THE MAXIMAL LENGTH OF CLOUD–ANTICHAINS

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1. Introduction

The notion of an antichain in a partially ordered set was generalized [2] and [3] to the seemingly natural notion of a "cloud–antichain" $\{\mathcal{A}_i\}_{i=1}^N$. Whereas in antichains *elements* of a partially ordered set are compared in cloud–antichains *sets of elements* take their role. Elements in different sets $\mathcal{A}_i$, called clouds, are required to be incomparable. Formally, for every two clouds $\mathcal{A}_i$ and $\mathcal{A}_j$ we have

$$A_i \not> A_j \text{ for } \textit{all } A_i \in \mathcal{A}_i \text{ and } \textit{all } A_j \in \mathcal{A}_j. \tag{1.1}$$

In [3] further notions of cloud–antichains were introduced. Whereas the logical structure of the formula (1.1) suggests to speak of an antichain of *type* $\underline{(\forall,\forall)}$, the new notions in [3] are of the types $(\forall,\exists)$, $(\exists,\forall)$, and $(\exists,\exists)$.

In the sequal we consider always the partially ordered set $\mathcal{P} = 2^{\Omega_n}$, the power set of $\Omega_n = \{1, 2, \dots, n\}$, with set theoretic containment as order relation. $\{\mathcal{A}_i\}_{i=1}^N$ is always a family of subsets of $\mathcal{P}$. It is said to be of type $(\exists,\forall)$, if for all $i \neq j$

$$\text{there } \textit{exists} \text{ an } A_i \in \mathcal{A}_i \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \text{ for } \textit{all } A_j \in \mathcal{A}_j \ , \tag{1.2}$$

it is of type $(\forall,\exists)$, if for all $i \neq j$

$$\text{for } \textit{all } A_i \in \mathcal{A}_i \text{ there } \textit{exists} \text{ an } A_j \in \mathcal{A}_j \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \tag{1.3}$$

and it is of type $(\exists,\exists)$, if for all $i \neq j$ there *exists* an $A_i \in \mathcal{A}_i$

$$\text{and there } \textit{exists} \text{ an } A_j \in \mathcal{A}_j \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \ . \tag{1.4}$$

The maximal cardinalities N of such systems as functions of n are denoted by $N_n(\exists,\forall)$, $N_n(\forall,\exists)$, and $N_n(\exists,\exists)$, resp.

Obviously, an analogously defined quantity $N_n(\forall,\forall)$ equals $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, because in an optimal configuration $|\mathcal{A}_i| = 1$ and Sperner's classical Theorem ([1]) applies. We also study systems with *disjoint* clouds. The maximal cardinalities are then denoted by $M_n(\exists,\forall)$, $M_n(\forall,\exists)$, and $M_n(\exists,\exists)$, resp.

We call two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ asymptotically equivalent and write $f(n) \sim g(n)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

All the six functions measuring maximal lengths of cloud–antichains in the cases described are determined up to asymptotic equivalence. Three of the functions are even determined exactly.

2. The Results

Theorem 1.

$$M_n(\exists,\forall) \sim 2^{n-1} \ .$$

Theorem 2.

$$N_n(\exists,\forall) = \binom{k}{\lfloor \frac{k}{2} \rfloor}, \ \ \textit{where} \ \ k = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Theorem 3.

$$M_n(\forall,\exists) = \begin{cases} 2, & \textit{if} \ \ n = 2 \\ 2^{n-1} - 1, & \textit{if} \ \ n \geq 3 \end{cases} .$$

Theorem 4.

$$N_n(\forall,\exists) \sim 2^{2^n - 2} \ .$$

Theorem 5.

$$M_n(\exists,\exists) = \binom{n}{\lfloor \frac{n}{2} \rfloor} + \left\lfloor \frac{2^n - 2 - \binom{n}{\lfloor \frac{n}{2} \rfloor}}{2} \right\rfloor \ .$$

Theorem 6.

$$N_n(\exists,\exists) \sim 2^{2^n} \ .$$

The proofs are delegated to the following sections. We begin with those for the exact estimates.

Throughout the paper we use a representation of the partically ordered set $(\mathcal{P},\subset)$ as sequence space $\left(\{0,1\}^n,\prec\right)$, where $A \in \mathcal{P}$ corresponds to $S(A) = (a_1,a_2,\ldots,a_n)$ with $a_t = \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{if } t \notin A \end{cases}$ and the inclusion $A \subset B$ translates into $S(A) \prec S(B) = (b_1,b_2,\ldots,b_n)$, which means that $a_t \leq b_t$ for $t = 1,2,\ldots,n$.

3. Proof of Theorem 2

We view the cloud–antichain $\{\mathcal{A}_i\}_{i=1}^N$ of type $(\exists,\forall)$ in $\{0,1\}^n$. For $x \in \{0,1\}^n$ let the weight $w(x)$ be the number of 1's in x . Let m be the maximal weight of members of $\bigcup_{i=1}^N \mathcal{A}_i$ and let $\{v_1, v_2, \dots, v_t\}$ be the set of members of $\bigcup_{i=1}^N \mathcal{A}_i$ with weight m . We assume first that $m > \lfloor \frac{n}{2} \rfloor$. It is known that in $\{0,1\}^n$ there exist pairwise different members $v'_1, v'_2, \dots, v'_t$ of weight $m-1$ with the property

$$v'_j \leq v_j \quad \text{for} \quad j = 1, 2, \dots, t. \tag{3.1}$$

For every i $(i = 1, \dots, N)$ we replace all members of $\{v_1, v_2, \dots, v_t\}$ in $\mathcal{A}_i$ by the corresponding members of $\{v'_1, v'_2, \dots, v'_t\}$ and call the new cloud $\mathcal{A}'_i$.

One readily verifies that $\{A'_i\}_{i=1}^N$ has again the $(\exists,\forall)$–property . Symmetrically, one can perform a transformation of the clouds via sequences of smallest weight, if this is smaller than $\lfloor \frac{n}{2} \rfloor$. Iteration of these two kinds of transformation results in a cloud–antichain $\{\mathcal{A}^*_i\}_{i=1}^N$ with the $(\exists,\forall)$–property involving only sequences of weight $\lfloor \frac{n}{2} \rfloor$. There are $k = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ such sequences and every $\mathcal{A}^*_i$ can be represented via the usual incidence relation as a binary vector u_i of length k .

Now observe that the $(\exists,\forall)$–property is equivalent to the following one: $u_i \not> u_j$ for all $i \neq j$. Sperner's Theorem [1] implies $N \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$.

Conversely, by choosing all clouds consisting of $\lfloor \frac{k}{2} \rfloor$ sets with $\lfloor \frac{n}{2} \rfloor$ elements each we achieve this bound.

4. Proof of Theorem 3

We make use of an auxiliary result. For $X \subset \{0,1\}^n$ let $\mathcal{C}_n(X)$ be the set of elements of $\{0,1\}^n$ which are comparable with at least one element in X .

Lemma 1. *If X is an (ordinary) antichain in $\{0,1\}^n$, $n \geq 4$, then* $\mid \mathcal{C}_n(X) \mid \geq 2|X| + 3$.

Proof Suppose that there is an $\alpha \in X$ with $w(\alpha) = 1$ (or $w(\alpha) = n-1$). Then necessarily $\mid \mathcal{C}_n(\{\alpha\}) \smallsetminus \{(0,\dots,0),(1,\dots,1)\} \mid = 2^{n-1} - 1$ and $\mathcal{C}_n(\{\alpha\}) \cap (X \smallsetminus \{\alpha\}) = \varnothing$, which implies $\mid \mathcal{C}_n(X) \smallsetminus \{(0,\dots,0),(1,\dots,1)\} \mid \geq \mid \mathcal{C}_n(\{\alpha\}) \smallsetminus$

$\{(0,\dots,0),(1,\dots,1)\}\ | +|X|-1 = 2^{n-1}-2+|X|$. Now $2^{n-1}-2+|X| > 2|X|$ holds for $n \geq 5$, because there $2^{n-1}-2 > \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq |X|$, and for $n=4$, because there $|X| \leq 4$ under the supposition $w(\alpha)=1$ for an $\alpha \in X$.

It remains to consider the case, where $2 \leq w(\alpha) \leq n-2$ for all $\alpha \in X$. There is a component, say the n–th , in which some $\beta \in X$ has a 1 . Define now $X^* = \big\{(a_1,a_2,\dots,a_{n-1},\overline{a}_n)|(a_1,a_2,\dots,a_n) \in X\big\}$ where the bar stands for complementation, and notice that $X, X^* \subset \mathcal{C}_n(X) - \big\{(0,\dots,0),(1,\dots,1)\big\}$ and that $X^* \cap X = \varnothing$, because X is an antichain.

Since $e_n = (0,\dots,0,1) \in \mathcal{C}_n(\{\beta\}) \subset \mathcal{C}_n(X)$ and since $e_n \notin X \cup X^*$, we have

$$| \,\mathcal{C}_n(X) \smallsetminus \big\{(0,\dots,0),(1,\dots,1)\big\}\, | \geq |X| + |X^*| + 1 = 2|X| + 1$$

and thus the result.

Now Theorem 3 is readily established. Suppose first that $1 = |\mathcal{A}_1| = |\mathcal{A}_2| = \cdots = |\mathcal{A}_s| < 2 \leq |\mathcal{A}_{s+1}| \leq \cdots \leq |\mathcal{A}_N|$ with $1 \leq s \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Define then $T = \mathcal{C}_n\big(\bigcup_{i=1}^s \mathcal{A}_i\big) \smallsetminus \big(\bigcup_{i=1}^s \mathcal{A}_i\big)$ and conclude with aid of Lemma 1 that $|T| \geq 2s+3-s$. Since by the $(\forall,\exists)$–property $T \cap \big(\bigcup_{i=1}^N \mathcal{A}_i\big) = \varnothing$, we have

$$2^n \geq \sum_{i=1}^N |\mathcal{A}_i| + |T| \geq s + 2(N-s) + s + 3$$

and thus $N \leq 2^{n-1}-2$ for $n \geq 4$.

Furthermore, since $\big\{(0,\dots,0),(1,\dots,1)\big\} \cap \bigcup_{i=1}^N \mathcal{A}_i = \varnothing$ in the remaining case $2 \leq |\mathcal{A}_1| \leq \cdots \leq |\mathcal{A}_N|$ we have $N \leq \frac{1}{2}(2^n-2)$ and thus again $N \leq 2^{n-1}-1$.

On the other hand there is a simple construction: every $\mathcal{A}_i$ consists of a sequence $\alpha_i \neq (0,\dots,0),(1,\dots,1)$ and its complement $\overline{\alpha}_i$. There are $2^{n-1}-1$ such clouds. The $(\forall,\exists)$–property holds.

Actually, for $n \geq 4$ this construction gives the only optimal configuration. Clearly, by the previous arguments an optimal configuration has clouds of cardinality 2 only. We shall exclude next clouds of the form $\mathcal{A} = \{a,b\}$ with $b \neq \overline{a}$. For such a cloud a and b have a component value in common, say 0 in the first component. But then $(0,1,\dots,1)$ cannot be in any other cloud, it has to be in $\mathcal{A}$ and equal, say, a . If now $w(b) \leq n-3$ then there is a c with $w(c) = w(b)+1$, $c \prec a$, $c \succ b$, and $c \notin \bigcup_{i=1}^N \mathcal{A}_i$.

This contradicts the equality $\bigcup_{i=1}^{N} \mathcal{A}_i = \{0,1\}^n \setminus \{(0,\dots,0),(1,\dots,1)\}$. If on the other hand $w(b) = n-2 \geq 2$ (since $n \geq 4$), then some d with $w(d) = w(b)-1$ and $d \prec b \prec a$ is not in $\bigcup_{i=1}^{N} \mathcal{A}_i$.

Finally the cases $n = 2,3$ go by inspection.

In case $n=2$ the only optimal configuration has clouds of cardinality 1. For $n=3$ there is (up to isomorphism) also the solution $\{\{110\},\{101\},\{011\}\}$ with clouds of cardinality 1 only. Furthermore, there are three non–isomorphic solutions, for instance $\{\{110,001\},\{101,010\},\{011,100\}\}$, $\{\{110,010\},\{101,001\},\{011,100\}\}$, and $\{\{110,010\},\{101,100\},\{011,010\}\}$, with clouds of cardinality 2.

5. Proof of Theorem 5

There are at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ clouds with 1 member and the sequences $(0,0,\dots,0)$ and $(1,1,\dots,1)$ can be eliminated from all clouds. Therefore

$$N \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} + \left\lfloor 2^{-1} \left(2^n - 2 - \binom{n}{\lfloor \frac{n}{2} \rfloor} \right) \right\rfloor .$$

We abbreviate the right hand side expression by R and construct now R clouds with the $(\exists,\exists)$–property .

<u>Case $n = 2\ell$:</u> For $i = 1,\dots,\binom{n}{\ell}$ choose $\mathcal{A}_i = \{a_i\}$ with $w(a_i) = \ell$. For $i = \binom{n}{\ell}+1,\dots,R$ choose $\mathcal{A}_i = \{b_i, \overline{b}_i\}$ with $1 \leq w(b_i) < \ell$.

<u>Case $n = 2\ell+1$:</u> For $n=3$ the choice $\mathcal{A}_1 = \{100\}$, $\mathcal{A}_2 = \{010\}$, $\mathcal{A}_3 = \{001\}$, $\mathcal{A}_4 = \{011,101,110\}$ works. For $n > 3$ there exists a partition of vectors of weight $\ell+1$ into $\lfloor \frac{\binom{2\ell+1}{\ell+1}}{2} \rfloor$ disjoint pairs $\mathcal{A}_i = \{c_i, d_i\}$ with Hamming distance $d_H(c_i,d_i) \geq 4$.

Further, for the next $\binom{2\ell+1}{\ell}$ indices we define $\mathcal{A}_i = \{a_i\}$ with $w(a_i) = \ell$ and for all the remaining indices we set $\mathcal{A}_i = \{b_i, \overline{b}_i\}$ with $1 \leq w(b_i) < \ell$. The $(\exists,\exists)$–property is readily verified.

6. Proof of Theorem 1

Since $M_n(\forall,\exists) \geq M_n(\exists,\forall)$ we conclude from Theorem 3 that $M_n(\exists,\forall) \leq 2^{n-1}-1$. The issue is to construct a cloud–antichain meeting asymptotically this bound.
We make use of the

General form of Baranyai's Theorem

Let $n_1,\ldots,n_t$ be natural numbers such that $\sum_{i=1}^{t} n_i = \binom{n}{k}$, then $\binom{\Omega_n}{k}$ can be partitioned into disjoint sets $P_1,\ldots,P_t$ such that $|P_i| = n_i$ and each $\ell \in \Omega_n$ is contained in exactly $\lceil \frac{n_i \cdot k}{n} \rceil$ or $\lfloor \frac{n_i \cdot k}{n} \rfloor$ members of P_i .

Our main auxiliary result is

Lemma 2. *For positive integers* n,k,λ *with* $2k-n \leq \lambda < k$ *the set* $\binom{\Omega_n}{k}$ *has a partition* $P(n,k,\lambda) = \left\{P_1,P_2,\ldots,P_{\lfloor \frac{1}{2}\binom{n}{k}\rfloor}\right\}$ *with* $P_i = \{a_i,b_i\}, |a_i \cap b_i| = \lambda$.

Proof For $\lambda = 0$ or $\lambda = 2k-n$, the statement follows from Baranyai's Theorem. We proceed by induction:

If at least one of the numbers $\binom{n-1}{k},\binom{n-1}{k-1}$ is even, then we can define (by forgetting the last element n)

$$P(n,k,\lambda) = P(n-1,k,\lambda) \cup P(n-1,k-1,\lambda-1).$$

If $\binom{n-1}{k} \equiv \binom{n-1}{k-1} \equiv 1 \mod 2$, then there remain 2 sets: $v = \binom{\Omega_{n-1}}{k} \setminus P(n-1,k,\lambda)$, $u = \binom{\Omega_{n-1}}{k-1} \setminus P(n-1,k-1,\lambda-1)$. Since the labelling of the elements in Ω_{n-1} does not matter, v can be any member of $\binom{\Omega_{n-1}}{k}$ and u can be any member of $\binom{\Omega_{n-1}}{k-1}$. Particularly, we can assume that $|v \cap u| = \lambda$.

For even $n = 2\ell$ as well as for odd $n = 2\ell+1$ we define the cloud–antichain

$$P = \bigcup_{s=\ell-\lfloor\frac{\ell-1}{7}\rfloor}^{s=\ell+\lfloor\frac{\ell-1}{7}\rfloor} P\big(n,s,\ell-s+3\lfloor\frac{\ell-1}{7}\rfloor\big) \text{ and calculate } |P| = \sum_{i=-\lfloor\frac{\ell-1}{7}\rfloor}^{\lfloor\frac{\ell-1}{7}\rfloor} \lfloor\frac{\binom{n}{\ell+i}}{2}\rfloor \sim \frac{1}{2}2^n.$$

It remains to be seen that P has the $(\exists,\forall)$–property . For this consider two clouds $\{a,b\}$ and $\{a',b'\}$ with $|a| = |b| = s$, $|a'| = |b'| = s'$ and w.l.o.g. $s < s'$ and $a \subset a'$. We claim that $b \not\subset a'$, because otherwise $a \cup b \subset a'$ in contradiction to

$$|a\cup b| = 2s - \big(\ell - s + 3\lfloor\frac{\ell-1}{7}\rfloor\big) = 3s - 3\lfloor\frac{\ell-1}{7}\rfloor - \ell \ge 3\big(\ell - \lfloor\frac{\ell-1}{7}\rfloor\big) - 3\lfloor\frac{\ell-1}{7}\rfloor - \ell$$
$$= 2\ell - 6\lfloor\frac{\ell-1}{7}\rfloor > \ell + \lfloor\frac{\ell-1}{7}\rfloor \ge s'.$$

We claim also that $b \not\subset b'$, because otherwise $a\cap b \subset a'\cap b'$ in contradiction to $|a\cap b| = \ell - s + 3\lfloor\frac{\ell-1}{7}\rfloor > \ell - s' + 3\lfloor\frac{\ell-1}{7}\rfloor$. $b' \not\subset b$ and $a' \not\subset b$ obviously holds, because $|a'| = |b'| = s' > s = |b|$. Finally, we claim that $a \not\subset b'$ because otherwise $a \subset a'\cap b'$ in contradiction to $|a| > |a\cap b| > |a'\cap b'|$. We have shown that $\{a,b\}$ and $\{a',b'\}$ are not comparable in the sense $(\exists,\forall)$.

Remark Bernhard Herwig [6] was the first to show that $\liminf_{n\to\infty} M_n(\forall,\exists)2^{-n} = c > 0$. By arguments based on the marriage theorem he actually proved that $c \ge \frac{1}{18}$.

7. Proof of Theorem 4

Since necessarily $(0,0,\dots,0),(1,1,\dots,1) \notin \bigcup_{i=1}^N \mathcal{A}_i$, we have $\{\mathcal{A}_i\}_{i=1}^N \subset \Omega' \triangleq \mathcal{P}\big(\{0,1\}^n \smallsetminus \{(0,0,\dots,0),(1,1,\dots,1)\}\big)$ and thus $N \le 2^{2^n-2}$. On the other hand let us consider $\{\mathcal{A}_i\}_{i=1}^{N^*} \subset \Omega'$, where each $\mathcal{A}_i$ contains a subset $\{\alpha,\overline{\alpha}\}$ and N^* is maximal. The $(\forall,\exists)$–property holds.

There are $2^{n-1}-1$ sets $\{\alpha,\overline{\alpha}\}$ and therefore

$$|\Omega'| - N^* = \sum_{k=0}^{2^{n-1}-1} \binom{2^{n-1}-1}{k} \cdot 2^k = 3^{2^{n-1}}.$$

This implies $N^* = 2^{2^n-2} - 3^{2^{n-1}} \sim 2^{2^n-2}$.

8. Proof of Theorem 6

Consider all clouds containing at least 2 sequences of weight $\lfloor\frac{n}{2}\rfloor$. This defines a cloud–antichain of type $(\exists,\exists)$ and length $N = 2^{2^n} - 2^{2^n-\binom{n}{\lfloor\frac{n}{2}\rfloor}}\left(\binom{n}{\lfloor\frac{n}{2}\rfloor}+1\right) \sim 2^{2^n}$. Clearly, $N_n(\exists,\exists) \le 2^{2^n}$.

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