5. *The structure of extremal sets*

Define $N(k,n) = \{m : m \leq n, \Omega(m) = k\}$. We call $A \subset [n]$ $N(k,n)$ typical if for fixed f and $n \to \infty$ $\delta(f, A \triangle N(k,n), n) = o(\delta(f,A,n))$.

Problem 9 Is it true that for "small" f an extremal set $N(k,n)$ is typical? For how fast decreasing f is this the case?

6. *Existence of many disjoint "larger" (almost extremal) primitive subsets*

For $f(m) = \frac{1}{m}$ there are "many" primitive disjoint subsets $A_1, \dots, A_k$ of $[n]$ with $\delta(f, A_i, n) > (1-\varepsilon)F(f,n)$, for $f(m) = 1$, $f(m) = \frac{1}{m \log m}$ there are no $A_1, \dots, A_k$ with these properties. Where is the limit?
We are grateful to A. Granville for having communicated to us in March 2004 a problem B. Poonen had once asked him, and which might be solvable using some of the results in [AKS04].

Suppose that $S \subset \mathbb{N}$ and $S(n) = S \cap [n]$. Consider the asymptotic density (known to exist) $dM(S(n))$ and the ratio $r(n) = \frac{|M(S(n)) \cap [n]|}{n}$.
Must

$$\lim_{n \to \infty} dM(S(n)) = \lim_{n \to \infty} r(n)? \tag{32}$$

One can prove that the answer is no by constructing sets S where the limit on the RHS does not exist; but, the limit on the LHS always exists because $dM(S(n))$ increases as $n \to \infty$, and is bounded from above by 1. An example of a set S for which the limit on the RHS does not exist is a union of dyadic integer intervals $\{x_i + 1, \dots, 2x_i\}$, where the x_i's are chosen to be very far apart. When $n = 2x_i$, for some i, there will be a higher proportion of integers $m \leq n$ divisible by some element of S than when $n = x_i$.
So, one can modify the question in the following way.

Problem 10 If we have that $\lim_{n \to \infty} r(n)$ exists, must it follow then that (32) holds?
Finally, we mention a conceivable sharpening of Theorem 56 of Lecture 16.

Conjecture (Ahlswede/Khachatrian; also Erdös) In Theorem 56 one can choose for every k $n(k) = cp_k^2$ for a suitable constant c. Presently, we have only $n(k) = \prod_{p \leq (p_{c_1 k})} p_{c_2 k}$.

Towards Combinatorial Algebraic Number Theory

After all these contributions to Combinatorial (Elementary) Number Theory, which in particular widens the area treated in [HT88], we open a new area of research by considering now seemingly basic extremal properties for algebraic number fields. We present with complete proofs our recent work. For its understanding knowledge about number fields is required.

We prove that for all sufficiently large N_0 a maximal set of ideals of the maximal order of an algebraic number field, such that any pair of ideals from this set is not coprime and the norm of each ideal does not exceed N_0, is of the form $E(N_0) = \{\theta : N(\theta) \le N_0,\ \theta = \eta_1 u\}$, where $\{\eta_1, \eta_2, \dots\}$ is the set of prime ideals of the maximal order and $N(\eta_3) > 2$.

In the paper [AK95b] the authors investigated the problem of finding the maximal sets of integers bounded from above by some number N_0 without $k+1$ coprimes. There it was proved that for all sufficiently large N_0 the unique maximal set is

$$E(N_0, k) = \{n \le N_0,\ n = p_i u,\ i = 1, \dots, k\}, \tag{33}$$

where $p_1 < p_2 < \dots$ is the sequence of prime numbers. Shortly before it was proved in [AK94b] that this assertion is not valid for all N_0, i.e., for small values of N_0 the set $E(N_0, k)$ is not maximal. These facts completely solved the problem of Erdős of determining the maximal sets of integers without $k+1$ coprimes.

It is natural to extend this problem to the case of algebraic numbers. Here we concentrate our attention on the problem when $k = 1$. It is a straightforward result that in the ring of integers the maximal set (for arbitrary N_0) is $E(N_0, 1)$. The answer is not so obvious in the case of ideals in the maximal order of an algebraic number field. Moreover, we can prove that the analogous result is true only for large enough N_0 and only when the norm satisfies $N(\eta_3) > 2$. We consider the maximal order $\mathcal{B}$ of the algebraic number field K, which is a finite extension of the rationals [1] R and $(K : R) = n$. Denote the set of integer ideals of the maximal order by Θ and the set of ideals whose norms do not exceed N_0 by $\Theta(N_0)$.

Let $\Omega = \{\eta_1, \eta_2, \dots\}$ be the set of prime ideals of the order $\mathcal{B}$, which are ordered in such a way that their norms do not decrease, i.e., $N(\eta_i) \le N(\eta_{i+1})$. Recall that for an arbitrary $\eta \in \Omega$, $N(\eta) = p^f$ for some prime p and positive integer f. We say that two ideals $\theta_1, \theta_2 \in \Theta$ are coprime if they do not have any common multiple in their prime ideal decomposition. The problem we are going to solve here is to determine for all sufficiently large N_0 the maximal set of ideals from $\Theta(N_0)$ such that it does not contain a pair of coprime ideals. The main problem here in comparison with the ring of integers, which was considered in [AK96a], is that the norm of prime ideals is not a strictly increasing function. We find that the solution of this problem is an interesting interaction between the methods of the work [AK96a] and a diametric problem. This interaction is based on the special properties of intersecting antichains, which we establish here.

Here is the main result.

Theorem 95 (Ahlswede and Blinovsky) *If $N(\eta_2) > 2$, then for sufficiently large N_0 any maximal set of ideals $O(N_0)$ without coprimes and with a norm not exceeding N_0 is one from*

$$E(N_0, \eta_i) \triangleq \{\theta \in \Theta(N_0) : \theta = \eta_i u\}, i = 1, \dots, k,$$

where k is the maximal number such that $N(\eta_i) = N(\eta_1),\ i \le k$.

[1] Here R denotes the field of rational numbers whereas the usually used letter Q denotes in this paper the alphabet $\{0, 1, \dots, q-1\}$.

If $N(\eta_2) = 2,\ N(\eta_3) > 2,$ *then the maximal set is one from* $E(N_0, \eta_1),\ E(N_0, \eta_2).$

Note that in this theorem we still have the condition (as in the ring of integers from [AK95b]), that N_0 must be sufficiently large, and one additional condition, that $N(\eta_3) > 2$. In the case $N(\eta_3) = 2$ we do not even have a conjecture what the maximal set of ideals with restricted norm and without coprimes is and we will show that the maximal density of such a set can be achieved on several sets of ideals.

Define $\mathcal{O}(N_0)$ as the family of maximal sets of ideals from Θ without coprime pairs whose norm does not exceed N_0. Next we assume that $N(\eta_3) > 2$. We say that two ideals $\theta_1, \theta_2 \in \Theta$ intersect in the ith position if $\eta_i | \theta_1,\ \eta_i | \theta_2$. We need the notion of left compressedness of $D \subset \Theta(N_0)$. We say that D is left compressed if for all $d \in D$ such that

$$d = \eta_\ell^i u,\ \eta_\ell \not| u,\ i \geq 1,$$

and all $\eta_k :\ k < \ell$, we have

$$\bar{d} = \eta_k^i u \in D.$$

Denote by $\mathcal{C}(N_0)$ the family of sets, which belong to the family $\mathcal{S}(\mathcal{N}_\prime)$ of sets of ideals without coprimes and with a norm not exceeding N_0 and which has the additional property that each set from this family is left compressed. Next we consider a set of ideals from $\mathcal{C}(N_0)$. It is easy to show (and it was done for example in Lemma 1 of [AK95b]) that

$$\mathcal{O}(N_0) \bigcap \mathcal{C}(N_0) \neq \emptyset.$$

Note that $D \in \mathcal{O}(N_0)$ is a downset, i.e., if $d = \eta_{i_1}^{\alpha_1} \dots \eta_{i_\ell}^{\alpha_\ell} \in D$, then $\bar{d} = \eta_{i_1} \dots \eta_{i_\ell} \in D$ and D is also an upset in the sense that

$$D = M(D) \bigcap \Theta(N_0),$$

where $M(\mathcal{A})$ is the set of multiples of $\mathcal{A} \subset \Theta$. For $D \subset \Theta$ we denote by $P(D) \subset \Theta$ the set of ideals such that for $\theta_1,\ \theta_2 \in P(D), \theta_1 \not| \theta_2$ and $D \subset M(P(D))$. It is easy to see that, if $D \in \mathcal{O}(N_0)$, then

$$D = M(P(D)) \bigcap \Theta(N_0)$$

and $P(D)$ is the set of square-free ideals.

Lemma 55 *For all sufficiently large* N_0 *there exists a* t *(which does not depend on* N_0 *and depends only on* K*) such that any two* $\theta_1, \theta_2 \in \mathcal{A} \in \mathcal{O}(N_0) \bigcap \mathcal{C}(N_0)$ *are* i*-intersecting for some* $i \leq t$.

Lemma 56 *The density of* $\mathcal{A} \in \mathcal{O}(N_0)$ *equals* $1/N(\eta_1)$.

The proof of Lemma 56 uses some results about intersecting antichains, which we introduce later.

Lemma 57 *If* $N(\eta_2) > 2$, *then the density* $1/N(\eta_1)$ *is achieved on one of the sets* $E(N_0, \eta_i);\ i = 1, \dots, k$. *If* $N(\eta_2) = 2$, *then the maximal density* $1/2$ *is achieved on two sets,* $E(N_0, \eta_1)$ *and* $E(N_0, \eta_2)$.

The statement of the next lemma is well known ([N74]).

Lemma 58 (Prime ideal theorem) *The following relation is valid*

$$\#\{\eta \in \Omega: \ N(\eta) \le z\} = \frac{z}{\log z}(1+o(1)), \ z \to \infty.$$

Proof of Lemma 55. Let

$$\pi(z) = \{\eta \in \Omega: \ N(\eta) \le z\}$$

be the number of prime ideals with norm not exceeding z. Our proof is based on the following statement, which was proved in [AK96a].

Proposition 24 *For all $\mathcal{A} \in \mathcal{O}(N_0) \bigcap \mathcal{C}(N_0)$ no $a \in P(\mathcal{A})$ has divisor η_i, $i \ge s$, where $s \ge 2$ is the minimal number, such that for $z \in R_+$ the following inequality is valid*

$$2\pi(z) \le \pi(\eta_s z). \tag{34}$$

Notice that this statement looks different from Lemma 4 in [AK96a] but the essential parts of the proofs coincide.

Now it is easy to see that for a given field K inequality (34) is always true for $s > s_0$, where s_0 is sufficiently large. Indeed, let us choose z_0 such that

$$\frac{1}{2}\frac{z}{\log z} \le \pi(z) \le 2\frac{z}{\log z}, \ z > z_0.$$

The possibility of such a choice follows from the mentioned Lemma 58. Then also

$$\frac{1}{2}\frac{p_s z}{\log(p_s z)} \le \pi(p_s z) \le 2\frac{p_s z}{\log(p_s z)}, \ z > z_0.$$

Now we choose s_0 such that for $s \ge s_0$

$$2\frac{2z}{\log z} \le \frac{p_s z}{\log(p_s z)}. \tag{35}$$

If $z < 2$, then $\pi(z) = 0$ and (34) is valid. If $2 \le z < z_0$, then we choose s_1 such that

$$\pi(p_s z) \ge \pi(p_s 2) \ge 2\pi(z_0) \ge 2\pi(z), \ s \ge s_1. \tag{36}$$

At last if $t = \max(s_0, s_1)$, then (35) and (36) imply (34). Lemma 55 is proved. □

Thus for some t, which is independent of N_0, each ideal from $P(\mathcal{A})$ has no divisors $\eta_j, \ j \ge t$. Hence we should consider only $P(\mathcal{A})$ such that $\theta = \eta_{i_1} \dots \eta_{i_r} \in P(\mathcal{A}), \ i_1 < \dots < i_r \le t$ for some t, which depends only on K. We assume that the square-free ideals $a_1, a_2, \dots \in P(\mathcal{A})$ are ordered colexicographically. Hence there exists a natural one-to-one correspondence between ideals from $P(\mathcal{A})$ and binary t-tuples. The set of $t-$tuples, which correspond to $P(\mathcal{A})$, is an intersecting antichain. Now we are going to investigate some properties of intersecting antichains. First of all note that a maximal $\mathcal{A}(N_0)$ must have maximal asymptotic density as $N_0 \to \infty$. The density $d\mathcal{A}(N_0)$ is equal to

$$d\mathcal{A}(N_0) = \sum_i dB^i, \tag{37}$$

where dB^i is the density of the set of ideals B^i from $\mathcal{A}$, which are divisible by $a_i \in P(\mathcal{A})$ and are not divisible by $a_j \in P(\mathcal{A}),\ j < i$. By left compressedness of the set $\mathcal{A}$ it follows that if $a_i = \eta_{j_1} \dots \eta_{j_{r_i}}$ $(j_1 < \dots < j_{r_i})$ and $N(\eta_j) = q_j$, then

$$B^i = \left\{ \theta \in \Theta(N_0) : \theta = \eta_{j_1}^{\alpha_1} \dots \eta_{j_{r_i}}^{\alpha_{r_i}} u,\ \alpha_j \geq 1,\ \left(u, \prod_{j \leq j_{r_i}} \eta_j \right) = 1 \right\}$$

and hence

$$dB^i = \sum_{\alpha_{j_p} \geq 1} \frac{1}{q_{j_1}^{\alpha_1} \dots q_{j_{r_i}}^{\alpha_{r_i}}} \prod_{j \leq j_{r_i}} \left(1 - \frac{1}{q_j} \right) = \prod_{j \leq j_{r_i},\ j \neq j_p;\ p=1,\dots,r_i} (q_j - 1) \left(\prod_{j=1}^{j_{r_i}} q_j \right)^{-1}$$

and

$$d\mathcal{A}(N_0) = \sum_i \prod_{j \neq j_p;\ p=1,\dots,r_i} (q_j - 1) \left(\prod_{j=1}^{j_{r_i}} q_j \right)^{-1}. \tag{38}$$

We now consider the t-tuples, whose jth element is chosen from the alphabet $\{0, 1, \dots, q_j - 1\}$, and consider sets $\mathcal{A}(t)$ of t-tuples such that every pair of t-tuples has a common unit in some position (possibly different for different pairs). As it was shown in [AK98], the cardinality of a maximal set of t-tuples from $Q^t = \{0, 1, \dots, q-1\}^t$ such that its diameter is d coincides with the maximal cardinality of a set of t-tuples from Q^t such that every pair of t-tuples from this set has $t-d$ common ones. The same is true for $Q^t = \prod_{i=1}^t Q_i = \{0, \dots, q_1 - 1\} \times \dots \times \{0, \dots, q_t - 1\}$. The characterization of all such maximal sets constitutes a diametric problem. In our case $d = t-1$. In [L79a] was proved (and it also follows from [AK98]) that maximal subsets from Q^t with diameter $d = t-1$ are of the form

$$\mathcal{A}_{ij} = \{(a_1, \dots, a_t) \in Q^t :\ a_i = j\},\ i = 1, \dots, t,\ j = 0, \dots, q-1.$$

Their cardinality is q^{t-1}. We use this result to show the validity of the following

Proposition 25 *Any maximal set from Q^t with diameter $d = t-1$ is one of the form*

$$\mathcal{A}_{ij} = \{(a_1, \dots, a_t) \in Q^t :\ a_i = j\},\ i = 1, \dots, k,\ j = 0, \dots, q-1$$

if $q_1 = \dots = q_k = q < q_{k+1},\ q_2 > 2$
and

$$\mathcal{A}_{ij} = \{(a_1, \dots, a_t) \in Q^t :\ a_i = j\},\ i = 1, 2,\ j = 0, 1$$

if $q_1 = q_2 = 2 < q_3$.

Proof. For $t = 1$ or $t = 2, q_1 = q_2 = 2$ the statement is obvious. Next we suppose that $t > 1$ and if $t > 2$, then $q_3 > 2$. The proof will use induction on t.

Suppose that $\mathcal{A} \subset Q^t$ is a maximal intersecting set. It can be easily seen that $|\mathcal{A}| = \prod_{i=2}^{t} q_i$. We set $\mathcal{A} = \bigcup_{j=0}^{q_t-1} \mathcal{A}_j$, where $\mathcal{A}_j = \{a \in \mathcal{A} : \ a = (a_1,\ldots,a_{t-1},j)\}$. Denote $\mathcal{A}'_j = \{x \in \mathcal{A}_j : \ \vec{x}^t$ intersects with all $\vec{y}^t, \ y \in \mathcal{A}\}$, where $\vec{x}^i = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_t)$. Denote also $\mathcal{T} = \bigcup_{j=0}^{q_t-1}\left(\mathcal{A}_j \setminus \mathcal{A}'_j\right)$.

We assume that $q_t > q_1$. Otherwise the proof of the lemma reduces to the proof of Theorem 2 from [L79a], which states the result for the case $Q_i = \{0,1,\ldots,q-1\}$ for $i = 1,\ldots,t$.

Consider two cases:

Case $\mathcal{T} = \mathcal{A}$. It is easy to see that for each $(a_1,\ldots,a_{t-1}) \in \{0,\ldots,q_1-1\} \times \ldots \times \{0,\ldots,q_{t-1}-1\}$ there exists not more than one $a_t \in \{0,\ldots,q_t-1\}$ such that $a = (a_1,\ldots,a_t) \in \mathcal{A}$. But in this case $|\mathcal{A}| \le \prod_{j=1}^{t-1} q_j$, which contradicts to the maximality of $\mathcal{A}$.

Case $\mathcal{T} \ne \mathcal{A}$. It can be easily seen that if $q_1 < q_t$, then $\mathcal{T} = \emptyset$. Indeed, consider the decomposition $\mathcal{T} = \bigcup_{j=0}^{q_1-1} \mathcal{T}_j$, where $\mathcal{T}_j = \{a = (a_1,\ldots,a_t) \in \mathcal{T} : \ a_1 = j\}$. With the pigeon-hole principle follows the existence of an $i \in \{0,\ldots q_1-1\}$ such that$|\mathcal{T}_i| \ge |\mathcal{T}|/q_1$. Then the set

$$\mathcal{A}' = \bigcup_{j=0}^{q_t-1} \mathcal{A}'_j \bigcup \{(a_1,\ldots,a_{t-1},m), \ m \in \{0,\ldots,q_t-1\} \\ \text{and } (a_1,\ldots,a_{t-1}) = \vec{a}^t \text{ for some } a \in \mathcal{T}_i\}.$$

is intersecting and $|\mathcal{A}'| > |\mathcal{A}|$, which is a contradiction.

Next, if $\mathcal{T} = \emptyset$, then $\mathcal{A} = \bigcup_{j=0}^{q_t-1} \mathcal{A}'_j$ and $\mathcal{B} = \{(a_1,\ldots,a_{t-1}) : \ (a_1,\ldots,a_t) \in \mathcal{A}\}$ is an intersecting set. By maximality of $\mathcal{A}$ we have $\mathcal{A} = \{(a_1,\ldots,a_{t-1},m), \ m \in \{0,\ldots,q_t-1\}, \ (a_1,\ldots,a_{t-1}) \in \mathcal{B}\}$. Hence to maximize $|\mathcal{A}|$ we should maximize the intersecting set $\mathcal{B}$, but this set consists of $(t-1)$-tuples and we can use induction. This completes the proof. □

Now we turn to some facts about intersecting antichains. We introduce several relations that have independent interest; however, for our proofs we only need Proposition 28.

Intersecting Antichains

Let us have an antichain $\mathcal{A} \subset 2^{[t]}$ satisfying at the same time for arbitrary $A_1, A_2 \in \mathcal{A}$, $A_1 \bigcap A_2 \ne \emptyset$. Such a set we call an intersecting antichain. Denote by $\mathcal{A}_i \subset \mathcal{A}$ the set of binary t-tuples such that i is the last position where every $A \in \mathcal{A}_i$ has a one. We start with a simple but interesting inequality.

Proposition 26 *If $\mathcal{A}$ is an intersecting antichain, then*

$$\sum_{i=1}^{t} \frac{|\mathcal{A}_i|}{2^i} \leq \frac{1}{2}. \tag{39}$$

(It is easy to see that bound (39) is tight; for example there is equality, when $|\mathcal{A}_i| = 0$ for $i \geq 2$, and $|\mathcal{A}_1| = 1$. There is equality also in many other cases, as we will show later).

Proof. Denote by $\mathcal{B}_i$ the set of vectors obtained from $\mathcal{A}_i$ by deleting the last $t-i$ zeros. The vectors from $\mathcal{B}_i$ have i components. $\mathcal{B} = \bigcup_{i=1}^{t} \mathcal{B}_i$ is a prefix-free code and $|\mathcal{B}_i| = |\mathcal{A}_i|$. Hence by the Kraft inequality we have

$$\sum_{i=1}^{t} \frac{|\mathcal{B}_i|}{2^i} \leq 1$$

and hence

$$\sum_{i=1}^{t} \frac{|\mathcal{A}_i|}{2^i} \leq 1.$$

Now for every i and every $b \in \mathcal{B}_i$ consider all possible continuations of i-tuple b to the length t. The number of such continuations is 2^{t-i}. This way we obtain a set of different t-tuples $\mathcal{C}$,

$$|\mathcal{C}| = \sum_{i=1}^{t} |\mathcal{B}_i| 2^{t-i} = \sum_{i=1}^{t} |\mathcal{A}_i| 2^{t-i}. \tag{40}$$

At the same time the set $\mathcal{C}$ is intersecting and hence

$$|\mathcal{C}| \leq 2^{t-1}. \tag{41}$$

Therefore

$$\sum_{i=1}^{t} |\mathcal{A}_i| 2^{t-i} \leq 2^{t-1}$$

and we obtain (39). Equality in (39) is achieved also on the intersecting antichain

$$\mathcal{A} = \begin{cases} \binom{[t]}{\frac{t+1}{2}}, & if\ 2 \not| t, \\ \left\{A \in \binom{[t]}{\frac{t}{2}+1} : 1 \notin A\right\} \cup \left\{A \in \binom{[t]}{\frac{t}{2}} : \ 1 \in A\right\}, & if\ 2|t. \end{cases} \tag{42}$$

This can be easily seen by the fact that the set $\mathcal{A}$ is an intersecting antichain whose sets $\mathcal{A}_i$ generate the sets $\mathcal{B}_i$ such that all possible continuations of the sets $\mathcal{B}_i$ to the length t form the intersecting set $\mathcal{C}$:

$$\mathcal{C} = \begin{cases} \bigcup_{j=(t+1)/2}^{t} \binom{[t]}{j}, & if\ 2 \not| t, \\ \left\{A \in \bigcup_{j=t/2+1}^{t} \binom{[t]}{j} : \ 1 \notin A\right\} \cup \left\{A \in \bigcup_{j=t/2}^{t} \binom{[t]}{j} : \ 1 \in A\right\}, & if\ 2|t \end{cases}$$

and this intersecting set has cardinality 2^{t-1}. Another proof of this fact can be done by induction (by proving relation (43) below). Consider for example the case $2 \nmid t$. We have

$$|\mathcal{A}_i| = a_i = \binom{i-1}{\frac{t-1}{2}}.$$

Hence

$$\sum_{i=1}^{t} \frac{a_i}{2^i} = \sum_{i=(t-1)/2}^{t-1} \frac{\binom{i}{\frac{t-1}{2}}}{2^{i+1}} = \frac{1}{2} \sum_{i=(t-1)/2}^{t-1} \frac{\binom{i}{\frac{t-1}{2}}}{2^i}.$$

We are done if we can show that

$$g(c) = \sum_{i=c}^{2c} \frac{\binom{i}{c}}{2^i} = 1. \tag{43}$$

We prove (43) by induction. For $c = 0, 1$ it is true. Then

$$\begin{aligned} g(c+1) &= \sum_{i=c+1}^{2c+2} \frac{\binom{i}{c+1}}{2^i} = \sum_{i=c+1}^{2c+2} \frac{\binom{i-1}{c}}{2^i} + \sum_{i=c+2}^{2c+2} \frac{\binom{i-1}{c+1}}{2^i} \\ &= \frac{1}{2}\left(g(c) + \frac{\binom{2c+1}{c}}{2^{2c+1}} + g(c+1) - \frac{\binom{2c+2}{c+1}}{2^{2c+2}} \right) = \frac{1}{2}(1 + g(c+1)). \end{aligned}$$

We can generalize inequality (39) to the case of r-intersecting antichains $\mathcal{A}$, i.e., when $|A_1 \bigcap A_2| \geq r$ for all $A_1, A_2 \in \mathcal{A}$. We use Katona's Lemma: if $\mathcal{C} \subset 2^{[t]}$ is an r-intersecting set, then

$$|\mathcal{C}| \leq K(t,r) = \begin{cases} \sum_{i=(t+r)/2}^{t} \binom{t}{i}, & 2|(t+r), \\ 2\sum_{i=(t+r-1)/2}^{t-1} \binom{t-1}{i}, & 2 \nmid (t+r). \end{cases}$$

and instead of inequality (41) we obtain

Lemma 59 *If $\mathcal{A}$ is an r-intersecting antichain, then*

$$\sum_{i=1}^{t} \frac{|\mathcal{A}_i|}{2^i} \leq \frac{K(t,r)}{2^t}. \tag{44}$$

Note that everywhere instead of the antichain condition we can consider the weaker condition that $\bigcup_{i=1}^{t} \mathcal{B}_i$ is a prefix-free code. However, when $r > 1$, equality in (44) is achieved only on the antichain $\mathcal{A}$ consisting of the minimal elements of Katona's set (about Katona's set see for example [AK05]), i.e., when

$$\mathcal{A} = \begin{cases} \binom{[t]}{\frac{t+r}{2}}, & if\ 2|(t+r), \\ \left\{A \in \binom{[t]}{\frac{t+r-1}{2}} :\ 1 \notin A\right\} \bigcup \left\{A \in \binom{[t]}{\frac{t+r+1}{2}} :\ 1 \in A\right\}, & if\ 2 \nmid (t+r) \end{cases}$$

We can find further generalizations of inequality (39), for example when the ground alphabet is $q-$ary. Note that the maximal number of intersecting $t-$tuples from

$Q^t = \{0,1,\dots,q-1\}^t$ is q^{t-1}. Hence if we consider $\mathcal{A}_i \subset \mathcal{A} \subset Q^t$ as the set of t-tuples such that i is the position of their rightmost nonzero symbol, we can write (39) with q instead of 2. However, more useful for our purpose will be the model, when we take into account only positions of t-tuples from $\mathcal{A}$, which contain ones. For $a^t = (a_1,\dots,a_t) \in Q^t$ define

$$B(a^t) = \{j:\ a_j = 1\}$$

and for $\mathcal{A} \subset Q^t$ denote $\mathcal{B}(\mathcal{A}) = \{B(a^t),\ a^t \in \mathcal{A}\}$. Let also $L(\mathcal{A})$ be the set of minimal elements of $\mathcal{B}(\mathcal{A})$. Denote by $\mathcal{A}_{i,\omega} \subset L(\mathcal{A})$ the set of t-tuples each having its last one in position i with the whole number of ones equal to ω. Then the following relation is valid:

$$\sum_{i=1}^{t}\sum_{\omega=1}^{i} \frac{|\mathcal{A}_{i,\omega}|(q-1)^{i-\omega}}{q^i} \le \frac{1}{q}. \tag{45}$$

The proof of this inequality involves similar counting arguments as the proof of (39).

To find a generalization of (45) for the case of r-intersecting sets we should know the formula for the maximal cardinality of a $q-$ary set $\mathcal{A}$ such that for every $A_1, A_2 \in \mathcal{A}$, $|A_1 \bigcap A_2| \ge r$, where intersection means the set of positions, where both A_1 and A_2 have ones.

Proposition 27 *If for $\mathcal{A} \subset Q^t, L(\mathcal{A})$ is an r-intersecting antichain, then*

$$\sum_{i=1}^{t} \frac{1}{q^i} \sum_{\omega=1}^{i} |\mathcal{A}_{i,\omega}|(q-1)^{i-\omega} \le \frac{N_q(t,r)}{q^t},$$

where $N_q(t,r)$ is the maximal cardinality of a set from $[q]^t$ whose diameter does not exceed $t-r$.

At last we need one, the most general case, when $\mathcal{A} \subset \prod_{i=1}^t Q_i = \{0,\dots,q_1-1\} \times \cdots \times \{0,\dots,q_t-1\}$. In this case, we have the following generalization of (45) (and correspondingly (39)), the proof of which we leave to the reader.

Proposition 28 *The following relation is valid:*

$$\sum_{i=1}^{t} \sum_{C \in L(\mathcal{C}):\ s^+(C)=i} \prod_{j \in [i]\setminus C} (q_j - 1) \prod_{j=i+1}^{t} q_j \le \prod_{j=2}^{t} q_j \tag{46}$$

where $s^+(C)$ is the position of the rightmost one of C.

Proof of Theorem 95. Now we summarize the facts that we have obtained and prove Theorem 95. Note that the expression on the LHS of (46) is equal to the number of t-tuples in some set $\mathcal{C} \subset \prod_{i=1}^t Q_i$ with intersecting $L(\mathcal{C})$ and if $L(\mathcal{C}) = P(\mathcal{A}(N_0))$, then it is proportional up to $\prod_{i=1}^t q_i$ to the density (37) of $\mathcal{A}(N_0) \subset \Theta(N_0)$ where we use the one-to-one correspondence between binary t-tuples and square-free ideals $\theta \in \Theta$, such that $\eta_\tau \not| \theta$ when $\tau > t$. Solving the diametric problem in this case, we see that the maximum of the LHS of (46) for left compressed sets and hence the maximum of the density of $\mathcal{A}(N_0)$ is achieved (only) when $L(\mathcal{C}) = \{(1,0,\ldots,0)\}$. As the number of possible $P(\mathcal{A}(N_0))$ such that for $\theta \in P(\mathcal{A}(N_0))$ we have $\eta_\tau \nmid \theta$ when $\tau > t$ is bounded from above independently of N_0, there exists N' such that when $N_0 > N'$ we have for the maximal $\mathcal{A}(N_0)$:

$$\mathcal{A}(N_0) = \{\theta \in \Theta(N_0) : \ \theta = \eta_1 u\}. \tag{47}$$

This maximal set is unique among left compressed sets. This proves Lemma 56. To prove Lemma 57 and Theorem 95 note that for not left compressed sets, in the case $N(\eta_2) > 2$ we have additional to (47) possibilities $\{\theta \in \Theta(N_0) : \ \theta = \eta_i u\}$, $i = 2,\ldots,k$ each of which is a maximal set and in the case $N(\eta_2) = 2$, $N(\eta_3) > 2$ we have one additional to (47) maximal set $\{\theta \in \Theta(N_0) : \ \theta = \eta_2 u\}$. This proves Theorem 95. □

Remark In the case, when $N(\eta_3) = 2$, the density $1/2$ is achieved besides the set (47) also on the set

$$\mathcal{A}''(N_0) = \{\theta \in \Theta(N_0) : \ \theta = \eta_1\eta_2 u, \ = \eta_1\eta_3 u, \ = \eta_2\eta_3 u\} \tag{48}$$

and at the present we are not able to determine in the general case when $N(\eta_3) = 2$ which set of ideals is maximal.

Note that the results that do not use the strict increase of the norm along the set of ideals $\eta_1, \eta_2, \ldots$ are still valid for the set of ideals as for the set of positive integers. Let us give an example. Write

$$\zeta(\mathcal{A}, s) = \sum_{\eta \in \mathcal{A}} \frac{1}{N(\eta)}, \ s > 1,$$

where $\mathcal{A}$ is some set of ideals. The lower Dirichlet density $\underline{D}(\mathcal{A})$ of the set $\mathcal{A}$ is defined as follows:

$$\underline{D}(\mathcal{A}) \triangleq \liminf_{s \to 1^+} \zeta(\mathcal{A}, s).$$

For an arbitrary pair of divisors η_1, η_2 denote by (η_1, η_2) ($[\eta_1, \eta_2]$) their greatest common divisor (least common multiple) and for two sets of ideals $\mathcal{A}, \mathcal{B}$ let

$$(\mathcal{A}, \mathcal{B}) = \{(\eta_1, \eta_2); \ \eta_1 \in \mathcal{A}, \ \eta_2 \in \mathcal{B}\},$$

$$[\mathcal{A}, \mathcal{B}] = \{[\eta_1, \eta_2]; \ \eta_1 \in \mathcal{A}, \ \eta_2 \in \mathcal{B}\}.$$

The following inequality is valid:

$$\underline{D}(\mathcal{A})\underline{D}(\mathcal{B}) \le \underline{D}([\mathcal{A}, \mathcal{B}])\underline{D}((\mathcal{A}, \mathcal{B})).$$

This inequality is from the class of correlation inequalities. The proof of this inequality is literally the same as when the sets $\mathcal{A}$ and $\mathcal{B}$ are sets of positive integers and this proof can be found in [AK97a].