Integral Inequalities for Increasing Functions.

Rudolf Ahlswede, Universität Bielefeld, Germany

and

David E. Daykin, University of Reading, England.

Abstract. For numbers of increasing real functions $f(x)$ with $\int_{-1}^{+1} f(x)\, dx \geq 0$ we give new integral inequalities. They generalize classical results. The proofs are short and simple being based on sequences.

1. Introduction.

Let E be the set of all real functions $f(x)$ defined and increasing for $-1 \leq x \leq 1$. Let F be the set of members of E with

$$0 \leq \int_{-1}^{1} f = \int_{x=-1}^{1} f(x)\, dx \tag{1}$$

Also let G be the subset of F with equality in (1). Our main results are:

THEOREM 1. *If* $f_1, \ldots, f_r \in E$ *and*

$$0 \leq f_i(0) + \int_0^1 f_i(x)\, dx \qquad \text{for} \quad 1 \leq i \leq r \tag{2}$$

then

$$0 \leq \left(\int_0^1 f_1\right) \cdots \left(\int_0^1 f_r\right) \leq \int_0^1 f_1 \cdots f_r \,.$$

Notice that for f_i defined and increasing for $0 \leq x \leq 1$ the condition (2) simply requires that f_i can be extended to lie in F.

THEOREM 2. *If* r *is even and* $f_1, \ldots, f_r \in G$ *then*

$$0 \leq \int_{-1}^{1} f_1 \cdots f_r \,.$$

THEOREM 3. If $f_1, \ldots, f_r, g_1, \ldots, g_s \in F$ and $0 \le \theta \le 1$ then

$$\left(\int_\theta^1 f_1 \cdots f_r\right)\left(\int_\theta^1 g_1 \cdots g_s\right) \le (1-\theta)\int_\theta^1 f_1 \cdots f_r g_1 \cdots g_s \ .$$

THEOREM 4. If r, s are odd and $f_1, \ldots, f_r, g_1, \ldots, g_s \in G$ then

$$\left(\int_{-1}^1 f_1 \cdots f_r\right)\left(\int_{-1}^1 g_1 \cdots g_s\right) \le \int_{-1}^1 f_1 \cdots f_r g_1 \cdots g_s \ .$$

These results will follow immediately from their analogues for sequences which we proceed to prove and discuss.

2. Finite increasing sequences. Let n be a fixed positive integer. Abusing our notation we now let E be the set of all real sequences $f(1) \le \ldots \le f(n)$. We let F be the members of E with

$$0 \le \Sigma f = f(1) + \ldots + f(n) \ . \tag{3}$$

Also we let G be the subset of F with equality in (3).

THEOREM 1'. If $f_1, \ldots, f_r \in E$ and

$$0 \le (n-1)\, f_i(1) + f_i(2) + f_i(3) + \ldots + f_i(n) \quad \text{for} \quad 1 \le i \le r$$

then $\quad 0 \le (n^{-1}\Sigma f_1) \ldots (n^{-1}\,\Sigma f_r) \le n^{-1}\,\Sigma f_1 \ldots f_r$.

The familiar Chebychev type inequality ([3], 2.17) says that if $f_1, \ldots, f_r \in E$ and are non-negative then for positive integers s

$$\left(n^{-1}\Sigma\, f_1^s\right)^{1/s} \ldots \left(n^{-1}\Sigma\, f_r^s\right)^{1/s} \le \left(n^{-1}\Sigma(f_1 \ldots f_r)^s\right)^{1/s} \ .$$

Clearly this inequality follows immediatley from Theorem 1'. It is more convenient to prove a slightly different form of Theorem 1' namely

<u>THEOREM 1".</u> <u>If</u> $f_1, \ldots, f_r \in F$ <u>and</u> t <u>is an integer in</u> $\frac{1}{2}n \le t \le n$ <u>then</u>

$$\mu_1 \ldots \mu_r \le m^{-1}\Sigma\, f_1(x) \ldots f_r(x) \tag{4}$$

<u>where</u> $0 \le \mu_i = m^{-1}\Sigma\, f_i(x)$ <u>for</u> $1 \le i \le r$,

<u>and</u> $m = n - t + 1$ <u>while summation is over</u> $t \le x \le n$.

<u>Proof.</u> We may assume that there is a smallest integer p in $t-1 \le p \le n$ such that for each i in $1 \le i \le r$ we have $0 < f_i(p+1) = \ldots = f_i(n) = g_i$ say. If $t-1 = p$ then (4) holds with equality. So assume $t-1 < p$ and put $q = n-p$ and $h_i = f_i(p)$ and $k_i = (qg_i + h_i)/(q+1)$. Then for each i because $f_i \in F$ we have $k_i \le g_i$ and $0 \le ph_i + qg_i$ so $|h_i| \le g_i$ so $h_i \le k_i$ and $0 < k_i$. We change f_i to a new function f_i^* by changing $f_i(x)$ to k_i for $p \le x \le n$. Then $f_i^* \in F$ and has the same μ_i as f_i . Further $\Sigma f_1^* \ldots f_r^* \le \Sigma f_1 \ldots f_r$ with summation over $t \le x \le n$, because ≤

$$(q + 1)\, \Pi k_i \le (q\, \Pi g_i) + \Pi h_i \tag{5}$$

with products over $1 \le i \le r$. The result (4) follows by repetition of this process. It is easy to prove (5) by induction on r .

If r is to be allowed to get large the condition $\frac{1}{2}n \le t$ of Theorem 1" is necessary. To see this let all $f_i \in G$ and be 1 for ~~$\frac{1}{2}(n-1)$~~ $\le x \le n$ and constant elsewhere. $\frac{1}{2}n$

<u>THEOREM 2'.</u> <u>If</u> r <u>is even and</u> $f_1, \ldots, f_r \in G$ then $0 \le \Sigma f_1 \ldots f_r$ <u>with summation over</u> $1 \le x \le n$.

Proof. Split the sum at $\frac{1}{2}n$ and apply Theorem 1" to each half.

Inversion and change of sign for each f_i shows there is no such result for r odd.

THEOREM 3'. *If* $f_1, \ldots, f_r, g_1, \ldots, g_s \in F$ *and* $\frac{1}{2}n \le t \le n$ *put*

$$A = \Sigma f_1 \ldots f_r , \quad B = \Sigma g_1 \ldots g_s , \quad C = \Sigma f_1 \ldots f_r g_1 \ldots g_s \tag{6}$$

with summation over $t \le x \le n$ *then* $AB \le (n - t + 1)C$.

Proof. We may assume $f_i(n) = g_j(n) = 1$ for all i, j . Then there will be a smallest integer p in $t-1 \le p \le n$ such that $f_i(x) = g_j(x) = 1$ for $p < x \le n$ and all i, j . If $t-1 = p$ the result holds with equality, so assume $t-1 < p$.

Now $-1 \le f_i(p), g_j(p) \le 1$ for all i, j . If say $f_1(p)$, $f_2(p) < 0$ then we change f_1, f_2 into two new functions f_1^*, f_2^* by changing $f_1(p), f_2(p)$ into $-f_1(p), -f_2(p)$ respectively. Clearly $f_1^*, f_2^* \in F$ and A, B, C do not change. So we may assume $0 \le f_2(p), \ldots, f_r(p)$ and that $c = f(p) < 1$ where f now denotes f_1 .

Put $q = n - p$ and $d = f_2(p) \ldots f_r(p)$ and $e = g_1(p) \ldots g_s(p)$ and $b = (q + cd)/(q + d)$. Notice that $0 \le d \le 1$ so $-1 \le c \le b$ and $0 \le b$, and trivially $-1 \le e \le 1$. We change f into a new function f^* by changing $f(x)$ to b for $p \le x \le n$. Let A^*, B^*, C^* denote the corresponding new values of A, B, C . Now f^* is increasing and the inequality $\Sigma f \le \Sigma f^*$ is equivalent to $0 \le q(1 - c)(1 - d)$ so $f^* \in F$. Observe that $A^* = A$ by definition of b, and trivially $B^* = B$. Finally the inequality $C^* \le C$ holds because it is equivalent to $qb + bde \le q + cde$ which is $0 \le qd(1 - c)(1 - e)$.

If $b = 0$ then $f^* = 0$ and the result holds. If $0 < b$ we divide f^* by b and go back to the beginning of the proof. The theorem follows by repetition of this process.

THEOREM 4'. *If* r, s *are odd and* $f_1, \ldots, f_r, g_1, \ldots, g_s \in G$ *and* A, B, C *are defined by* (6) *with summation over* $1 \le x \le n$ *then* $AB \le \tfrac{1}{2}nC$.

Proof. Suppose first that n is even. We use (6) to define A_1, B_1, C_1 with summation over $1 \le x \le \tfrac{1}{2}n$ and A_2, B_2, C_2 with summation over $\tfrac{1}{2}n < x \le n$. Thus $A = A_1 + A_2$ and similarly for B, C. Now Theorem 3' says that $A_2B_2 \le \tfrac{1}{2}nC_2$. If we multiply all f_i, g_j by -1 it also says that $A_1B_1 \le \tfrac{1}{2}nC_1$. Similarly from Theorem 1" we find that $A_1, B_1 \le 0 \le A_2, B_2$. It is now clear that $AB \le \tfrac{1}{2}nC$. This case n even of this theorem yields Theorem 4 which in turn contains the case n odd of this theorem.

We now give an example to show that the constant $\tfrac{1}{2}n$ in Theorem 4' is best possible. We let all f_i be $-1, \ldots, -1, 0, \ldots, 0, p$ and all g_j be $a, \ldots, a, 1, \ldots, 1$ with $a = -(\tfrac{1}{2}n - 1)/(\tfrac{1}{2}n + 1)$ then $A \sim p^r$ and $B \sim \tfrac{1}{2}n - 1$ while $C \sim p^r$. Examples of the form $-1, \ldots, -1, n-1$ and $-n+1, 1, \ldots, 1$ indicate that there are no other inequalities between AB or $|A||B|$ and C or $|C|$ with summation over $1 \le x \le$ ~~m~~. n

<u>DEFINITION</u>. We say non-negative real numbers $w(t), \ldots, w(n)$ are <u>good weights</u> if $\frac{1}{2}n \le t$ and for all $f_1, \ldots, f_r \in F$ we have

$$0 \le \Sigma w\, f_1 \ldots f_r \tag{7}$$

with summation over $t \le x \le n$.

Thus good weights are related to Theorems 1, 1', 1'' . We could not find weights for the other theorems.

Let H be the set of all $f \in G$ of the form $-p/q, \ldots, -p/q, 0, \ldots, 0, 1, \ldots, 1$ where $q, n-p-q, p$ terms have the value $-p/q, 0, 1$ respectively and the positive integers p, q have $p+q \le n$. It is easy to see that H is a basis for G . If we adjoin the function $1, \ldots, 1$ to H we get a basis for F .

<u>THEOREM 5</u>. <u>The non-negative reals</u> $w(t), \ldots, w(n)$ <u>with</u> $\frac{1}{2}n \le t$ <u>are good weights iff</u> (7) <u>holds whenever</u> $r = 1$ <u>and</u> $f_1 \in H$.

<u>Proof</u>. Necessity is obvious, so to show sufficiency let $f_1, \ldots, f_r \in F$. By linearity we may assume $f_1, \ldots, f_r \in H$. There is a least p in $t-1 \le p \le n$ such that $f_i(x) = 1$ for all i and $p < x \le n$. If $t-1 = p$ then (7) clearly holds, so assume $t-1 < p$ and $f_1(p) < 1$. Then by inspection of the functions in H we see that $w(x)\, f_1(x) \le w(x)\, f_1(x) \ldots f_r(x)$ for $t \le x \le m$ and the theorem is proved.

<u>3. Remarks on Lattices</u>. The FKG and GKS inequalities of physics have many applications (see [1, 2, 4, 5]). It was trying to generalise them that led to this paper. Let L be the lattice of subsets of a

finite set. Examples show that our above results do not generalise to L. For $\alpha, \lambda \in L$ let $\sigma_\alpha(\lambda)$ be 1 if $\alpha \subset \lambda$ but -1 otherwise. The case $|\alpha| = 1$ of these functions σ_α is used in physics. We do not allow $|\alpha| = 0$. Then it is easy to see that $\Sigma\sigma_\alpha \leq 0 \leq \Sigma\sigma_\alpha\sigma_\beta$ where summation is over $\lambda \in L$. We have proved that $\Sigma\sigma_\alpha\sigma_\beta\sigma_\gamma$ is >0 if $1 = |\alpha| < |\beta|$ and $\alpha \not\subset \beta$ and $\alpha \cup \beta \subset \gamma$, is $=0$ if $\alpha = \{1, 2\}$, $\beta = \{2, 3\}$, $\gamma = \{1, 3\}$, but is <0 otherwise. Elementary arguments show that $0 \leq (-1)^r\Sigma\sigma_{\alpha_1} \ldots \sigma_{\alpha_r}$ if $r = 4$ and $|\alpha_i| = 2$ or if $r \leq 2^{s-1}$ and $s \leq |\alpha_i|$. We omit the proofs.

REFERENCES.

1. R. Ahlswede, D. E. Daykin, An inequality for the weights of two families of sets, their unions and intersections, Z. Wahrscheinlichkeitstheorie und Verw. Gebeite 43(1978) 183-185.

2. R. Ahlswede, D. E. Daykin, Inequalities for a pair of maps $S \times S \to S$ with S a finite set. Math. Z. 165(1979) 267-289.

3. G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press (1959).

4. D. G. Kelley, S. Sherman, General Griffith's inequalities on correlations in Ising ferromagnets, J. Math. Phys. 9(1968) 466-484.

5. P. D. Seymour, D. J. A. Welsh, Combinatorial applications of an inequality from statistical mechanics, Math. Proc. Camb. Phil. Soc. 77(1975) 485-495.