left(C\left(r - \frac{1}{r}\right)\right) - \beta = 0 \tag{77}$$

From the condition (74) follows that

$$M(C, r, \beta) \triangleq \int_{1/r}^{r} y(x)dx = \frac{1}{C^2}\int_0^{C(r-1/r)} \psi(x)dx - \beta\left(r - \frac{1}{r}\right) \leq 1. \tag{78}$$

Also we have

$$L^r(y) = \frac{1}{C}\int_0^{C(r-1/r)} N(-\psi'(x))dx$$

and $\psi(x)$, $-\psi'(x) \searrow 0$ as $x \to \infty$.

Next $(L^r)'_C(y) < 0$ and using the differentiation of (75), (77) and (78) it is easy to see that $M'_C(C, r, \beta) < 0$.

Hence we should choose $C$ as small as possible such that condition (78) is still valid i.e we should have

$$\frac{1}{C^2}\int_0^{C(r-1/r)} \psi(x)dx = \beta(C)\left(r - \frac{1}{r}\right) + 1. \tag{79}$$

Equations (77), (79) together with (75) define $C$ as a function of $r$: it is always possible to find a unique $C$ which satisfies these relations, because for given $r$ $M(C, r, \beta(C))$ is a continuous, monotone function of $C$ and tends to 0 as $C \to \infty$ and

$$M(C, r, \beta(C)) \to \infty \text{ as } C \to 0. \tag{80}$$

It is left to show that

$$C(r) \to \sqrt{\int_0^\infty \psi(x)dx} \text{ as } r \to \infty. \tag{81}$$

Assume for the moment that (81) is true. Then since

$$N(-\psi'(x)) = \psi(x) - x\psi'(x)$$

with the validity of (81) we have

$$\lim_{r\to\infty} L^r(y) = \frac{2}{C}\int_0^\infty \psi(x)dx = 2C.$$

Taking into account (59) we obtain the inequality

$$\limsup_{\delta\to 0}\limsup_{n\to\infty} \frac{\ln \#^{n,\delta}_{\mathcal{A},\mathcal{B}}}{\sqrt{n}} \le \lim_{r\to\infty} L^r(y) = 2C.$$

From this inequality and (57) follows Theorem 3.

Now inequality

$$\max_{f\in\mathcal{C}_2} L(f) \le 2C$$

follows from Theorems 1 and 3. Also

$$\max_{f\in\mathcal{C}_2} L(f) \ge L(f_{\max}) = 2C$$

and this completes the proof of (2). Theorem 2 is also proved. Hence $\lim_{r\to\infty} C(r)$ exists.

Next, taking into account that $\psi(x)$ is monotone decreasing and $\int_0^\infty \psi(x)dx$ converges, it is easy to see that $\beta(C(r))r \to 0$ as $r\to\infty$. Taking $\lim_{r\to\infty}$ from both sides of (79) we obtain

$$\frac{1}{C^2}\int_0^\infty \psi(x)dx = 1.$$

This proves (81). All proves are complete.

# References


[1] A.Vershik, Statistical Analysis of Combinatorial Partitions and Their Limit Shapes, Functional Anal. Appl., Vol.30, N2, 1996, 90–105

[2] K.Chandrasekharan, Arithmetical Functions, Springer–Verlag, Berlin, 1970

[3] V. Blinovsky, Large Deviations Principle for the Boarder of Random Young Diagram, Problems of Information Transmission, Vol.34, No 1, 52–62, 1998.

[4] V. Blinovsky, Large Deviations Problem for Random Young diagram, in book: Numbers, Information and Complexity, Special volume in honour of R. Ahlswede on occasion of his 60th birthday, Kluwer acad. Publ., Boston, 473–488, 2000.

[5] A. Dembo, A. Vershik and O. Zeitouni, Large Deviations Principle for Integer Partitions, Markov Processes and related fields, Vol.6, No.1, 147–180, 2000.

[6] A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications, Boston: Jones and Barlett Publishers, 1993.

[7] I. Gelfand and S. Fomin, Calculus of Variations, Prentice–Hall, Inc., 1963.

[8] A. Kolmogorov and S. Fomin, Elements of the Theory of Functions and Functional Analysis, Vol.2, Graylock Press, 1961.