RICH COLORINGS WITH LOCAL CONSTRAINTS

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Abstract

A new hypergraph coloring problem is introduced by defining $N(H,e)$ as the maximal number of colors in a vertex coloring of a hypergraph $H=(V,\mathcal{E})$, which has not more than e different colors in every edge. Our main results concern the asymptotic behaviour of this quantity for the uniform hypergraph $H(n,\ell,k)=(\mathcal{V}(n,\ell),\mathcal{E}(n,\ell,k))$ with vertex set $\mathcal{V}(n,\ell)=\binom{\Omega_n}{\ell}$ for $\Omega_n=\{1,2,\dots,n\}$ and edge set $\mathcal{E}(n,\ell,k)=\left\{E=\binom{A}{\ell}:A\in\binom{\Omega_n}{k}\right\}$. In case $\ell=2$ there are connections to Turan's graph.

1. Introduction

As a natural generalization of the concept of a chromatic number of a graph (which includes also several of its generalizations suggested by others (see ch. 19 of [6])), Erdös and Hajnal [1] introduced the chromatic number $\psi(H)$ of a hypergraph $H = (V, \mathcal{E})$ as the minimal number of colors needed to color the vertices such that no edge $E \in \mathcal{E}$ with $|\mathcal{E}| > 1$ has all its vertices with the same color.

A stronger notion requires that all vertices in an edge get different colors. However, this is equivalent to coloring the graph with vertex set V and any two vertices x, y joined by an edge, if for some $E \in \mathcal{E} \quad x, y \in E$.

Related, but weaker, notions of hypergraph coloring where introduced in [2]. There it was demonstrated that the essence of many multi–user source coding problems is a statement about vertex colorings of hypergraphs, which assign to the vertices of every edge E a certain percentage, that is, $\varepsilon|E|$ different colors. Another notion requires that in no edge E a colour occurs more than k resp. $\delta|E|$ times.

In the study of memories, which we introduced in [3], we encountered still another hypergraph coloring problem. $H = (V, \mathcal{E})$ is said to carry M colors, if there is a vertex coloring with M colors, that is, a surjective map $\varphi : V \to \{1, 2, \dots, M\}$, such that all M colors occur in every edge.

Let $M(V, \mathcal{E})$ be the maximal number of colors carried by $(V, \mathcal{E})$. Clearly, $M(V, \mathcal{E}) \leq \min_{E \in \mathcal{E}} |E|$. A simple probabilistic argument yields a lower bound.

Coloring Lemma AZ [3].

If $|V| \geq 3$, then $M(V, \mathcal{E}) \geq (\ell n |V|)^{-1} \min_{E \in \mathcal{E}} |E|$.

Since in typical applications in Information Theory the quantities $|V|$ and $|\mathcal{E}|$ grow exponentially in the blocklength n we have there $M(V, \mathcal{E}) \sim \min_{E \in \mathcal{E}} |E|$.

If numbers are not in this range it is much harder to derive bounds. The determination of $M(V, \mathcal{E})$ for any hypergraph $(V, \mathcal{E})$ is a problem of considerable generality. It includes problems of

To see this, let us define for positive integers n, k, ℓ with $n > k > \ell$ a hypergraph $H(n, \ell, k) = (\mathcal{V}(n, \ell), \mathcal{E}(n, \ell, k))$ with vertex–set $\mathcal{V}(n, \ell) = \binom{\Omega_n}{\ell}$ for $\Omega_n = \{1, 2, \dots, n\}$ and edge–set $\mathcal{E}(n, \ell, k) = \left\{ E = \binom{A}{\ell} : A \in \binom{\Omega_n}{k} \right\}$. Now the classical Ramsey number $n(k, \ell)$ is the smallest integer such that for $n \geq n(k, \ell) \qquad M(\mathcal{V}(n, \ell), \mathcal{E}(n, \ell, k)) = 1$.

The work [3] on write–efficient memories has been continued in [4] to include cases where several persons use the same storage medium subject to certain priority rules. This has led us to several novel extremal problems. Most of them are very complex. Among the accessible one's there is the following coloring problem, which seems to be basic. As in the previous coloring problem one wants many colors "in V ". However now one wants only "a few" colors in every edge. The formal description follows.

We denote the cardinality of the range of a function by $\parallel g \parallel$. For the hypergraph $H = (V, \mathcal{E})$ a map $f : V \to \mathbb{N}$ is called <u>e–coloring</u> , if

$$\parallel f_{|E} \parallel \leq e \text{ for all } E \in \mathcal{E} \ , \tag{1.1}$$

where $f_{|E}$ is the restriction of f to E .

We call an e–coloring f an <u>(N, e)–coloring</u> , if

$$\parallel f \parallel = N \ . \tag{1.2}$$

As a basic quantity we introduce $N(H, e)$ as the maximal N for which an (N, e)–coloring of H exists. In particular we are interested here in the hypergraph $H(n, \ell, k)$. The set of its e–colorings is $\Phi(n, k, \ell, e)$.

With the abbreviation $N(n, k, \ell, e) \stackrel{\text{def}}{=} N(H(n, \ell, k), e)$ we can thus write

$$N(n, k, \ell, e) = \max\{\parallel \varphi \parallel : \varphi \in \Phi(n, k, \ell, e)\} \ . \tag{1.3}$$

Since obviously

$$N(n, k, 1, e) = \begin{cases} e & \text{for } e < k \\ n & \text{for } e \geq k \end{cases}$$

we study cases with $\ell \geq 2$. Our best results are for $\ell = 2$.

They are formulated in terms of the following threshold functions indexed by $i = 1, 2$; $s = 0$ and $s = 1, 2, \dots$.

$$e_i(0, k) = \sup\{e : \lim_{n \to \infty} \frac{1}{n^i} N(n, k, 2, e) = 0\} \ , \tag{1.4}$$

$$e_i(s, k) = \sup\{e : \lim_{n \to \infty} \frac{1}{n^i} [N(n, k, 2, e) - N(n, k, 2, e_2(s-1, k) + 1)] = 0\}. \tag{1.5}$$

They were found to be appropriate tools for catching the structure of this coloring problem. Since the total number of vertices is $\frac{n(n-1)}{2}$ the order of the number of colors cannot exceed n^2 . So we ask how big e has to be when this magnitude occurs. Also a linear growth is interesting.

In the analysis of these functions we use another usefull concept. For fixed $\ell \in \mathbb{N}$ we call $F : \mathbb{N} \to \mathbb{N}$ an ℓ–local– global function, if for all $n > m > \ell$

$$N(n, m, \ell, F(m)) \leq F(n) \ . \tag{1.6}$$

For $\ell = 2$ there are local–global functions, which are closely related to the Turán function $t_p(n) = \sum_{0 \le i < j < p} \left\lfloor \frac{n+i}{p} \right\rfloor \cdot \left\lfloor \frac{n+j}{p} \right\rfloor$, which counts the number of edges in the Turán graph $C_p(n)$.

This is a complete p–partite graph with r vertex sets of size $q+1$ and $p-r$ vertex sets of size q , when $n = pq + r, 0 \le r < p$.

It is convenient to denote these sets by $Q_i (i = 1, 2, \dots, p)$ and to let Q_1 contain the first numbers in Ω_n, Q_2 the next and so on.

For any graph $G = (\Omega_n, \mathcal{E})$ we denote by $V(m)$ a set with m vertices, by $\mathcal{T}_m$ the set of $V(m)$, which are vertex sets of complete subgraphs, and we denote $|\mathcal{T}_m|$ by $T_m = T_m(G)$.

Turán's result is that up to relabelling of the vertices $C_p(n)$ is the only graph with a number of edges equal to $\max\{|\mathcal{E}| : T_{p+1}(\Omega_n, \mathcal{E}) = 0\}$.
The paper is organized as follows:

<u>In Section 2</u> we give in Theorem 1 a sufficient condition for $F : \mathbb{N} \to \mathbb{N}$ to be ℓ–local–global . This condition can be expressed in terms of the decrement of density

$$\triangle_\ell^n(F) = \frac{F(n+1)}{\binom{n+1}{\ell}} - \frac{F(n)}{\binom{n}{\ell}} \ .$$

Next we upper bound in Theorem 2 this decrement for $N(n, k, \ell, e)$.

<u>In Section 3</u> rather exact results are derived for the 2–local–global functions $t_p(n)$ (Theorem 3).

<u>In Section 4</u> this result in conjunction with a probabilistic argument are used to express threshold functions in terms of Turán's function.

<u>In Section 5</u>, finally, the method of proof for Theorem 1 is used to derive very general statements of Turán–type. Instead of excluding certain complete subgraphs a more general constraint on the number of edges in certain subgraphs is imposed.

2. Coloring properties of $H(n, l, k)$ in terms of local–global functions

With every coloring $\varphi \in \Phi(n, k, \ell, e)$ we associate a system of distinct representatives $\mathcal{R}_\varphi = \{E_i : 1 \le i \le \| \varphi \|\}$, where $E_i \in \varphi^{-1}(i) \subset \mathcal{V}(n, \ell)$ and is arbitrary otherwise. We need below the hypergraph $I_\varphi = (\Omega_n, \mathcal{R}_\varphi)$.

Theorem 1. *Let* $\ell \in \mathbb{N}$ *be fixed.*

If $F : \mathbb{N} \to \mathbb{N}$ *satisfies for every* $k = \ell + 1, \ldots, u$

$$\frac{F(k+1)+1}{\binom{k+1}{\ell}} + \frac{2}{(k+1)\binom{k}{\ell}} > \frac{F(k)}{\binom{k}{\ell}} \text{ and } F(k) < \binom{k}{\ell}$$

then for all n *and* m *with* $\ell < m \leq n \leq u + 1$

$$N(n, m, \ell, F(m)) \leq F(n) \ .$$

Proof. We proceed by induction on n . Clearly, the statement is true for $n = m$. Assume now that is true for $k \geq m$, but that it is not true for $k+1 \leq s+1$, that is, $N(k+1, m, \ell, F(m)) > F(k+1)$. This means that a $\varphi \in \Phi(k+1, m, \ell, F(m))$ exists with $\| \varphi \| = F(k+1) + x$ for some $x \in \mathbb{N}$.

Removing now the element j from Ω_{k+1} we get a subhypergraph $H_j(\mathcal{V}_j(k+1, \ell), \mathcal{E}_j(k+1, \ell, m))$, where

$$\mathcal{V}_j(k+1, \ell) = \binom{\Omega_{k+1} - \{j\}}{\ell} \text{ and } \mathcal{E}_j(k+1, \ell, m) = \left\{ E = \binom{A}{\ell} : A \in \binom{\Omega_{k+1} - \{j\}}{m} \right\}.$$

We also get a subhypergraph $I_{\varphi j}$ of I_φ , which has the vertex set $\Omega_{k+1} - \{j\}$ and edge set $\mathcal{R}_{\varphi j} = \{R : R \in \mathcal{R}_\varphi \text{ and } j \notin R\}$. Let φ_j be the restriction of φ on H_j . By the definition of $I_{\varphi j}$ we have

$$|\mathcal{R}_{\varphi j}| \leq \| \varphi_j \| \ . \tag{2.1}$$

Consequently, the degree $d(j)$ of j in I_φ satisfies

$$\begin{aligned} d(j) &= |\{R : R \in \mathcal{R}_\varphi \text{ and } j \in R\}| = |\mathcal{R}_\varphi| - |\mathcal{R}_{\varphi j}| \\ &= \| \varphi \| - |\mathcal{R}_{\varphi j}| \geq \| \varphi \| - \| \varphi_j \| = F(k+1) + x - \| \varphi_j \| \ . \end{aligned} \tag{2.2}$$

Furthermore, replacing Ω_k by $\Omega_{k+1} - \{j\}$, φ_j induces an isomorphic map φ_j in $\Phi(k, n, \ell, F(m))$ and thus by induction hypothesis $\| \varphi_j \| \leq F(k)$. Hence

$$d(j) \geq F(k+1) + x - F(k) \tag{2.3}$$

and therefore

$$(k+1)[F(k+1) + x - F(k)] \leq \sum_{j=1}^{k+1} d(j) = |\mathcal{R}_\varphi| \cdot \ell = \| \varphi \| \ \ell = \ell(F(k+1) + x) \ . \tag{2.4}$$

We show next that (2.4) can be improved to

$$(k+1)[F(k+1)+x-F(k)]+2 \leq \ell[F(k+1)+x] \ . \tag{2.5}$$

From here one readily calculates that

$$\begin{aligned}\frac{F(k)}{\binom{k}{\ell}} &\geq \frac{1}{(k+1)\binom{k}{\ell}}[(k+1)-\ell)(F(k+1)+x)+2] \\ &\geq \frac{F(k+1)+1}{\binom{k+1}{\ell}}+\frac{2}{(k+1)\binom{k}{\ell}}\end{aligned}$$

a contradiction to the assumption on F .

We shall prove (2.5) by showing that for at least two j strict inequality holds in (2.3). Since by assumption $F(m) < \binom{m}{\ell}$, we know that for at least one color $c \in \varphi(\mathcal{V}(k+1,\ell))$ $|\varphi^{-1}(c)| \geq 2$. We distinguish two cases. If $\bigcap_{V\in\varphi^{-1}(c)} V = \phi$, then for $R \in \mathcal{R}_\varphi$ with $\varphi(R) = c$ for all $j \in R$ there is a $V' \in \varphi^{-1}(c)$ such that $j' \notin V'$. Therefore we get

$$|\mathcal{R}_{\varphi j}|+1 \leq \parallel \varphi_j \parallel \tag{2.6}$$

and since $|R| = \ell \geq 2$ strict inequality holds in (2.3) for at least two j .

If $\bigcap_{V\in\varphi^{-1}(c)} V \neq \phi$, then another color c' with $|\varphi^{-1}(c')| \geq 2$ exists, because otherwise for any $j \in \bigcap_{V\in\varphi^{-1}(c)} V$ the subhypergraph H_j has distinct colors for its vertices and thus in particular for any edge in H_j there are $\binom{m}{\ell} > F(m)$ colors in contradiction to $\parallel \varphi \parallel \leq F(m)$.

Select now $V_1, V_2 \in \varphi^{-1}(c)$, $V_1', V_2' \in \varphi^{-1}(c')$ such that $V_1 = R_c$ and $V_1' = R_{c'}$ are two edges in I_φ . Two subcases arise.

Subcase 1. $(V_1 \setminus V_2) \cap (V_1' \setminus V_2') \neq \phi$.
Here for $j \in (V_1 \setminus V_2) \cap (V_1' \setminus V_2')$ we have

$$|\mathcal{R}_{\varphi j}|+2 \leq \parallel \varphi_j \parallel \ , \tag{2.7}$$

since R_c and $R_{c'}$ are not in $I_{\varphi j}$ but V_2 (with color c) and V_2' (with color c') are in H_j .

Subcase 2. $(V_1 \setminus V_2) \cap (V_1' \setminus V_2') = \phi$.
There are different j_1, j_2 such that $j_1 \in V_1 \setminus V_2$ and $j_2 \in V_1' \setminus V_2'$. For each of them we conclude as in the first case above that (2.6) holds and again (2.5).

Theorem 2.

(i) $\dfrac{N(n,k,\ell,e)}{\binom{n}{\ell}} - \dfrac{N(n+1,k,\ell,e)}{\binom{n+1}{\ell}} \geq \dfrac{\ell-1}{(k+1)\binom{k}{\ell}}$ *for* $e < \binom{k}{\ell}$

(ii) $\dfrac{N(n,k,\ell,e)}{\binom{n}{\ell}}$ *is strictly decreasing in* n *and less than* $\dfrac{e}{\binom{k}{\ell}}$.
("global density" is smaller than "local density").

(iii) $\lim_{n\to\infty} \dfrac{N(n,k,\ell,\alpha\binom{k}{\ell})}{\binom{n}{\ell}} \leq \alpha$,

(iv) $N(n,k,2,k-1) = n-1$.

Proof. (i) Just replace $F(k+1)+x$ and $F(k)$ in the proof of Theorem 1 by $N(n+1,k,\ell,e)$ and $N(n,k,\ell,e)$.

(ii) is an immdiate consequence of (i) .

(iii) is a direct consequence of (ii) .

(iv) Let $\varphi(x,y) = \min(x,y)$ for $x,y \in \Omega_n$ and notice that $\varphi \in \Phi(n,k,2,k-1)$ and $\parallel \varphi \parallel = n-1$. Therefore we have $N(n,k,2,k-1) \geq n-1$ and the reverse inequality can be derived by applying Theorem 1 with $F(k) = k-1$. The hypothesis holds, because

$$\frac{k+1}{\binom{k+1}{2}} - \frac{k-1}{\binom{k}{2}} = \frac{2}{(k+1)k(k-1)}[(k+1)(k-1)-(k-1)(k+1)] = 0 \ > \ -\frac{2}{(k+1)\binom{k}{2}} \ .$$

3. Turán's function as 2–local–global function

It if often convenient to use the function

$$g_p(n) = t_p(n) + 1 \ . \tag{3.1}$$

Theorem 3.

(i) $N(n,k,2,g_p(k)+\alpha)=g_p(n)+\alpha$ *for* $n\geq k$
when $0\leq\alpha\leq\lfloor\frac{k}{\lfloor p\rfloor/2}\rfloor-1, k\geq 4$ *and* $p\geq 2$.

(ii) $N(n,k,2,\alpha)=\alpha$ *when* $\alpha\leq\lfloor\frac{k}{2}\rfloor$ *and* $k\geq 4$.

(iii) $N(n,3,2,2)=n-1, N(n,3,2,1)=1, N(n,3,2,3)=\binom{n}{2}$.

Remark The ranges for α are sufficiently large for our purposes, but not necessarily optimal.

Proof.
(i) Direct part

Recall the definition of $C_p(n)$ and its vertex sets $Q_1,\dots,Q_p$ in Section 1. Define φ for $H(n,2,k)$ as follows:
The edges in the subgraph $C_p(n)$ get colors $1,2,\dots,t_p(n)$. For $i=1,\dots,\alpha$ the edge $(i,i+1)$ gets color $t_p(n)+i$ and all other edges get color 0 . φ is an $(g_p(n)+\alpha,\ g_p(k)+\alpha)$–coloring , because by assumption $k\geq 2p$. We conclude that $N(n,k,2,g_p(k)+\alpha)\geq g_p(n)+\alpha$ for $k\leq n$.

Converse part

It is sufficient to verify the hypothesis of Theorem 1 for $m=pq+r, 0\leq r<p$ and $F(m)=g_p(m)+\alpha$. We first notice that

$$\begin{aligned} g_p(p(q'+1)) &= \binom{p}{2}(q'+1)^2+1 \\ &= \binom{p}{2}(q'+1)^2+\binom{p-p}{2}q^2+p(p-p)q(q+1)+1\ . \end{aligned}$$

Thus, for the values as specified

$$
\begin{aligned}
&(m-1)[F(m+1)+1]-(m+1)F(m)\\
&=(pq+r-1)[g_p(pq+r+1)-g_p(pq+r)+1]-2[g_p(pq+r)+\alpha]\\
&=(pq+r-1)[\binom{r+1}{2}(q+1)^2+\binom{p-r-1}{2}q^2+(r+1)(p-r-1)q(q+1)+1\\
&-\binom{r}{2}(q+1)^2-\binom{p-r}{2}q^2-r(p-r)q(q+1)]\\
&-2[\binom{r}{2}(q+1)^2+\binom{p-r}{2}q^2+r(p-r)q(q+1)+1]-2\alpha\\
&=(pq+r-1)[r(q+1)^2]-(p-r-1)q^2+(p-2r-1)q(q+1)]+(pq+r-1)\\
&-[p(p-1)q^2+2r(p-1)q+r(r-1)+2]-2\alpha\\
&=(pq+r-1)[(p-1)q+r]-[p(p-1)q^2+2r(p-1)q+r(r-1)+2]+(pq+r-1)-2\alpha\\
&=(r-p+1)q+pq+r-3-2\alpha=(r+1)q+r-3-2\alpha\\
&\overset{\text{def}}{=}L(r,q)-2\alpha\ .
\end{aligned}
$$

Therefore

$$\frac{F(m+1)+1}{\binom{m+1}{2}}-\frac{F(m)}{\binom{m}{2}}=\frac{L(r,q)-2\alpha}{(m+1)\binom{m}{2}} \tag{3.2}$$

and since, $\frac{DL}{Dr}\geq 0$, where $q\geq 1$, and since $L(0,q)=q-3$, we have $L(r,q)-2\alpha\geq q-3-2\alpha\geq q-3-2\lfloor k/p\rfloor/2+2\geq \frac{m}{p}-\frac{k}{p}-1\geq -1>-2$.

Furthermore, $F(k)\leq g_p(k)+\lfloor\frac{\lfloor\frac{k}{p}\rfloor}{2}\rfloor-1<\binom{k}{2}$ and thus the hypotheses of Theorem 1 hold and the proof is complete.

(ii) Direct part

The coloring function

$$\varphi(x,y)=\begin{cases} i \text{ for } (x,y)=(i,i+1)\\ \text{and } \alpha=1,2,\ldots,\alpha-1\\ 0 \text{ otherwise}\end{cases}$$

yields $N(n,k,2,\alpha)\geq\alpha$.

Converse part

For $F\equiv\alpha$ we have $\frac{\alpha+1}{\binom{m+1}{2}}-\frac{\alpha}{\binom{m}{2}}=\frac{m-1-2\alpha}{(m+1)\binom{m}{2}}>-\frac{2}{(m+1)\binom{m}{2}}$, $F(k)=\alpha<\binom{k}{2}$ and again by Theorem 1 $N(n,k,2,\alpha)\leq\alpha$.

(iii) Here, in the case $k=3$, only the first equation is non–obvious. It is the answer, if no three colors are in a triangle. However, this is covered by Theorem 2, (iv).

4. The threshold functions

Theorem 4. *For* $s = 0, 1, 2, \ldots$ *and* $k > 2(s+2)$

(i) $e_2(s,k) = g_{s+2}(k) - 1$

(ii) $e_1(s,k) = g_{s+1}(k) + \left\lfloor \frac{k}{\lfloor s+1/2 \rfloor} \right\rfloor - 1$.

We begin with an auxiliary result with a simple probabilistic proof.

Lemma.
If $(\Omega_n, \mathcal{E}_n)_{n=1}^{\infty}$ *is a sequence of graphs with* $T_k(\Omega_n, \mathcal{E}_n) > \delta\ n^k$ *for some* $\delta > 0$ *, then for* $m \in \mathbb{N}$ *there exists an* $n_0 = n_0(\delta, m, k)$ *and a* $\beta > 0$ *such that for* $n > n_0$ *there are* m *vertices* $x_1, \ldots, x_m \in \Omega_n$ *and* $M = \lfloor \beta n^{k-1} \rfloor$ $(k-1)$*–subsets of* Ω_n *, say,* $A_1, \ldots, A_n$ *with*

$$\{x_i\} \cup A_j \in \mathcal{T}_k \textit{ for } 1 \le i \le m \ ;\ 1 \le j \le M \ . \tag{4.1}$$

Proof. For $A \in \mathcal{T}_{k-1}$ define

$$J_k(A) = \{x \in \Omega_n : \{x\} \cup A \in \mathcal{T}_k\}\ , \tag{4.2}$$

$$\mathcal{T}'_{k-1} = \{A : |J_n(A)| \ge \frac{\delta k}{2} n\}\ . \tag{4.3}$$

Then

$$|\mathcal{T}'_{k-1}| \cdot n + \frac{\delta k}{2} n \cdot n^{k-1} > \sum_{A \in \mathcal{T}_{k-1}} |J_k(A)| = kT_k > k\ \delta\ n^k$$

and hence

$$|\mathcal{T}'_{k-1}| > \frac{\delta k}{2} n^{k-1}\ . \tag{4.4}$$

Also, for $x \in \Omega_n$ define $\mathcal{L}_k(x) = \{B \in \mathcal{T}_{k-1} : \{x\} \cup B \in \mathcal{T}_k\}$.

Furthermore, let $X_1, \ldots, X_m$ be i.i.d. RV's with uniform distribution on Ω_n . Then $\mathcal{L} = \cap_{i=1}^m \mathcal{L}_k(X_i)$ is a random system of elements of $\mathcal{T}_{k-1}$ which extend to elements of $\mathcal{T}_k$ for every $X_i (i = 1, \ldots, m)$. With the indicator function $1_{\mathcal{L}}$ we can write

$$\begin{aligned} E|\mathcal{L}| &= \sum_B E\ 1_{\mathcal{L}}(B) \ge \sum_{B \in \mathcal{T}'_{k-1}} E 1_{\mathcal{L}}(B) \ge |\mathcal{T}'_{k-1}| \frac{\binom{\frac{1}{2}\delta k n}{m}}{n^m} \\ &> \frac{\delta k}{2} n^{k-1} \frac{\binom{\frac{1}{2}\delta k n}{m}}{n^m} \ge \beta\ n^{k-1} \text{ for some } \beta > o \end{aligned}$$

when $n > n_0(\delta, m, k)$.

Proof of Theorem 4.

(i) From Theorem 3 we know that

$$N(n,k,2,q_p(k)) = q_p(n) \text{ for } k \geq 4 \ .$$

So it is sufficient to prove that for any sequence of coloring functions $(\varphi_n)_{k=1}^{\infty}$, where φ_n is a coloring of $H(n,2,k)$ with $\| \varphi_n \| \geq \frac{s}{2(s+1)} n^2 + \alpha n^2 = g_{s+1}(n) + \alpha' \ n^2 \ (\alpha, \alpha' > 0)$, there exists a sequence of edges $(E_n)_{n=1}^{\infty}$ with $E_n \in \mathcal{E}(n,2,k)$ and such that $\| \varphi_n(E_n) \| \geq g_{s+2}(k)$, when n is large enough. As in Section 2 we define the graph $I_{\varphi n} = (\Omega_n, \mathcal{R}_{\varphi_n})$.

Then by the Corollary in [9] $T_{s+2} \geq \alpha^{s+1} n^{s+2}$.

Define now for $k = (s+2)q + r, \ 0 \leq r < s+2$

$$\ell_i = \begin{cases} q & \text{for } i = 1, \ldots, s+2-r \\ q+1 & \text{for } i = s+3-r, \ldots, s+2 \ . \end{cases}$$

With the Lemma we get $\{x_1, \ldots, x_{\ell_{s+2}}\} \overset{\text{def}}{=} L_{s+2} \subset \Omega_n$ and $A_1(s+2), \ldots, A_{M_{s+2}}(s+2)$ such that for all $x_i \in L_{s+2}$ and $A_j(s+2)$

$\{x_i\} \cup A_i(s+2) \in \mathcal{T}_{s+2}$ and $M_{s+2} \geq \delta_{s+1} n^{s+1}$.

Let G^*_{s+1} be the minimal subgraph of $G^*_{s+2} \overset{\text{def}}{=} I_{\varphi_n}$ containing $\bigcup_j \ A_j(s+2)$. Then $T_{s+1}(G^*_{s+1}) \geq \delta_{s+1} n^{s+1}$. Repeating this procedure to $G^*_{s+1}, G^*_s, \ldots, G^*_2$ one can get $L_{s+1}, G^*_s, L_s, G^*_{s-1}, \ldots, L_2, G^*_1$ such that $|L_j| = \ell_j + 1$ and the vertex set of G^*_1 has cardinality greater than $\delta_1 n > \ell_1$ for some $\delta > 0$, when n is big enough. Thus, to be specific, we can take a subset L_1 with cardinality $\ell_1 + 1$ from G^*_1 . We can easily see that for all $y_i \in L_i, y_j \in L_j (i \neq j) \quad (y_i, y_j) \in \mathcal{R}_{\varphi_n}$.

Thus by the definition of I_{φ_n} for all $y_i, y'_i \in L_i$, $y_j, y'_j \in L_j (i \neq j)$

$$y_i \neq y'_i \ \text{ or } \ y_j \neq y'_j \quad \varphi_n(y_i, y_j) \neq \varphi_n(y'_i, y'_j) \ .$$

Now we select any pair (a,b) with $a, b \in L_1$ and consider $\varphi_n(a,b)$. We find that there exists at most one pair $(i,j) \ (i < j)$ and one pair (y_i, y_j) with $y_i \in L_i, y_j \in L_j$ such that $\varphi_n(y_i, y_j) = \varphi_n(a,b)$.

We choose now an ℓ_m–subset L^*_m of L_m such that $y_j \notin L^*_j$ and $a, b \in L^*_1$. Now for $E_n \overset{\text{def}}{=} \bigcup_j \ L^*_j$

$$|\varphi_n(E_n)| \geq g_{s+2}(k) \ .$$

(ii) By (i) and (i), (ii) in Theorem 3

$$e_1(s,k) \geq g_{s+1}(k) + \left\lfloor \frac{k}{\lfloor s + 1/2 \rfloor} \right\rfloor - 1 \ .$$

On the other hand we define φ by

$$\varphi(i,2i) = i \text{ for } i = 1,2,\ldots,\lfloor n/2\rfloor$$
$$\varphi(x,y) = 0 \text{ otherwise, when } s = 0$$

and in case $s > 0$:

(a) Each pair $(i,2i)$ with $i = 1,2,\ldots,\lfloor\lfloor n/s\rfloor/2\rfloor$ has a unique color.

(b) Each pair (x,y) with $x \in L_i, y \in L_j \quad (i \neq j)$ has a unique color.

(c) All other pairs have the same color.

Thus $e_1(s,k) < g_{s+1}(k) + \lfloor\lfloor k/s+1\rfloor/2\rfloor$.

5. An extension of Turán's Theorem

The method of proof for Theorem 1 can also be used to derive Turán's Theorem. Actually a more general result can be obtained, which has other interesting consequences and leads to a remarkable conjecture. The result is for a sequence of families of hypergraphs with restrictions on the number of edges. The restrictions are specified by a sequence $\left(\overrightarrow{\varepsilon(n)}\right)_{n=2}^{\infty}$ of vectors $\vec{\varepsilon}(n) = (\varepsilon_2(n),\ldots,\varepsilon_n(n))$ with components satisfying $0 \leq \varepsilon_\ell(n) \leq \binom{n}{\ell}$ for $\ell = 2,3,\ldots,n$.

The sequence of hypergraphs is $\left(\mathcal{H}_n\left(\overrightarrow{\varepsilon(n)}\right)\right)_{n=2}^{\infty}$, where

$$\mathcal{H}_n\left(\overrightarrow{\varepsilon(n)}\right) = \{\text{hypergraph } H(\Omega_n,\mathcal{E}) : \mathcal{E} \text{ has not more than } \varepsilon_\ell(n) \text{ edges of cardinality } \ell\} . \tag{5.1}$$

We are interested in its subset

$|\mathcal{H}_n\left(\overrightarrow{a\varepsilon(n)},k,K\right) = \{H(\Omega_n,\mathcal{E}) \in \mathcal{H}_n\left(\overrightarrow{\varepsilon(n)}\right)$: all its subhypergraphs on k vertices have not more than K edges $\}$ and investigate

$$\mathcal{T}\left(n,\overrightarrow{\varepsilon(n)},k,K\right) = \max\{|\mathcal{E}| : H(\Omega_n,\mathcal{E}) \in \mathcal{H}_n\left(\overrightarrow{\varepsilon(n)},k,K\right)\} .$$

This function specializes to Turán's function in case $\varepsilon_2(n) = \binom{n}{2}, \varepsilon_\ell(n) = 0$ for $\ell \neq 2$ and $K = \binom{k}{2} - 1$.

It is convenient to use for a function $\psi : \mathbb{N} \to \mathbb{N}$ the increment of density $\triangle_\ell^n(\psi) = \dfrac{\psi(n+1)}{\binom{n+1}{\ell}} - \dfrac{\psi(n)}{\binom{n}{\ell}}$.

We say that $\left(\overrightarrow{\varepsilon(n)}\right)_{n=2}^{\infty}$ is a uniform restriction, if $\triangle_\ell^n(\varepsilon_\ell) = 0$ for all n,ℓ .

Theorem 5.

(i) Suppose that $\left(\overrightarrow{\varepsilon(n)}\right)_{n=2}^{\infty}$ *and* $g : \mathbb{N} \to \mathbb{N}$ *satisfy*

(1) $0 \le g(n) \le \binom{n}{\ell_0}$ *for some* ℓ_0

(2) For

$$\varepsilon_\ell^*(n) = \begin{cases} \varepsilon_\ell(n) & \text{for } \ell > \ell_0 \\ g(n) & \text{for } \ell = \ell_0 \end{cases}$$

$$\frac{1}{\binom{N}{\ell_0}} \sum_{\ell \ge \ell_0} \binom{N}{\ell} \triangle^{N}_{\ell}(\varepsilon_\ell^*) > -\frac{1}{\binom{N+1}{\ell_0}} \text{ for } N \le n \ ,$$

then for $F(N) = \sum_{\ell \ge \ell_0} \varepsilon_\ell^*(N)$ *we have*

$$T\big(n, \overrightarrow{\varepsilon(n)}, k, F(K)\big) \le F(n).$$

Particularly,

(ii) If $\left(\overrightarrow{\varepsilon(n)}\right)_{n=2}^{\infty}$ *is uniform and* g *satisfies*

(3) $\triangle_\ell^N(g) > -\frac{1}{\binom{N+1}{\ell}}$

then $T\big(n, \overrightarrow{\varepsilon(n)}, k, F(k)\big) \le F(n)$.

Proof. For $n = k$ (i) holds by definition of $\mathcal{T}$. Induction step from n to $n+1$: Assume that for some $H \in \mathcal{H}_{n+1}\big(\overrightarrow{\varepsilon(n+1)}, k, F(k)\big)$ $|\mathcal{E}| = F(n+1)+x$ for a positive x . By omitting edges, if necessary, we can achieve $x = 1$. Further, we can assume that $\varepsilon_\ell(n+1)$ equals the number of ℓ–size edges.

Removal of $v \in \Omega_{n+1}$ from H leads to the subhypergraph H_v with n vertices and $F(n+1)+1-d(v)$ edges. By induction hypotesis $F(n+1)+1-d(v) \le F(n)$ or equivalently $\sum_{\ell \ge \ell_0} \varepsilon_\ell^*(n+1) + 1 - d(v) \le \sum_{\ell \ge \ell_0} \varepsilon_\ell^*(n)$. With the concept $d_\ell(v) = \{E \in \mathcal{E} | v \in E, |E| = \ell\}$ this can be written in the form

$$\sum_\ell d_\ell(v) = d(v) \ge \sum_{\ell \ge \ell_0} \varepsilon_\ell^*(n+1) + 1 - \sum_{\ell \ge \ell_0} \varepsilon_\ell^*(n)$$

and summation over all v $\ell \ge \ell_0$ gives

$$\sum_{v \in \Omega_{n+1}} \sum_\ell d_\ell(v) \ge (n+1) \left[\sum_{\ell \ge \ell_0} \varepsilon_\ell^*(n+1) + 1 - \sum_{\ell \ge \ell_0} \varepsilon_\ell^*(n) \right] \tag{5.2}$$

Furthermore, since by our restriction on the number of edges for $\ell > \ell_0$

$$\frac{1}{\ell} \sum_v d_\ell(v) = \varepsilon_\ell(n+1) = \varepsilon_\ell^*(n+1) \ , \tag{5.3}$$

we also have

$$\sum_v d_{\ell_0}(v) \geq (n+1)\left[\sum_{\ell\geq\ell_0} \varepsilon^*_\ell(n+1)+1-\sum_{\ell\geq\ell_0}\varepsilon^*_\ell(n)\right]-\sum_{\ell>\ell_0}\ell\varepsilon^*_\ell(n+1)\ . \tag{5.4}$$

Now we use the identity

$$\sum_\ell \frac{1}{\ell}\sum_{v\in\Omega_{n+1}} d_\ell(v)=|\mathcal{E}|=F(n+1)+1\ . \tag{5.5}$$

Here the left hand side expression is minimal if equality holds in (5.4).
Thus,

$$\begin{aligned}&\sum_{\ell>\ell_0}\varepsilon^*_\ell(n+1)+\frac{n+1}{\ell_0}\left[\sum_{\ell\geq\ell_0}\varepsilon^*_\ell(n+1)+1-\sum_{\ell\geq\ell_0}\varepsilon^*_\ell(n)\right]\\ &-\frac{1}{\ell_0}\sum_{\ell>\ell_0}\ \ell\ \varepsilon^*_\ell(n+1)\leq F(n+1)+1\\ &=\sum_{\ell\geq\ell_0}\varepsilon^*_\ell(n+1)+1 \text{ (by definition)}\end{aligned}$$

and equivalently

$$\frac{n+1-\ell_0}{\ell_0}\left(\varepsilon^*_\ell(n+1)+1\right)-\varepsilon^*_{\ell_0}(n)+\sum_{\ell>\ell_0}\frac{n+1-\ell}{n+1}\varepsilon^*_\ell(n+1)\sum_{\ell>\ell_0}\varepsilon^*_\ell(n)\geq 0$$

or

$$\begin{aligned}&\frac{n+1-\ell_0}{n+1}+\sum_{\ell\geq\ell_0}\binom{n}{\ell}\left[\frac{\frac{n+1-\ell}{n+1}}{\binom{n}{\ell}}\varepsilon^*_\ell(n+1)-\frac{\varepsilon^*_\ell(n)}{\binom{n}{\ell}}\right]\\ &=\frac{\binom{n}{\ell_0}}{\binom{n+1}{\ell_0}}+\sum_{\ell\geq\ell_0}\binom{n}{\ell}\triangle^n_\ell\left(\varepsilon^*_\ell\right)\leq 0\end{aligned}$$

in contradiction to (i).

Corollary 2. (Turán):

Suppose that $G=(\Omega_n,\mathcal{E})$ is a graph without k–complete subgraph, then $|\mathcal{E}|\leq t_{k-1}(n)-1$ (Sect. 4) and the bound is best possible.

Proof. Just take $\varepsilon_\ell(N)=0\quad\forall\ell\neq 2$ and $\varepsilon_2(N)=\binom{N}{2}, g(N)=\varepsilon^*_2(N)=g_{k-1}(N)$.

By the proof of Theorem 3 $\triangle^N_2(t_{k-1})>-\frac{1}{\binom{N+1}{2}}$ as can be see by taking $\alpha=-1$ and by noticing $g_{k-1}(N)-1=g(N)$.

If G has no k complete subgraph, then every k–subgraph has not more than $\binom{k}{2} - 1 = \binom{k-1}{2} + \binom{k-1}{1} = 1 = \binom{k-2}{2} + (k-2) + (k-1) - 1 = \binom{k-2}{2} + 1 \cdot (k-2) \cdot 2 = g_{k-1}(k) - 1$ edges and our proof is complete.

We call the hypergraph $H = (\Omega_k, \mathcal{E})$ with $\mathcal{E} = \bigcup_{\ell \geq 2} \binom{\Omega_k}{\ell}$ k–complete .

Corollary 3. If $H = (\Omega_n, \mathcal{E})$ has no k–complete subhypergraph, then

$$|\mathcal{E}| \leq \sum_{\ell=3}^{n} \binom{n}{3} + t_{k-1}(n) - 1$$

and this bound is best possible.

Proof. Choose $\varepsilon_\ell(n) = \binom{n}{\ell}$ and $g = t_k(n) - 1$ and proceed as in the proof of Corollary 1. An optimal configuration is Turán's graph together with all subsets of cardinality > 2 .

<u>A conjectured extension of Turán's Theorem</u>

For $k < n$ and arbitrary K let

$$G_{n,k,K} = \max\{g(n) : g \text{ with } g(k) = K \text{ and } \triangle_2^N (g) > -\frac{1}{\binom{N+1}{2}} \text{ for } k \leq N \leq n\} \ ,$$

then

$$\max\{|\mathcal{E}| |G = (\Omega, \mathcal{E}) \text{ without more than } K \text{ edges in any } k\text{–subgraph}\} = G_{n,k,K} \ .$$

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