CLASSICAL RESULTS ON PRIMITIVE AND RECENT RESULTS ON CROSS–PRIMITIVE SEQUENCES

Rudolf Ahlswede and Levon H. Khachatrian

Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany

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When the kind invitation of Ron Graham and Jaroslav Nesetril, to write in honour of Paul Erdős about aspects of his work, reached us, our first reaction was to follow it with great pleasure. Our second reaction was not as clear: Which one among the many subjects in mathematics, to which he has made fundamental contributions, should we choose?

Finally we just followed the most natural idea to write about an area which just had started to fascinate us: Density Theory for Integer Sequences.

More specifically we add here to the classical theory of primitive sequences and their sets of multiples results for cross–primitive sequences, a concept, which we introduce. We consider both, density properties for finite and infinite sequences. In the course of these investigations we naturally come across the main theorems in the classical theory and the predominance of results due to Paul Erdős becomes apparent. Several times he had exactly proved the theorems we wanted to prove! Many of them belong to his earliest contribution to mathematics in his early twenties.

Quite luckily our random approach led us to the perhaps most formidable period in Erdős' work. It reminds us about a statement, which K. Reidemeister ([18], ch. 8) made about Carl Friedrich Gauss:

" . . . . Aber das Epochale ist doch die geniale Entdeckung des Jünglings: die Zahlentheorie."

1. Classical results

At first we set up our notation.

$\mathbb{N}$ denotes the set of positive integers and $\mathbb{P} = \{p_1, p_2, \dots\} = \{2, 3, 5, \dots\}$ denotes the set of all primes. For the number $u, v \in \mathbb{N}$ we write $u|v$, if u divides v . Further (u, v) stands for the largest common divisor and $< u, v >$ denotes the smallest common multiple of u and v .

In case $(u, v) = 1$ u and v are said to be relatively prime (or coprimes). The greatest prime factor of u is written as $p^+(u)$. For $i \le j$ $[i, j]$ equals $\{i, i+1, \dots, j\}$ and $(i, j]$ equals $\{i+1, \dots, i\}$. Any set $A \subset \mathbb{N}$ can also be viewed as an increasing sequence $(a_i)_{i=1}^{\infty}$, where $A = \{a_i : i \in \mathbb{N}\}$, and vice versa. We reserve the letter A for such sets or sequences. It is convenient to use the abbreviations $A(x) = A \cap [1, x]$ and $|B|$ for the cardinality of any set B . We also use $\phi(x, y) = \left|\left\{n \in [1, x] : P^t(n) > y\right\}\right|$.

The lower and upper asymptotic density of A are

$$\underline{\mathbf{d}}\ A = \liminf_{x \to \infty} \frac{|A(x)|}{x} \qquad \text{and} \qquad \overline{\mathbf{d}}\ A = \limsup_{x \to \infty} \frac{|A(x)|}{x} \ . \tag{1.1}$$

If $\underline{\mathbf{d}}\ A = \overline{\mathbf{d}}\ A$, then A possesses the asymptotic density $\mathbf{d}\ A = \underline{\mathbf{d}}\ A = \overline{\mathbf{d}}\ A$. Related quantities are

$$\underline{\delta}\ A = \liminf_{x \to \infty} \frac{1}{\log x} \sum_{a_i \le x} \frac{1}{a_i} \quad \text{and} \quad \overline{\delta}\ A = \limsup_{x \to \infty} \frac{1}{\log x} \sum_{a_i \le x} \frac{1}{a_i}, \tag{1.2}$$

the logarithmic lower and upper density of A . If $\underline{\delta}\ A = \overline{\delta}\ A$, then A possesses logarithmic density $\delta\ A = \underline{\delta}\ A = \overline{\delta}\ A$.

In the first half of the century there was noticeable interest in the study of density properties of the *set of multiples*

$$M(A) = \{m \in \mathbb{N} : \quad \text{for some} \quad a \in A \quad a|m\} \tag{1.3}$$

of infinite sequences A of positive integers. This naturally relates to the study of primitive sequences.

A sequence $A = (a_i)_{i=1}^{\infty}$ is *primitive*, if

$$a_i \nmid a_j \quad \text{for} \quad i \ne j \ . \tag{1.4}$$

One readily verifies that every A contains a unique subsequence $P(A)$ which is primitive and satisfies

$$M\big(P(A)\big) = M(A) \ . \tag{1.5}$$

Actually,

$$P(A) = \{a \in A : \nexists\, b \in A, b \neq a, \quad \text{and} \quad b|a\} \ . \tag{1.6}$$

One question of Chowla (see [2]) opened the subject: Does $\mathbf{d}\ M(A)$ exist for every $A \subset \mathbb{N}$?

This can readily be shown to be the case for all finite A , however, was open for a longer time and finally settled in the negative by Besicovitch [2] in the infinite case.

Theorem I. *(Besicovitch [2])*

For every $\varepsilon > 0$ *there is an* $A \subset \mathbb{N}$ *with*

$$\overline{\mathbf{d}}\ M(A) \geq \frac{1}{2} \quad \textit{and} \quad \underline{\mathbf{d}}\ M(A) \leq \varepsilon \ .$$

Actually, the A's are constructed as unions of suitable intervals. The primitive sequence $P(A)$ generating the $M(A)$ of Theorem I gives the next famous result.

Theorem II. *(Besicovitch [2])*

For every $\varepsilon > 0$ *there is primitive sequence* A' *with*

$$\overline{\mathbf{d}}\ A' \geq \frac{1}{2} - \varepsilon \quad \textit{and} \quad \underline{\mathbf{d}}\ A' \leq \varepsilon \ .$$

This shows that a question of Davenport and Erdős (see [7], [13]), whether every primitive sequence has asymptotic density 0 , has a negative answer.

We derive next an upper bound on $\overline{\mathbf{d}}\ A$ because it is instructive and beautiful. For any primitive $A = \{a_1, \dots, a_\alpha\} \subset [1, 2n]$ let d_i denote the greatest odd devisor of a_i . Then necessarily $d_1, \dots, d_\alpha$ are all distinct and hence

$$|A| = \alpha \leq n \ . \tag{1.7}$$

Theorem III. *(Behrend [3])*

For every primitive A $\overline{d}\ A \leq \frac{1}{2}$.

Example IV (Everybody)

$\{n+1, \dots, 2n\}$ is primitive and has density $\frac{1}{2}$.

This simple fact is very relevant in the analysis of infinite primitive sequences.

Now Paul Erdős enters the scene.

Theorem V. *(Erdős [5])*

For a primitive $A \not\supset \{1\}$

$$\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i} < \infty \ .$$

It is an open problem of Erdős whether $\displaystyle\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i} \leq \sum_{i=1}^{\infty} \frac{1}{p_i \log p_i}$.

By Abel summation it can be shown that for any set $B \subset \mathbb{N}$

$$0 \leq \underline{\mathbf{d}}\ B \leq \underline{\delta}\ B \leq \overline{\delta}\ B \leq \overline{\mathbf{d}}\ B \leq 1 \ . \tag{1.8}$$

Since $\displaystyle\frac{1}{\log n} \sum_{N < a_i \leq n} \frac{1}{a_i} \leq \sum_{N < a_i \leq n} \frac{1}{a_i \log a_i}$, by Theorem V $\delta\ A = 0$ for primitive A .

Also by (1.8) $\underline{\mathbf{d}}\ A = 0$. We state this result.

Theorem VI. *(Erdős [5])*

For every primitive sequence A $\underline{\mathbf{d}}\ A = \delta A = 0$ *or (equivalently)*

$$\frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} = o(1) \quad \textit{as} \quad n \to \infty \ . \tag{1.9}$$

Logarithmic density has turned out to be an appropriate measure!

Also, what can be said about the speed in (1.9)?

Theorem VII. *(Behrend [3])*

There is a constant γ *such that for every primitive sequence* A

$$\frac{1}{\log n} \sum_{a_i \leq n} \frac{1}{a_i} \leq \gamma \frac{1}{(\log \log n)^{\frac{1}{2}}} \quad \textit{for} \quad n \geq 3 \ . \tag{1.10}$$

In the proof the general case is reduced to A's consisting entirely of square–free integers and their analysis is based on Sperner's Lemma [1]! This gave a strong impetus also to combinatorial extremal theory starting with [12] and continuing with $[24], \ldots, [30]$ and many, many others.

Theorem VII is best possible in the sense that γ cannot be replaced $o(n)$.

Theorem VIII. *(S. Pillai [8])*

There exists a positive constant c *, such that to every* $x \geq 3$ *corresponds a primitive set* A_x *with*

$$\frac{1}{\log x} \sum_{a_i \leq x} \frac{1}{a_i} > \frac{c}{(\log \log x)^{\frac{1}{2}}} \ .$$

Subsequently Erdős, Sárközy, and Szemerédi [20] showed that c can be choosen as $(2\pi)^{-\frac{1}{2}} - \varepsilon$ for any $\varepsilon > 0$ and that this is best possible.

The last three theorems concern in essence only finite primitive sequences. Related to infinite primitive sequences in the true sense is

Theorem IX. *(Erdős, Sárközy, Szemerédi [21])*

For every infinite primitive sequence A

$$\sum_{a_i \leq x} \frac{1}{a_i} = o\left(\frac{\log x}{(\log \log x)^{\frac{1}{2}}}\right)$$

and this bound is best possible.

We draw attention also to a survey paper of Erdős, Sárközy, and Szemerédi [22] and to a related paper of Pomerance and Sárközy [23].

Concerning $\overline{\mathbf{d}}\, A$ there is the following improvement of Theorem III.

Theorem X. *(Erdős [14])*

Let A *be an infinite primitive sequence, then for every* $a \in A$ *of the form* $a = 2^u(2v+1) \leq n$ *;* $u, v \geq 0$ *,*

$$|A(n)| \leq n - \left\lfloor \frac{1}{2} n \right\rfloor - \left\lfloor \frac{1}{2}\left(\frac{n}{3^u(2v+1)} - 1\right)\right\rfloor \ .$$

Hence, $\overline{\mathbf{d}}\, A < \frac{1}{2}$.

After Besicovitch's negative answer to Chowla's question, it is natural to adress the next question:

Under which conditions on A does $\underline{\mathbf{d}}\, M(A)$ or $\mathbf{d}\, M(A)$ or $\delta\, M(A)$ exist?

Davenport/Erdős and Erdős answered all these questions:

We derive here the simplest and most transparent result, Theorem XI below, in order to explain the important role of a quantity, which we consider to be a density concept for sets of multiples and denote as μ .

Since A is fixed, we write M for $M(A)$. Furhter we denote by $M_m = M_m(A)$ the set of multiples of the first m elements of A , namely $a_1, a_2, \dots, a_m$. M_m can be represented as the union of a finite number of congruence classes, and therefore possesses asymptotic density. If we denote by $M^{(i)}(n)$ the natural numbers, not exceeding n , which are divisible by a_i but not diversible by any one of $a_1, \ldots, a_{i-1}$, then we have

$$M_m(n) = \dot{\bigcup_{i=1}^{m}} M^{(i)}(n) \ . \tag{1.11}$$

By inclusion–exclusion for every $i = 1, 2, 3, \ldots,$

$$|M^{(i)}(n)| = \left\lfloor \frac{n}{a_i} \right\rfloor - \sum_{j<i} \left\lfloor \frac{n}{< a_i, a_j >} \right\rfloor + \sum_{k<j<i} \left\lfloor \frac{n}{< a_k, a_i, a_j >} \right\rfloor - \ldots$$

and

$$\mathbf{d}\ M^{(i)} = \lim_{n\to\infty} \frac{|M^{(i)}(n)|}{n} = \frac{1}{a_i} - \sum_{j<i} \frac{1}{< a_j, a_i >} + \ldots \ .$$

Therefore by (1.11)

$$\mathbf{d}\ M_m = \sum_{i=1}^{m} \mathbf{d}\ M^{(i)} \ . \tag{1.12}$$

Since $0 < \sum_{i=1}^{m} \mathbf{d}\ M^{(i)} < 1$ and $\mathbf{d}\ M^{(i)} \geq 0$, $\lim_{m\to\infty} \mathbf{d}\ M_m = \sum_{i=1}^{\infty} \mathbf{d}\ M^{(i)}$ exists.
We define now the "density" μ by

$$\mu\ A = \lim_{m\to\infty} \mathbf{d}\ M_m(A), \quad A \subset \mathbb{N} \ . \tag{1.13}$$

Since $M_m(A) \subset M(A)$, we see immediately that

$$\mu\ A \leq \underline{\mathbf{d}}\ M(A) \ . \tag{1.14}$$

Suppose now that $\sum_{i=1}^{\infty} a_i^{-1} < \infty$. Then $\overline{\mathbf{d}}\ M(A) \leq \mathbf{d}\ M_m(A) + \sum_{i=m+1}^{\infty} \frac{1}{a_i}$ and thus $\overline{\mathbf{d}}\ M(A) \leq \mu\ A \leq \underline{\mathbf{d}}\ M(A)$.

Theorem XI. *([17])*

If $\sum_{i=1}^{\infty} a_i^{-1} < \infty$ *, then* $\mathbf{d}M(A)$ *exists and equals* μA *.*

Here are the high–lights.

Theorem * XII. *(Davenport/Erdős [7], also [13])*
For any $A \subset \mathbb{N}$ $M(A)$ *has logarithmic density and*

$$\delta M(A) = \underline{\mathbf{d}}\ M(A) = \mu A \ .$$

Theorem * XIII. *(Erdős [11])*
A necessary and sufficient condition for $\mathbf{d}\ M(A)$ *to exists is*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \sum_{n^{1-\varepsilon} < a_i \le n} |M^{(i)}(n)| = 0 \ . \tag{1.15}$$

Eventhough condition (1.15) looks complicated, it yields a useful sufficient condition.

Theorem * XIV. *(Erdős [11])*
If $A \subset \mathbb{N}$ *satisfies for some constant* c $|A(n)| \le \frac{cn}{\log n}$ *for* $n \ge 2$ *, then* $dM(A)$ *exists.*

The case $A = \mathbb{P}$ is included here.

The result is best possible in the following sense.

Theorem XV. *(Erdős [12])*
For any monotonically increasing function $\Psi : \mathbb{N} \to \mathbb{R}_+$ *with* $\lim\limits_{n \to \infty} \Psi(n) = \infty$ *there exsists an* $A \subset \mathbb{N}$ *such that*

$$|A(n)| \le const \frac{n\ \Psi(n)}{\log n} \quad \text{for large} \quad n,$$

but $\mathbf{d}M(A)$ *does not exist.*

We present now two further results with many applications.

The first of them was probably motivated by Example IV. It shows how the set of multiples of certain intervals behaves in density. This is the key idea in Besicovitch's construction [2]. Erdős improved the length of the intervals.

Theorem * XVI. *(Erdős [5])*
The intervals $(T^{1-\varepsilon}, T] \subset \mathbb{N}$ *satisfy* $\lim\limits_{\substack{\varepsilon \to 0 \\ T \to \infty}} \mathbf{d}M\big((T^{1-\varepsilon}, T]\big) = 0$.

The second result is the famous Behrend Lemma in a dual formulation, that is, for $X \subset \mathbb{N}$ we use $M(X)$ instead of $\mathbb{N} \setminus M(X)$.

Lemma XVII *. *(Behrend [10])*
Let $A, B \subset \mathbb{N}$ *be finite, then*

$$\mathbf{d}M(A) \cdot \mathbf{d}M(B) \leq \mathbf{d}\big(M(A) \cap M(B)\big) \ .$$

Moreover, equality holds exactly if the primitive sets $P(A)$ *and* $P(B)$ *satisfy* $(a, b) = 1$ *for all* $a \in P(A), b \in P(B)$ *.*

Finally there are also several papers concerning the growth of $\phi(x, y)$ ([15]). We use later only the following result.

Theorem XVIII *. *(Chowla and Vijayaraghavan [15])*

$$\lim_{x \to \infty} \frac{\phi(x, x^\theta)}{x} = \log \frac{1}{\theta}, \quad \textit{for} \ \ \frac{1}{2} \leq \theta < 1.$$

Remark:

We apologize for not including in our sketch several results of basic nature such as Rogers inequality [17] and others. Our selection is guided by our present research interest. The reader may consult the books by Halberstam and Roth [17] and Hall and Tenenbaum [19].

Results marked with a star are applied in subsequent sections.

2. New results

We introduce a seemingly basic and new concept.

The pair of sets (or sequences) (A, B) with $A, B \subset \mathbb{N}$ is called *cross–primitive*, if

$$a \nmid b \quad \text{and} \quad b \nmid a \quad \text{for all} \quad a \in A, b \in B \ . \tag{2.1}$$

It is convenient to denote the set of all cross–primitive pairs (A, B) with $A, B \subset \mathbb{N}(x)$ (resp. $\mathbb{N}$) by $\text{Cross}(x)$ (resp. $\text{Cross}(\infty)$). We are again interested in density properties.

We begin with the finite case and define

$$c(x) = \max_{(A,B) \in \text{Cross}(x)} \frac{|A| \cdot |B|}{x^2}.$$

Theorem 1.

For all $x \in \mathbb{N}$ $\quad c(x) < \frac{1}{4}$ *and*

$$\lim_{x \to \infty} c(x) = \frac{1}{4} \ .$$

Remark: As analogue for a primitive sequence see the simple Example IV and (1.7). We believe that our construction is optimal for large x . Erdős thinks that the deviation of $\max\limits_{(A,B) \in \text{Cross}(x)} |A||B|$ from $\frac{x^2}{4}$ is of the order x^α for some $\alpha > 1$.

The infinite case shows more complex behaviour and that's the case where also several classical results on primitive sequences are used.

Theorem 2.

$$\max_{(A,B) \in \text{Cross}(\infty)} \underline{\mathbf{d}}\ A \cdot \underline{\mathbf{d}}\ B = \frac{1}{16} \ .$$

The maximum is assumed for a pair with densities.

One auxiliary result for proving this Theorem deserves special attention. It is an infinite form of Behrend's Lemma XVII, but by no means an easy extension. On the opposite, it involves the essence of the Davenport/Erdős Theorem XIV and expresses it in an elegant way.

Lemma 1. *For arbitrary* $A, B \subset \mathbb{N}$

$$\underline{\mathbf{d}}\ M(A) \cdot \underline{\mathbf{d}}\ M(B) \leq \underline{\mathbf{d}}\big(M(A) \cap M(B)\big) \ .$$

We use another auxiliary result, which should be known to the experts, but we could not find stated in the literature.

Lemma 2. *For any* $0 \leq \lambda \leq 1$ *, and any* $q_1 \in \mathbb{P}$ *,* $\frac{1}{q_1} \leq \lambda$ *there exists a set of primes* $Q = \{q_1 < q_2 \dots\} \subset \mathbb{P}$ *with*

$$\mathbf{d}\ M(Q) = \lambda \ .$$

Finally, we enter the world of pathologies discovered by Besicovitch.

Theorem 3.

$$\sup_{(A,B)\in \mathrm{Cross}(\infty)} \overline{\mathbf{d}}\, A \cdot \overline{\mathbf{d}}\, B = 1\ .$$

The supremum cannot be assumed: For $A=\{a_1,\dots\}$ $\overline{\mathbf{d}}\, B \le 1-\frac{1}{a_1}$.

Remark: The construction in the proof of this Theorem can be used to show that in Lemma 1 $\underline{\mathbf{d}}$ cannot be replaced by $\overline{\mathbf{d}}$.

3. Proof of Theorem 1

Since for $(A,B)\in \mathrm{Cross}(x)$ A and B must be disjoint and $1\notin A\cup B$, necessarily $|A|+|B|<x$ and $\frac{|A|\cdot|B|}{x^2}<\frac{1}{4}$. Therefore also $c(x)<\frac{1}{4}$ for all $x\in\mathbb{N}$.

To complete the proof, we have to construct a sequence $(A_x,B_x)_{x=1}^{\infty}$ with $(A_x,B_x)\in \mathrm{Cross}(x)$ and

$$\lim_{x\to\infty}\frac{|A_x|\cdot|B_x|}{x^2}=\frac{1}{4}\ . \tag{3.1}$$

We define for a $\theta, \frac{1}{2}\le 0<1$, which we adjust later,

$$\begin{aligned} A_x &= \{a\in\mathbb{N}: x^{1-\theta}\le a\le x \ \text{ and } \ p^+(a)\le x^\theta\},\\ B_x &= \{b\in\mathbb{N}: b\le x \ \text{ and } \ p^+(b)> x^\theta\}, \end{aligned}$$

and observe that for a θ in the specified interval $(A_x,B_x)\in \mathrm{Cross}(x)$. Hence

$$|A_x|\ge x-x^{1-\theta}-|B_x|\ . \tag{3.2}$$

Now Theorem XIII * says that $\displaystyle\lim_{x\to\infty}\frac{|B_x|}{x}=\log\frac{1}{\theta}$ and since $x^{1-\theta}=o(x)$

$$|B_x|\sim x\log\frac{1}{\theta}, |A_x|\gtrsim x(1-\log\frac{1}{\theta})\ . \tag{3.3}$$

We choose nor θ such that $\log\frac{1}{\theta}=1-\log\frac{1}{\theta}=\frac{1}{2}$, that is, $\theta=e^{-\frac{1}{2}}\sim 0.6065>\frac{1}{2}$. Clearly, (3.3) implies now (3.1).

Remark: A good estimate of $|B_x|$ is possible, because B_x can be partitioned according to the biggest prime in the decomposition of its members. These biggest primes are essentially known in magnitude by the Prime Number Theorem. Our first proof followed this line. Then we learnt about [15].

4. Proof of Lemma 1

Behrend's Lemma implies that for every $n \in \mathbb{N}$

$$\mathbf{d}M\big(A(n)\big) \cdot \mathbf{d}M\big(B(n)\big) \leq \mathbf{d}\Big(M\big(A(n)\big) \cap M\big(B(n)\big)\Big) \ . \tag{4.1}$$

Since $A(n)$, $B(n)$ and thus also $A(n) \cap B(n)$ are monotonically increasing in n , we have

$$M\big(A(n)\big) \cap M\big(B(n)\big) \subset M(A) \cap M(B)$$

and therefore

$$\underline{\mathbf{d}}\Big(M\big(A(n)\big) \cap M\big(B(n)\big)\Big) \leq \underline{\mathbf{d}}\big(M(A) \cap M(B)\big) \ . \tag{4.2}$$

Since $\mathbf{d}\Big(M\big(A(n)\big) \cap M\big(B(n)\big)\Big) = \underline{\mathbf{d}}\Big(M\big(A(n)\big) \cap M\big(B(n)\big)\Big)$, (4.1) and (4.2) imply

$$\mathbf{d}M\big(A(n)\big) \cdot \mathbf{d}M\big(B(n)\big) \leq \underline{\mathbf{d}}\big(M(A) \cap M(B)\big) \ . \tag{4.3}$$

Now by Theorem XII

$$\lim_{n\to\infty} \mathbf{d}M\big(A(n)\big) = \underline{\mathbf{d}}M(A), \lim_{n\to\infty} \mathbf{d}M\big(B(n)\big) = \underline{\mathbf{d}}M(A)$$

and therefore

$$\underline{\mathbf{d}}M(A)\underline{\mathbf{d}}M(B) \leq \underline{\mathbf{d}}\big(M(A) \cap M(B)\big) \ .$$

5. Proof of Lemma 2

For any $Q = \{q_1 < q_2 < \dots\} \subset \mathbb{P}$ by the Prime Number Theorem (or weaker versions)

$$|Q \cap [1, n]| < \text{const} \cdot \frac{n}{\log n} \ .$$

Therefore by Theorem XIV $M(Q)$ possesses asymptotic density and by Theorem XII

$$\underline{\mathbf{d}}M(Q) = \mathbf{d}M(Q) = \sum_{i=1}^{\infty} q^{(i)},$$

where

$$q^{(i)} = \frac{1}{q_i} - \sum_{j<i} \frac{1}{q_j q_i} + \sum_{k<j<i} \frac{1}{q_k q_j q_i} - \dots, \tag{5.1}$$

because $Q \subset \mathbb{P}$. Therefore

$$\sum_{i=1}^{\infty} q^{(i)} = 1 - \prod_{i=1}^{\infty} \left(1 - \frac{1}{q_i}\right)$$

and now the statement follows from $\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty$, because $-\log\left(1-\frac{1}{q_i}\right) > \frac{1}{q_i}$ and for any nullsequence $\{a_i\}_{i=1}^{\infty}$ of positive numbers with $\sum_{i=1}^{\infty} a_i = \infty$ any real number $r > 0$ equals $\sum_{j=1}^{\infty} a_{ij}$ for a suitable subsequence $\{a_{ij}\}_{j=1}^{\infty}$.

6. Proof of Theorem 2

We show first that for $(A,B) \in \mathrm{Cross}(\infty)$

$$\underline{\mathbf{d}}A \cdot \underline{\mathbf{d}}B \le \frac{1}{16} \ . \tag{6.1}$$

We associate with (A,B) the sets

$$A^* = M(A) \setminus \big(M(A) \cap M(B)\big),$$
$$B^* = M(B) \setminus \big(M(A) \cap M(B)\big),$$

and observe that also $(A^*, B^*) \in \mathrm{Cross}(\infty)$.
Moreover, we notice that

$$A \subset A^* \qquad \text{and} \qquad B \subset B^* \tag{6.2}$$

and that

$$M(A) \cap M(B) = M(C), \tag{6.3}$$

where

$$C = \{< a, b >: a \in A, b \in B\} \ . \tag{6.4}$$

By our definitions and properties (6.2) and (6.3) we have also

$$A \cap [1,x] \subset \big(M(A) \cap [1,x]\big) \setminus \big(M(C) \cap [1,x]\big) \tag{6.5}$$

and therefore

$$|A \cap [1,x]| \le |M(A) \cap [1,x]| - |M(C) \cap [1,x]| \ . \tag{6.6}$$

Let now $(x_i)_{i=1}^{\infty}$ be an increasing sequence of positive integers with

$$\lim_{i\to\infty} \frac{|M(A) \cap [1,x_i]|}{x_i} = \underline{\mathbf{d}}M(A) \ . \tag{6.7}$$

Then by (6.6) and (6.7)

$$\underline{\mathbf{d}}A \le \liminf_{i\to\infty} \frac{|A \cap [1,x_i]|}{x_i} \le \underline{\mathbf{d}}M(A) - \liminf_{i\to\infty} \frac{|M(C) \cap [1,x_i]|}{x_i} \le \underline{\mathbf{d}}M(A) - \underline{\mathbf{d}}M(C) \ . \tag{6.8}$$

Now we lower bound $\mathbf{d}M(C)$ by Lemma 1:

$$\underline{\mathbf{d}}M(C) = \underline{\mathbf{d}}\big(M(A) \cap M(B)\big) \geq \underline{\mathbf{d}}M(A) \cdot \underline{\mathbf{d}}M(B)$$

and conclude that

$$\underline{\mathbf{d}}A \leq \underline{\mathbf{d}}M(A) - \underline{\mathbf{d}}M(A) \cdot \underline{\mathbf{d}}M(B) = \underline{\mathbf{d}}M(A)\big(1 - \underline{\mathbf{d}}M(B)\big) \ . \tag{6.9}$$

Symmetrically

$$\underline{\mathbf{d}}B \leq \underline{\mathbf{d}}M(B)\big(1 - \mathbf{d}M(A)\big) \tag{6.10}$$

and thus finally

$$\underline{\mathbf{d}}A \cdot \underline{\mathbf{d}}B \leq \underline{\mathbf{d}}M(A)\big(1 - \underline{\mathbf{d}}M(A)\big) \cdot \underline{\mathbf{d}}M(B)\big(1 - \underline{\mathbf{d}}M(B)\big) \leq \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \ .$$

We construct now $(A,B) \in \text{Cross}(\infty)$ with

$$\mathbf{d}\ A \cdot \mathbf{d}\ B = \frac{1}{16} \ . \tag{6.11}$$

By Lemma 2 there is a $Q = \{q_1 \subset q_2 \dots\} \subset \mathbb{P}$ with

$$\mathbf{d}\ M(Q) = \frac{1}{2} \qquad \text{and} \qquad q_1 > 2 \ . \tag{6.12}$$

Set

$$\begin{aligned} A &= \{a \in \mathbb{N} : 2 \mid a \quad \text{and} \quad q_i \nmid a \quad \text{for all} \quad q_i \in Q\}, \\ B &= \{b \in \mathbb{N} : 2 \nmid b \quad \text{and} \quad b \in M(Q)\}. \end{aligned}$$

Equivalently

$$A = M(\{2\}) \smallsetminus \big(M(\{2\}) \cap M(Q)\big) \quad \text{and} \quad B = M(Q) \smallsetminus \big(M(\{2\}) \cap M(Q)\big) \ .$$

Also, it is clear that $(A,B) \in \text{Cross}(\infty)$ and that $M(\{2\}) \cap M(Q) = M(C)$, where

$$C = \{2q_i : q_i \in Q\} \ .$$

Obviously $|C \cap [1,n]| \leq \text{const}\frac{n}{\log n}$ and again by Theorem XIV $M(C)$ has asymptotic density and is given by the formula $\mathbf{d}M(C) = \sum\limits_{i=1}^{\infty} q_*^{(i)}$, where

$$q_*^{(i)} = \frac{1}{2q_i} - \sum_{j<i} \frac{1}{2q_j q_i} + \sum_{k<j<i} \frac{1}{2q_k q_j q_i} \cdots = \frac{q^{(i)}}{2} \ .$$

Hence $\mathbf{d}M(C) = \frac{1}{2}\sum\limits_{i=1}^{\infty} q^{(i)} = \dfrac{\mathbf{d}M(Q)}{2} = \dfrac{1}{4}$.

Therefore

$$\mathbf{d}A = \mathbf{d}M(\{2\}) - \mathbf{d}(C) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \mathbf{d}B = \mathbf{d}M(Q) - \mathbf{d}(C) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

and (6.11) holds.

7. Proof of Theorem 3

Let us fix $\delta > 0$, $\delta_i > 0$ for $i \in \mathbb{N}$; $\sum_{i=1}^{\infty} \delta_i = \delta$ and $0 < \theta < 1$.

By Theorem XVI for a $\delta_i > 0$ there are positive numbers $T(\delta_i)$ and $\lambda(\delta_i)$ such that

$$\mathbf{d}\ M\big([T^{1-\lambda_i}, T]\big) < \delta_i \quad \text{for} \quad T > T(\delta_i), \lambda_i < \lambda(\delta_i)\ . \tag{7.1}$$

We fix arbitrary $\lambda_i \in \big(0, \lambda(\delta_i)\big)$ for $i \in \mathbb{N}$ and $\lambda^* \in (0,1)$.
Now (7.1) and the definition of density tell us that for $T_j > T(\delta_j)$ $(j = 1, 2, \ldots, i)$ and $S > S(\lambda^*, T_1, T_2, \ldots, T_i, \lambda_1, \ldots, \lambda_i, \delta_1, \ldots, \delta_i)$ (suitable) simultaneously

$$\frac{|M\big([T_j^{1-\lambda_j}, T_j]\big) \cap [S^{1-\lambda^*}, S]|}{S - S^{1-\lambda^*}} < \delta_j \quad \text{for} \quad j \le i\ . \tag{7.2}$$

Now let R_1 be an integer with the properties

$$R_1 > T(\delta_1) \qquad \text{and} \qquad \frac{1}{R_1^{\lambda_1}} < \theta\ . \tag{7.3}$$

We fix the interval $[R_1^{1-\lambda_1}, R_1]$.
Let L_1 be an integer with the properties

$$L_1^{1-\lambda_1} > R_1 \qquad \text{and} \qquad L_1^{1-\lambda_1} > S(\lambda_1, R_1, \lambda_1, \delta_1)\ . \tag{7.4}$$

We choose the interval $[L_1^{1-\lambda_1}, L_1]$.
Furthermore, we choose R_2, L_2 such that $L_1 < R_2^{1-\lambda_2}, R_2 < L_2^{1-\lambda_2}$,
$R_2^{1-\lambda_2} > \max\big\{T(\delta_2), S(\lambda_2, L_1, \lambda_1, \delta_1)\big\}, \frac{1}{R_2^{\lambda_2}} < \theta$ and $L_2^{1-\lambda_2} > S(\lambda_2, R_1, R_2, \lambda_1, \lambda_2, \delta_1, \delta_2)$.

We fix now intervals $[R_2^{1-\lambda_2}, R_2]$ and $[L_2^{1-\lambda_2}, L_2]$. Continuing this procedure, for every $i \in \mathbb{N}$ we choose R_i, L_i such that $L_{i-1} < R_i^{1-\lambda_i} < L_i^{(1-\lambda_i)^2}$,
$R_i^{1-\lambda_i} > \max\big\{T(\delta_i), S(\lambda_i, L_1, \ldots, L_{i-1}, \lambda_1, \ldots, \lambda_{i-1}, \delta_1, \ldots, \delta_{i-1})\big\}$, $\frac{1}{R_i^{\lambda_i}} < \theta$,

and $L_i^{1-\lambda_i} > S(\lambda_i, R_1, \ldots, R_i, \lambda_1, \ldots, \lambda_i, \delta_1, \ldots, \delta_i)$.

We fix intervals $[R_i^{1-\lambda_i}, R_i]$ and $[L_i^{1-\lambda_i}, L_i]$.

By our construction one has for every $i \in \mathbb{N}$

$$\frac{|M\big([R_j^{1-\lambda_j}, R_j]\big) \cap [L_i^{1-\lambda_i}, L_i]|}{L - L^{1-\lambda_i}} < \delta_j \quad \text{for all} \quad j \leq i \tag{7.5}$$

and analogously

$$\frac{|M\big([L_j^{1-\lambda_j}, L_j]\big) \cap [R_i^{1-\lambda_i}, R_i]|}{R_i - R_i^{1-\lambda_i}} < \delta_j \quad \text{for all} \quad j \leq i-1 \ . \tag{7.6}$$

Now we introduce (disjoint) sets

$$A^* = \bigcup_{i=1}^{\infty} [R_i^{1-\lambda_i}, R_i], B^* = \bigcup_{i=1}^{\infty} [L_i^{1-\lambda_i}, L_i] \tag{7.7}$$

and consider the sets

$$A = A^* \smallsetminus M(B^*), B = B^* \smallsetminus M(A^*) \ . \tag{7.8}$$

It is clear from this definition that $(A, B) \in \text{Cross}(\infty)$. The upper densities $\overline{\mathbf{d}}A$ and $\overline{\mathbf{d}}B$ are lower bounded now with the help of (7.5) and (7.6).

For every $i \in \mathbb{N}$ the number of integers in A , which do not exceed R_i is at least

$$\begin{aligned}
&|[R_i^{1-\lambda_i}, R_i] \smallsetminus \big(M(B^*) \cap [R_i^{1-\lambda_i}, R_i]\big)| \\
&\geq (R_i - R_i^{1-\lambda_i}) - \sum_{j=1}^{i-1} |M\big([L_j^{1-\lambda_j}, L_j]\big) \cap [R_i^{1-\lambda_i}, R_i]| \\
&> (R_i - R_i^{1-\lambda_i}) - (R_i - R_i^{1-\lambda_i}) \cdot \sum_{j=1}^{i-1} \delta_j > (R_i - R_i^{1-\lambda_i})(1-\delta) \ .
\end{aligned}$$

Therefore, for every $i \geq 1$

$$\frac{|A \cap [1, R_i]|}{R_i} > \frac{R_i - R_i^{1-\lambda_i}}{R_i}(1-\delta) = \left(1 - \frac{1}{R_i^{\lambda_i}}\right)(1-\delta) > (1-\theta)(1-\delta) \ ,$$

because $\dfrac{1}{R_i^{\lambda_i}} < \theta$ for all $i \in \mathbb{N}$.

Hence, $\overline{\mathbf{d}}A \geq (1-\theta)(1-\delta)$. Similarly $\overline{\mathbf{d}}B \geq (1-\theta)(1-\delta)$ and therefore

$$\overline{\mathbf{d}}A \cdot \overline{\mathbf{d}}B \geq (1-\theta)^2(1-\delta)^2 \ .$$

The result follows, because θ and δ can be made arbitrarily small.

8. Concluding remarks

The notion of cross–primitive pairs can be generalized to that of cross–primitive k–tuples of sets $(A_1, \ldots, A_k)$. The understanding here is that any pair (A_i, A_j) $(i \neq j)$ is cross–primitive. $\mathrm{Cross}(x)$ then becomes $\mathrm{Cross}_k(x)$.

We guess that

1. $$\lim_{x\to\infty} \max_{(A_1,\ldots,A_k)\in \mathrm{Cross}_k(x)} \prod_{i=1}^{k} \frac{|A_i|}{x} = \left(\frac{1}{k}\right)^k$$

2. $$\max_{(A_1,\ldots,A_k)\in \mathrm{Cross}_k(\infty)} \prod_{i=1}^{k} \underline{\mathbf{d}} A_i = \left(\frac{1}{k}\right)^k \left(\frac{k-1}{k}\right)^{k(k-1)}$$

3. $$\sup_{(A_1,\ldots,A_k)\in \mathrm{Cross}_k(\infty)} \prod_{i=1}^{k} \overline{\mathbf{d}} A_i = 1 \ .$$

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