THE NUMBER OF VALUES OF COMBINATORIAL FUNCTIONS

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1. *Introduction*

An important discovery of Marica–Schönheim [5] is

THEOREM 1. *Any m distinct sets have at least m distinct differences.*

This result has various generalisations [1–4] and one of a new kind is

THEOREM 2. *If* $S_1, \ldots, S_m$ *are distinct sets, and* $T_1, \ldots, T_n$ *are sets such that each* S_i *has some* T_j *as a subset, then there are at least m distinct differences* $S_i \setminus T_j$.

The form of this theorem led us to conjecture that m sets can be partitioned into $\mathfrak{S}$ and $\mathfrak{T}$ having $m-1$ differences $S \setminus T$ with $S \in \mathfrak{S}$, $T \in \mathfrak{T}$. Given an ordered sequence $S_1, \ldots, S_m$ of sets let d be the number of differences $S_i \setminus S_j$ with $i < j$ and e be the largest n for which there are $S_{i_1} \subset \ldots \subset S_{i_n}$ with $1 \leqslant i_1 < \ldots < i_n \leqslant m$. We believe m, d, e are related.

The generalisation of Theorem 1 by Daykin–Lovász [4] is

THEOREM 3. *Any non-trivial Boolean function takes at least m distinct values when evaluated over m distinct sets.*

We give a generalisation of this theorem, which also yields a new proof.

2. *Proof of Theorem* 2

We may assume all the sets S_i, T_j are subsets of $\{1, 2, \ldots, r\}$ and use induction on r. The case $r = 1$ is trivial. Put $\mathfrak{S} = \{S_1, \ldots, S_m\}$, $\mathfrak{T} = \{T_1, \ldots, T_n\}$, $\mathfrak{A} = \{S \setminus r : S \setminus r \in \mathfrak{S}$ and $S \cup r \in \mathfrak{S}\}$, $\mathfrak{B} = \{S \setminus r : S \in \mathfrak{S}\}$, $\mathfrak{C} = \{T : r \notin T \in \mathfrak{T}\}$ and $\mathfrak{D} = \{T \setminus r : T \in \mathfrak{T}\}$. Then $m = |\mathfrak{A}| + |\mathfrak{B}|$, where $|.|$ denotes cardinality. Also $\mathfrak{A}, \mathfrak{C}$ and $\mathfrak{B}$, $\mathfrak{D}$ satisfy the hypothesis on $\{1, 2, \ldots, r-1\}$ so $|\mathfrak{A}| \leqslant |\mathfrak{A} \setminus \mathfrak{C}|$ and $|\mathfrak{B}| \leqslant |\mathfrak{B} \setminus \mathfrak{D}|$. If $E \in \mathfrak{A} \setminus \mathfrak{C}$ then $E = A \setminus C$ for some $A \in \mathfrak{A}$, $C \in \mathfrak{C}$. Thus $A \setminus r$, $A \cup r \in \mathfrak{S}$ and $r \notin C \in \mathfrak{T}$ so $E \setminus r$, $E \cup r \in \mathfrak{S} \setminus \mathfrak{T}$. On the other hand if $E \in \mathfrak{B} \setminus \mathfrak{D}$ then clearly either $E \setminus r$ or $E \cup r$ is in $\mathfrak{S} \setminus \mathfrak{T}$. Hence $|\mathfrak{A} \setminus \mathfrak{C}| + |\mathfrak{B} \setminus \mathfrak{D}| \leqslant |\mathfrak{S} \setminus \mathfrak{T}|$ and the result follows.

3. *Generalisation of Theorem* 3

Let c be a fixed positive integer. If S is a set then S^c denotes the set of all c-dimensional vectors with elements in S, and a c-ary operation f on S is a mapping $f : S^c \to S$. Given such a map f for $A_1, \ldots, A_c \subset S$ put

$$f(A_1, \ldots, A_c) = \{f(a_1, \ldots, a_c) : a_i \in A_i \quad \text{for} \quad 1 \leqslant i \leqslant c\}.$$

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Call f *expansive* if

$$|A| \leqslant |f(A, \ldots, A)| \quad \text{for all} \quad A \subset S.$$

Call f *c-expansive* if

$$|A_1| \leqslant |f(A_1, \ldots, A_c)| \quad \text{for all} \quad A_1, \ldots, A_c \subset S \quad \text{with} \quad |A_1| = \ldots = |A_c|.$$

Notice that when $|S| = 2$ expansive is the same as c-expansive and simply means non-constant Boolean function.

If S, T are sets and $f : S^c \to S$ while $g : T^c \to T$ we define the *direct product* h *of* f and g to be the map $h : (S \times T)^c \to S \times T$ such that

$$h((s_1, t_1), \ldots, (s_c, t_c)) = (f(s_1, \ldots, s_c), g(t_1, \ldots, t_c)) \text{ for all } s_i \in S \text{ and } t_j \in T.$$

The direct product of expansive maps is not expansive, for example let $c = 2, S = \{0, 1, 2\}$, $f(a, b) = \max\{0, a-b\}$, take the direct product of f with itself and $A = (S \times S) \setminus \{(0, 0), (2, 2)\}$. It would be interesting to have more results like

THEOREM 4. *In the above notation, if f is expansive and g is c-expansive then h is expansive.*

Proof. If $B \subset S \times T$ and m is a positive integer let B_m be the set of all $s \in S$ such that $(s, t) \in B$ for at least m different $t \in T$. Let $A \subset S \times T$ be given and $x \in f(A_m, \ldots, A_m)$. Thus there are $s_1, \ldots, s_c \in A_m$ with $x = f(s_1, \ldots, s_c)$. For $1 \leqslant i \leqslant c$ there are distinct $t_{i1}, \ldots, t_{im} \in T$ with $(s_i, t_{ij}) \in A$ for $1 \leqslant j \leqslant m$. By hypothesis on g we have

$$m \leqslant |\{g(t_{1j_1}, \ldots, t_{cj_c}) : 1 \leqslant j_1, \ldots, j_c \leqslant m\}|$$

and this means that $x \in (h(A, \ldots, A))_m$. Finally

$$|A| = \sum |A_m| \leqslant \sum |f(A_m, \ldots, A_m)| \leqslant \sum |(h(A, \ldots, A))_m| = |h(A, \ldots, A)|$$

and the proof is complete.

Now let $|S| = 2$ and $f_1, \ldots, f_n : S^c \to S$. Further let $\mathfrak{N}$, $\mathfrak{P}$ be the set of all matrices of order $n \times c$, $n \times 1$ respectively with elements in S. Define $e : \mathfrak{N} \to \mathfrak{P}$ by

$$e(a_{ij}) = \begin{pmatrix} f_1(a_{11}, & \ldots, & a_{1c}) \\ & \vdots & \\ f_n(a_{n1}, & \ldots, & a_{nc}) \end{pmatrix} \quad \text{for all} \quad (a_{ij}) \in \mathfrak{N}.$$

By induction on n we immediately get from Theorem 4 that if $f_1, \ldots, f_n$ are non-constant then e is expansive. The case of this with $f_1 = \ldots = f_n$ is Theorem 3.

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