A Recursive Bound for the Number of Complete K-Subgraphs of a Graph

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Abstract

The following inequality was conceived as a tool in determining coloring numbers in the sense of Ahlswede, Cai, Zhang ([1]), but developed into something of a seemingly basic nature.

Theorem For any graph $G = (\Omega_n, \mathcal{E})$ with n vertices let T_k be the number of complete k–subgraphs of G. Then for $k \geq 2$

$$T_k \geq \frac{T_{k-1}}{k \cdot n}[2(k-1)|\mathcal{E}| - (k-2)n^2] . \tag{1}$$

Proof of the Theorem stated in the abstract. By its definition we have $T_1 = n$. We show (1) by induction on k.

For $k = 2$ $T_2 = |\mathcal{E}| = \frac{n}{2 \cdot n}[2|\mathcal{E}|] = |\mathcal{E}|$, so (1) holds even with equality. For the induction step from $k-1$ to k we need some notation.

$V(m)$ denotes a set with m vertices and $\mathcal{T}_m$ stands for the set of all those sets, which are the vertex set of a complete m–subgraph. We also set

$$\mathcal{E}_v = \{v' : (v, v') \in \mathcal{E}\} , \tag{2}$$

$$\mathcal{E}_{V(m)} = \cap_{v \in V(m)} \mathcal{E}_v , \tag{3}$$

$$\mathcal{T}_m(V(m-1)) = \{V(m) \in \mathcal{T}_m : V(m) \supset V(m-1)\} , \tag{4}$$

and start now with

$$T_k = \frac{1}{k} \sum_{V(k-1) \in \mathcal{T}_{k-1}} |\mathcal{E}_{V(k-1)}| . \tag{5}$$

Next we bound $|\mathcal{E}_{V(k-1)}|$ from below with the help of the identity

$$\left| \bigcup_{V(k-2) \subset V(k-1)} \mathcal{E}_{V(k-2)} \right| = \sum_{V(k-2) \subset V(k-1)} |\mathcal{E}_{V(k-2)}| - (k-2)|\mathcal{E}_{V(k-1)}| , \tag{6}$$

which holds, because vertices from the union which are counted more than once in the sum are actually counted $k-1$ times and they are exactly the vertices in $\mathcal{E}_{V(k-1)}$.

Since the union has a cardinality not exceeding n, we get

$$|\mathcal{E}_{V(k-1)}| \geq \frac{1}{k-2} \left(\sum_{V(k-2) \subset V(k-1)} |\mathcal{E}_{V(k-2)}| - n \right) . \tag{7}$$

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Substituting this in (5) yields

$$\begin{aligned}
k(k-2)T_k &\geq \sum_{V(k-1)\in\mathcal{T}_{k-1}} \left(\sum_{V(k-2)\subset V(k-1)} |\mathcal{E}_{V(k-2)}| - n \right) \\
&= \sum_{V(k-2)\in\mathcal{T}_{k-2}} \sum_{V(k-1)\in\mathcal{T}_{k-1}(V(k-2))} |\mathcal{E}_{V(k-2)}| - n|\mathcal{T}_{k-1}| \\
&= \sum_{V(k-2)\in\mathcal{T}_{k-2}} |\mathcal{T}_{k-1}(V(k-2))|^2 - n\, T_{k-1} \\
&\geq T_{k-2}\left(\frac{(k-1)T_{k-1}}{T_{k-2}}\right)^2 - n\, T_{k-1} \quad \text{(by convexity of } x^2\text{)} \\
&= \frac{T_{k-1}}{T_{k-2}}((k-1)^2 T_{k-1} - n\, T_{k-2}) \\
&\geq \frac{T_{k-1}}{T_{k-2}}\left((k-1)\frac{T_{k-2}}{n}(2(k-2)|\mathcal{E}| - (k-3)n^2) - n\, T_{k-2}\right) \\
&= \frac{T_{k-1}}{n}(2(k-1)(k-2)|\mathcal{E}| - ((k-1)(k-3)+1)n^2)
\end{aligned}$$

and therefore (1).

The following consequence is useful.

Corollary If for some $\alpha > 0$ $|\mathcal{E}| \geq \frac{k-1}{2k}n^2 + \alpha n^2$, then

$$T_{k+1} \geq \alpha^k\, n^{k+1} \; . \tag{8}$$

Proof: Since $\frac{k-1}{2k} \geq \frac{\ell-1}{2\ell}$ for $\ell = 1, 2, \ldots, k$, the assumption implies

$$2\ell|\mathcal{E}| - (\ell-1)n^2 \geq 2\ell\cdot\alpha n^2$$

and therefore by (1) and since $T_1 = n$

$$T_{\ell+1} \geq \frac{1}{\ell+1}\frac{T_\ell}{n} 2\ell\, \alpha n^2 \geq \alpha n T_\ell \; ,$$

which implies (8).

Remark Our result falls into the context of paragraph VI.1 of [2]. A well–known result by Turan ([3]) concerns the determination of the maximal number $t_k(n)$ of edges in an n–graph such that $T_{k+1} = 0$.

The optimal graphs have the following structure:

For $n = km + r, r < k$, partition Ω_n into r sets with $m+1$ vertices and $k-r$ sets with m vertices and include exactly all edges connecting vertices of different sets.

Therefore one has for Turan's function

$$t_k(n) = \binom{r}{2}(m+1)^2 + \binom{k-r}{2}m^2 + r(k-r)(m+1)m \ . \tag{9}$$

It is remarkable that our quite general inequality almost implies this identity. In fact, in an optimal graph clearly $T_k \geq 1$, because otherwise an edge could be added. Therefore from the inequality we conclude

$$|\mathcal{E}| \leq \frac{n^2(k-1)}{2 \cdot k} \tag{10}$$

and if n is a multiple of k , that is, $n = m \cdot k$, then (10) takes the form $|\mathcal{E}| \leq m^2\binom{k}{2}$ and thus the bound in (9) follows.

For general $n = km + r$ an easy calculation shows that the bound in (10) is tight, if $\frac{(k-r)r}{2k} < 1$. This is for instance always the case also for $r = 1, 2$.

[1] R. Ahlswede, N. Cai and Z. Zhang, "Rich colorings with local constraints", Preprint in SFB Diskrete Strukturen, Bielefeld 1989.

[2] B. Bollobás, "Extremal Graph Theory", Acad. Press, 1978.

[3] P. Turan, "An extremal problem in graph theory (Hungarian)", Mat.Fiz. Lapok, 48, 436–452, 1941.