A Graph–Theoretic Approach to Searching with Lies

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We consider searching for an integer $m \in \{1, \ldots, M\}$ by asking questions when at most e lies are allowed in the answers : the question $\mathcal{Q}$ is a subset of the set $\{1, \ldots, M\}$; the answer is either 0 or 1, and the answers 0 if $m \in \mathcal{Q}$ and 1 if $m \notin \mathcal{Q}$ are considered as lies. The problem can be formulated as one of finding a strategy that allows us to discover any of M integers with the minimum number of questions for any possible combination of lies.

We attack this problem by introducing a tree structure in such a way that the nodes "contain" integer–valued vectors, and the vector contained in each node is uniquely determined by the arithmetic operation applied to the vectors contained in its sons. More precisely, we notice that the identity

$$\sum_{i=0}^{e} \binom{n-e+i-1}{i} 2^{e-i} = V_e^{(n)}$$

where $V_e^{(n)} = \sum_{i=0}^{e} \binom{n}{i}$ denotes the size of a ball of radius e in the Hamming space $\{0,1\}^n$, can be used in the following way. We construct a regular rooted binary tree having $k_{e-i}^* = \binom{n-e+i-1}{i}$ terminal nodes at level i and associate a vector of length $e+1$ having 1 in the i–th position and 0's in all other positions with each of them, where $i = 0, \ldots, e$. If two nodes located at some level t of the tree contain the vectors $\mathbf{a} = (a_0, \ldots, a_e)$ and $\mathbf{b} = (b_0, \ldots, b_e)$ and these nodes have a common parent at level $t+1$, then we specify that the parent must contain the vector $\mathbf{c} = (c_0, \ldots, c_e)$ having components $c_e = a_e + b_e$ and $c_j = a_j + b_j - c_{j+1}$, $j = e-1, \ldots, 0$. It can be easily checked that if the tree is constructed in accordance with these rules and has the distribution of terminal nodes $(k_0^*, \ldots, k_e^*)$, then the root contains the vector having 1 in the e–th position and 0's in all other positions.

In a more general context, we cross algebra and topology and notice that the relationships between the distribution of terminal nodes of the trees under considerations and the vectors contained in their roots relate trees to coverings of the Hamming space by the sets whose cardinalities coincide with the sizes of the Hamming balls. One of the conclusions that follows from these observations is that it is possible to establish an equivalence between the Kraft inequality for regular rooted binary trees having terminal nodes only at levels $0, \ldots, e$ and the Hamming bound on the cardinality of a block code having the minimal distance $2e+1$. Furthermore, application of this construction to the searching problem represents many of the known results as corollaries from the properties of the trees and leads to a new upper bound on the number of questions that differ from the known lower bound by $3e$. Namely, we describe the searching strategy that requires at most $n(M) + 3e$ questions, where $n(M)$ is the minimal integer satisfying the inequality $MV_e^{(n(M))} \leq 2^{n(M)}$.