COMMUNICATION COMPLEXITY IN LATTICES

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1. INTRODUCTION

Let $\mathcal{X}$ denote a finite lattice and let $\hat{f} : \mathcal{X} \to \mathcal{Z}$ be a function mapping $\mathcal{X}$ into some set $\mathcal{Z}$. In this note we determine the communication complexity of functions $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ defined by

$$f(x,y) := \hat{f}(x \wedge y) \text{ for all } x, y \in \mathcal{X}. \tag{1}$$

The communication complexity of a function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ (where $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are finite sets), denoted as $C(f)$, is the number of bits that two processors, P_1 and P_2 say, have to exchange in order to compute the function value $f(x,y)$, when initially P_1 only knows $x \in \mathcal{X}$ and P_2 only knows $y \in \mathcal{Y}$.
More specifically, let $\mathcal{Q}$ denote the set of protocols computing f such that finally both processors know the result and let $l_P(x,y)$ be the number of bits transmitted for the input (x,y), when the protocol $P \in \mathcal{Q}$ is used. Then the (worst-case) *communication complexity* is

$$C(f) := \min_{P \in \mathcal{Q}} \max_{(x,y) \in \mathcal{X} \times \mathcal{Y}} l_P(x,y). \tag{2}$$

A *protocol* P is a pair of mappings $\phi_1 : \mathcal{X} \times \{0,1\}^* \to \{0,1\}^*$, $\phi_2 : \mathcal{Y} \times \{0,1\}^* \to \{0,1\}^*$. So on input (x,y) the processors, starting with P_1, say, alternatively send binary messages N_1, N_2, N_3, etc., until they both know the result. Each message depends on the previous messages and on the current processor's input, hence $N_1 = \phi_1(x)$, $N_2 = \phi_2(y, \phi_1(x))$, $N_3 = \phi_1(x, \phi_1(x)\phi_2(y, \phi_1(x)))$, etc. . It is required that the set of messages a processor is allowed to send at an arbitrary moment in the course of the protocol is *prefix-free*, i. e., no possible message is the beginning (prefix) of another one. This property assures that the other processor immediately recognizes the end of the message and can hence start the transmission of its next message without delay.
An upper bound on $C(f)$ for any function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ (w. l. o. g. $|\mathcal{X}| \leq |\mathcal{Y}|$) is always obtained from the following *trivial protocol*: P_1 transmits all the bits of its input $x \in \mathcal{X}$. P_2 now is able to compute the function value and returns the result $f(x,y) \in \mathcal{Z}$. Hence

$$C(f) \leq \lceil log|\mathcal{X}| \rceil + \lceil log|\mathcal{Z}| \rceil. \tag{3}$$

Throughout this paper the logarithm is always taken to the base 2.
The following lower bound is due to Mehlhorn and Schmidt [1]:

$$C(f) \geq \lceil log(\sum_{z \in \mathcal{Z}} \text{rank } M_z(f)) \rceil, \tag{4}$$

where $M_z(f) := (m_{xy})_{x,y \in \mathcal{X}}$ is a Boolean matrix with $m_{xy} = 1$ exactly if $f(x,y) = z$.

2. THE MAIN RESULTS

In the following, we denote by $\preceq$ the underlying order of the lattice $\mathcal{X}$ and by μ the associated *Möbius function*. Further let

$$\mathcal{X}_z := \{x \in \mathcal{X} : \text{ there is some } \hat{x} \preceq x \text{ with } \hat{f}(\hat{x}) = z\}. \tag{5}$$

MAIN THEOREM: The communication complexity of the function f defined as in (1) is bounded from above and below as follows:

$$\lceil log(\sum_{z\in\mathcal{Z}} |\{x \in \mathcal{X} : \sum_{\hat{x}\preceq x, \hat{f}(\hat{x})=z} \mu(\hat{x},x) \neq 0\}|)\rceil \leq C(f) \leq \lceil log(\sum_{z\in\mathcal{Z}} |\mathcal{X}_z|)\rceil + 1. \tag{6}$$

If, additionally, $\sum_{\hat{x}\preceq x, \hat{f}(\hat{x})=z} \mu(\hat{x},x) \neq 0$ for all possible $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, i.e., for all x and z for which there exists some $\hat{x} \preceq x$ with $\hat{f}(\hat{x}) = z$, then upper and lower bound differ by at most one bit, namely

$$\lceil log(\sum_{z\in\mathcal{Z}} |\mathcal{X}_z|)\rceil \leq C(f) \leq \lceil log(\sum_{z\in\mathcal{Z}} |\mathcal{X}_z|)\rceil + 1. \tag{7}$$

The lower bounds are based on the following theorem, which was discovered by Wilf [2] (see also Lindström [3]) and first used in the study of communication complexity by Lovasz [4]. We shall present Wilf's short proof from which the succeeding corollary is immediate, since the incidence matrix of a poset is nonsingular.

THEOREM (WILF): Let $\mathcal{X}$ be a finite lattice with order $\preceq$ and Möbius function μ. Further, let $\{a_x : x \in \mathcal{X}\}$ be a set of arbitrary numbers. Then

$$\det\ (a_{x\wedge y})_{x,y\in\mathcal{X}} = \det\ \mathrm{diag}\ (\sum_{\hat{x}\preceq x} \mu(\hat{x},x)\cdot a_{\hat{x}})_{x\in\mathcal{X}} = \prod_{x\in\mathcal{X}} (\sum_{\hat{x}\preceq x} \mu(\hat{x},x)\cdot a_{\hat{x}}).$$

PROOF: For arbitrary numbers $\{b_x : x \in \mathcal{X}\}$ consider the matrix $\zeta^T \cdot \mathrm{diag}(b_x)_{x\in\mathcal{X}} \cdot \zeta$, where $\zeta := (\zeta_{\hat{x}x})_{\hat{x},x\in\mathcal{X}}$, with $\zeta_{\hat{x}x} = 1$ exactly if $\hat{x} \preceq x$ ($\zeta_{\hat{x}x} = 0$ else) is the incidence matrix of $(\mathcal{X}, \preceq)$. By the rules for matrix multiplication this is just the matrix $(\sum_{\hat{x}\preceq x\wedge y} b_{\hat{x}})_{x,y\in\mathcal{X}}$.
Now let $a_x := \sum_{\hat{x}\preceq x} b_{\hat{x}}$ for all $x \in \mathcal{X}$. By the Möbius inversion formula then $b_x = \sum_{\hat{x}\preceq x} \mu(\hat{x},x)\cdot a_{\hat{x}}$ for all $x \in \mathcal{X}$ and the theorem follows.

COROLLARY: Let $\mathcal{X}$ and $\{a_x : x \in \mathcal{X}\}$ be as in the preceding theorem. Then

$$\mathrm{rank}\ (a_{x\wedge y})_{x,y\in\mathcal{X}} = \mathrm{rank}\ \mathrm{diag}\ (\sum_{\hat{x}\preceq x} \mu(\hat{x},x)\cdot a_{\hat{x}})_{x\in\mathcal{X}}. \tag{8}$$

PROOF OF THE LOWER BOUNDS IN THE MAIN THEOREM:
Observe that the function value matrices $M_z(f)$ are just of the form $(a_{x\wedge y})_{x,y\in\mathcal{X}}$ with

$$a_{x\wedge y} = 1 \text{ exactly if } \hat{f}(x\wedge y) = z.$$

With the above corollary for all $z \in \mathcal{Z}$ it is

$$\mathrm{rank} M_z(f) = |\{x \in \mathcal{X} : \sum_{\hat{x}\preceq x, \hat{f}(\hat{x})=z} \mu(\hat{x},x) \neq 0\}|\ .$$

The lower bound in (6) follows by application of the Mehlhorn - Schmidt lower bound (4).

PROOF OF THE UPPER BOUND IN THE MAIN THEOREM:
The upper bound in the Main Theorem is obtained via a natural and useful improvement of the trivial protocol, which was first introduced by Ahlswede and Cai [5]. As the trivial protocol, it consists of two rounds. In the first round the processor P_1 encodes its input $x \in \mathcal{X}$. The processor P_2 then knows both values x and y and hence is able to compute the result $f(x,y)$, which is returned to P_1. However, now the set of possible function values is reduced to

$$\hat{F}(x) := \{\hat{f}(\hat{x}) : \hat{x} \preceq x\}, \tag{9}$$

since the second processor already knows $x \in \mathcal{X}$.
Hence, only $\lceil log|\hat{F}(x)|\rceil$ bits have to be reserved for the transmission of the result $f(x,y)$ such that the first processor can assign longer messages (code words) to elements with few predecessors

in the poset. So, in contrast to the trivial protocol, the messages $\{\phi_1(x) : x \in \mathcal{X}\}$ are now of *variable* length. Since the prefix property has to be guaranteed, Kraft's inequality for prefix codes yields a condition, from which the upper bound can be derived.
Specifically, we require that to each $x \in \mathcal{X}$ there corresponds a message $\phi_1(x) \in \{0,1\}^*$ of (variable) length $l(x)$, say, with the property that for all $x \in \mathcal{X}$ the sum $l(x) + \lceil log|\hat{F}(x)|\rceil$ takes a fixed value, L say.
Kraft's inequality then states that a prefix code exists, if $\sum_{x\in\mathcal{X}} 2^{-l(x)} \leq 1$. This is equivalent to $\sum_{x\in\mathcal{X}} 2^{-(L-\lceil log|\hat{F}(x)|\rceil)} \leq 1$ and to

$$\sum_{x\in\mathcal{X}} 2^{\lceil log|\hat{F}(x)|\rceil-1} \leq 2^{L-1}. \tag{10}$$

Now, let us choose

$$L := \lceil log(\sum_{z\in\mathcal{Z}} |\mathcal{X}_z|)\rceil + 1. \tag{11}$$

Then (10) holds, since

$$\begin{aligned} &\sum_{x\in\mathcal{X}} 2^{\lceil log|\hat{F}(x)|\rceil-1} \leq \sum_{x\in\mathcal{X}} 2^{log|\hat{F}(x)|} = \sum_{x\in\mathcal{X}} |\hat{F}(x)| = \sum_{x\in\mathcal{X}} |\{\hat{f}(\hat{x}) : \hat{x} \preceq x\}| \\ &= \sum_{z\in\mathcal{Z}} |\{x \in \mathcal{X} : \exists\hat{x} \text{ with } \hat{f}(\hat{x}) = z \text{ and } \hat{x} \preceq x\}| = \sum_{z\in\mathcal{Z}} |\mathcal{X}_z| \leq 2^{L-1} \end{aligned}$$

by definition of $\mathcal{X}_z$ and L.

Remark: Observe that in the proof of the Main Theorem we do not exploit the property that $\mathcal{X}$ is a lattice. It suffices to assume that $(\mathcal{X}, \preceq)$ is a poset in which the meet $x \wedge y$ is well defined for all $x, y \in \mathcal{X}$.

The first function of this type studied in this context is the function $f_1 : \mathcal{X} \times \mathcal{X} \to \{0,1\}$ defined by $\hat{f}_1(x) = 1$, exactly if $x = x_{min}$, where x_{min} denotes the minimal element in the lattice $\mathcal{X}$. Hence $f_1(x,y) = 1$, exactly if $x \wedge y = x_{min}$.

Corollary 1: The communication complexity of the function f_1 is bounded as follows:

$$\lceil log(2 \cdot |\{x \in \mathcal{X} : \mu(x_{min}, x) \neq 0\}| - 1)\rceil \leq C(f_1) \leq \lceil log|\mathcal{X}|\rceil + 1. \tag{12}$$

If, additionally $\mu(x_{min}, x) \neq 0$ for all $x \in \mathcal{X}$, then

$$C(f_1) = \lceil log|\mathcal{X}|\rceil + 1. \tag{13}$$

Proof: Observe that $\sum_{\hat{x}\preceq x, \hat{f}(\hat{x})=1} \mu(\hat{x}, x) = \mu(x_{min}, x)$ for all $x \in \mathcal{X}$ and that $\sum_{\hat{x}\preceq x, \hat{f}(\hat{x})=0} \mu(\hat{x}, x) = -\mu(x_{min}, x)$ if $x \neq x_{min}$ (0 if $x = x_{min}$). Hence, the lower bound in (12) holds.
Further, $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_0 = \mathcal{X} - \{x_{min}\}$. With the additional fact in mind that $\lceil log(2s-1)\rceil = \lceil log(2s)\rceil$ for all positive integers s, it is also clear by (7) that $C(f_1) \geq \lceil log|\mathcal{X}|\rceil + 1$, whenever $\mu(x_{min}, x) \neq 0$ for all $x \in \mathcal{X}$. The upper bound $C(f_1) \leq \lceil log|\mathcal{X}|\rceil + 1$ follows from the trivial protocol (3).

The communication complexity of the function f_1 was first determined by Hajnal, Maass, and Turan [6]. They considered a different model, in which communication already stops, when one processor knows the result. So, the bit for the transmission of $f_1(x,y)$ will not be sent in this case.
Hajnal, Maass, and Turan [6] also introduced the Möbius function in the study of lower bounds for the communication complexity. In this context, Lovasz [4] used the Theorem of Wilf [2] concerning the rank of matrices of the form $(a_{x\wedge y})_{x,y\in\mathcal{X}}$. Björner, Karlander, and Lindström [7] determined $C(f_1)$ for two special lattices.

Ahlswede and Cai [5] considered the function $f_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, defined by $f_2(x,y) = x \wedge y$ and obtained the following result:

COROLLARY 2: Let

$$I(\mathcal{X}) := \{(\hat{x}, x) \in \mathcal{X}^2 : \hat{x} \preceq x\}, \tag{14}$$

then

$$\lceil log|\{(\hat{x},x) \in \mathcal{X}^2 : \mu(\hat{x},x) \neq 0\}|\rceil \leq C(f_2) \leq \lceil log|I(\mathcal{X})|\rceil + 1. \tag{15}$$

If, additionally, $\mu(\hat{x},x) \neq 0$ for all $(\hat{x},x) \in \mathcal{X}^2$ with $\hat{x} \preceq x$, then upper and lower bound differ by one bit only, namely

$$\lceil log|I(\mathcal{X})|\rceil \leq C(f_2) \leq \lceil log I(\mathcal{X})\rceil + 1. \tag{16}$$

PROOF: Observe that $\sum\limits_{\hat{x} \preceq x, \hat{f}_2(\hat{x}) = z} \mu(\hat{x},x) = \mu(z,x)$ if $z \preceq x$. Further, here $\mathcal{X}_z = I(z) := \{x \in \mathcal{X} : z \preceq x\}$, and since $I(\mathcal{X}) = \sum\limits_{z \in \mathcal{X}} I(z)$, Corollary 2 is an immediate consequence of the Main Theorem.

Especially for the Boolean lattice, Ahlswede and Cai [5] demonstrated that upper and lower bound coincide (see also the subsequent section).

In our last example, we assume that the lattice $\mathcal{X}$ is equipped with a *rank function* r. Recall that the *Whitney numbers* $W(t)$ count the elements of rank t in $\mathcal{X}$.
We consider the function f_3 where $f_3(x,y) = r(x \wedge y)$ for all $x, y \in \mathcal{X}$. The following result is an immediate consequence of the Main Theorem.

COROLLARY 3: Let $\mathcal{X}$ be a finite lattice with rank function r and maximum rank n. Then

$$\lceil log|\{x \in \mathcal{X} : \sum_{\hat{x} \preceq x, r(\hat{x}) = t} \mu(\hat{x},x) \neq 0\}|\rceil \leq C(f_3) \leq \lceil log \sum_{t=0}^{n} (t+1) \cdot W(t)\rceil + 1. \tag{17}$$

If $\sum\limits_{\hat{x} \preceq x, r(\hat{x}) = t} \mu(\hat{x},x) \neq 0$ for all $x \in \mathcal{X}$ and $t \leq r(x)$, then

$$\lceil log \sum_{t=0}^{n} (t+1) \cdot W(t)\rceil \leq C(f_3) \leq \lceil log \sum_{t=0}^{n} (t+1) \cdot W(t)\rceil + 1. \tag{18}$$

3. COMMUNICATION COMPLEXITY IN GEOMETRIC LATTICES

The condition under which upper and lower bound differ by at most one bit in the Main Theorem is usually hard to check. However, it is well known that in geometric lattices $\mu(\hat{x},x) \neq 0$ whenever $\hat{x} \preceq x$. This is just the condition required in Corollary 2. Especially, then $\mu(x_{min},x) \neq 0$ for all $x \in \mathcal{X}$, which guarantees the coincidence of upper and lower bound in Corollary 1.
Now, additionally, we require that in a geometric lattice the Möbius function is of the form

$$\mu(\hat{x},x) = (-1)^{r(x)-r(\hat{x})} \cdot \nu(\hat{x},x), \text{ where } \nu(\hat{x},x) > 0 \text{ if } \hat{x} \preceq x. \tag{19}$$

For instance, this holds in the Boolean lattice and in the vector space lattice. In this case, obviously $\sum\limits_{\hat{x} \preceq x, r(\hat{x}) = t} \mu(\hat{x},x) \neq 0$ for all $x \in \mathcal{X}$ and $t \leq r(x)$, since all the summands have the same sign. Hence, the condition of Corollary 3 is fulfilled. Let us summarize our findings

THEOREM 4: In a geometric lattice $\mathcal{X}$ with maximum rank n

$$C(f_1) = \lceil log|\mathcal{X}|\rceil + 1, \tag{20}$$

$$\lceil log|I(\mathcal{X})|\rceil \leq C(f_2) \leq \lceil log|I(\mathcal{X})|\rceil + 1. \tag{21}$$

If, additionally (19) holds, then

$$\lceil log \sum_{t=0}^{n} (t+1) \cdot W(t) \rceil \leq C(f_3) \leq \lceil log \sum_{t=0}^{n} (t+1) \cdot W(t) \rceil + 1. \tag{22}$$

Geometric lattices have a further useful property concerning the Whitney numbers $W(0), \ldots, W(n)$, where n is the maximum rank in the lattice. This property was first discovered by Dowling and Wilson [8] (see also [9]):

$$W(0) + W(1) + \ldots + W(i) \leq W(n-i) + \ldots + W(n-1) + W(n) \text{ for all } i = 0, \ldots, n. \tag{23}$$

We shall use this inequality in the proof of the next theorem, which demonstrates that the lower bound in (22) differs by at most two bits from the upper bound obtained by the trivial protocol.

THEOREM 5: In a geometric lattice $\mathcal{X}$ with maximum rank n, in which (22) holds, always

$$\lceil log|\mathcal{X}| + log(n+2) \rceil - 1 \leq C(f_3) \leq \lceil log|\mathcal{X}| \rceil + \lceil log(n+1) \rceil. \tag{24}$$

If, additionally, $\mathcal{X}$ is modular, then, compared to the trivial protocol, one bit of transmission can be saved for the computation of f_3, if

$$\lceil log|\mathcal{X}| + log(n+2) \rceil = \lceil log|\mathcal{X}| \rceil + \lceil log(n+1) \rceil - 1. \tag{25}$$

PROOF: The upper bound in (24) is the one obtained from the trivial protocol (3). Concerning the lower bound, observe that

$$\begin{aligned} (n+1) \cdot |\mathcal{X}| &= \sum_{t=0}^{n} W(t) \cdot (t+1) + \sum_{t=0}^{n} W(t) \cdot (n-t) = \sum_{t=0}^{n} W(t) \cdot (t+1) + \sum_{t=0}^{n} \sum_{i=0}^{t} W(i) \\ &\leq \sum_{t=0}^{n} W(t) \cdot (t+1) + \sum_{t=0}^{n} \sum_{i=0}^{t} W(n-i) \qquad \text{(by (23))} \\ &\leq \sum_{t=0}^{n} W(t) \cdot (t+1) + \sum_{t=0}^{n} W(t) \cdot t = 2 \cdot \sum_{t=0}^{n} W(t) \cdot (t+1) - |\mathcal{X}|. \end{aligned} \tag{26}$$

We know from (22) that

$$C(f_3) \geq \lceil log \sum_{t=0}^{n} W(t) \cdot (t+1) \rceil \geq \lceil log \frac{(n+2)}{2} \rceil \qquad \text{(by (26))},$$

from which the lower bound in (24) is immediate.

Especially for modular lattices, like the Boolean lattice and the vector space lattice, we know that $W(i) = W(n-i)$ for all $i = 0, \ldots, n$ (see e. g. [9]) and hence equality holds in (26). So, in this case, we can also compare the upper bound obtained from the trivial protocol (3) with the one obtained with the Ahlswede - Cai protocol. This proves (25).

As an application we now shall study the Boolean lattice (set intersection) and the partition lattice. For the Boolean lattice the following results have been obtained in [5] and [10] by different methods.

COROLLARY 6: For the Boolean lattice with maximum rank n

$$C(f_2) = \lceil n \cdot log3 \rceil, \tag{27}$$

$$n + \lceil log(n+1) \rceil - 1 \leq C(f_3) \leq n + \lceil log(n+1) \rceil. \tag{28}$$

Here

$$C(f_3) = \begin{cases} n + \lceil log(n+1) \rceil & \text{for } n = 2^m - 1 \\ n + \lceil log(n+1) \rceil - 1 & \text{for } n = 2^m, m \geq 2, \end{cases}$$

where m is a positive integer.

PROOF: In order to prove (27), observe that in (7) $|\mathcal{X}_z| = |I(z)| = |\{x \in \mathcal{X} : z \preceq x\}| = 2^{n-r(z)}$ is a power of 2 for all $z \in \mathcal{X}$ and hence Kraft's inequality in this case yields $\sum_{z \in \mathcal{X}} 2^{-\lceil n \cdot log 3 \rceil - log|I(z)|} \leq 1$ such that upper and lower bound coincide for $C(f_2)$.

Since $|\mathcal{X}| = 2^n$ for the Boolean lattice, upper and lower bound in (24) here differ by at most one bit and (28) is obvious. Further, upper and lower bound coincide for $n = 2^m - 1$. From (25) we know that for $n = 2^m, m \geq 2$ the Ahlswede - Cai protocol uses one bit of transmission less than the trivial protocol.

COROLLARY 7: For the partition lattice with maximum rank n

$$C(f_2) \leq \lceil log(B_{n+1} - B_n) \rceil + 1, \tag{29}$$

where B_n denotes the n-th Bell number.

PROOF: The partition lattice is geometric, hence the Möbius function does not vanish on any interval in it. The same property then holds for the partition lattice 'turned upside down' (cf. Lovasz [4], p. 234). In this lattice the Whitney numbers are just the Stirling numbers of the second kind, S_t^n say. By the well known recursion $S_t^{n+1} = S_{t-1}^n + t \cdot S_t^n$ we then have

$$\sum_{t=0}^{n} (t+1) \cdot W(t) = \sum_{t=0}^{n} (t+1) \cdot S_{t+1}^n = \sum_{t=1}^{n+1} t \cdot S_t^n = \sum_{t=1}^{n+1} (S_t^{n+1} - S_{t-1}^n) = B_{n+1} - B_n$$

Now the right-hand side of (22) gives (29). Here we cannot obtain a lower bound via (22), because (19) does not hold in the partition lattice.

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