Solution of Ulam's Searching Game with Three Lies or an Optimal Adaptive Strategy for Binary Three-Error-Correcting Codes

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Abstract

In this paper we determine the minimal number of yes-no queries that are needed to find an unknown integer between 1 and N, if at most three of the answers are lies. This strategy is also an optimal adaptive strategy for binary three-error-correcting codes.

Key words: Searching with Lies; Transmission with Feedback; Ulam's Game

1 Introduction

In 1976 Ulam [29] suggested an interesting two-person search game in his autobiography (pp. 281-282), which can be formalized as follows:

> Person 1 thinks of a number between one and one million. Person 2 is allowed to ask questions to which Person 1 is supposed to answer only yes or no. Person 2 asks for subsets of the set $\{1, \ldots, 1000000\}$. The difficulty is that Person 1 is allowed to lie l times. Now we want to know: How many questions does Person 2 have to ask in order to get the correct answer?

In [25] the game with a fixed fraction of lies was discussed first. The problem is solved for one lie in [24]. The solution for $l = 2$ and $|\mathcal{X}| = 10^6$ can be found in [11]. The solution for $|\mathcal{X}| = 2^m$ and $l = 2$ is presented in [9] and its generalization for arbitrary $|\mathcal{X}|$ is given in [14]. The case of three lies if $N = 1000000$ is solved in [20]. The case of three lies if $N = 2^k$ is solved in [21]. In [15] it is shown how to solve Ulam's game for $l = 1, 2, 3, 4$ and $N = 2^{20}$. The generalization for $N = 1000000$ and arbitrary l is given in [7].

In this paper we solve Ulam's game for $l = 3$ and arbitrary N. In Section 2 we present some properties of the volume, which is an important auxiliary function of the game. If N is small ($N \leq 265$), then we get the result by exhaustive computer search. In the case $N > 265$ we can give a special strategy. The main idea of the strategy is as follows. If N is big enough, then we can divide the game into three parts:

- The first three questions
- Typical questions
- The last 15 questions

In Section 3 we show how to ask the first three questions in an optimal way. It turns out that after these questions we can decide how many further questions are necessary for Person 2 to get the correct answer. There exists a strategy that forms the next question until there are 15 left. We call these typical and present them in Section 4. Unfortunately, we cannot give a similar strategy for the last 15 questions. In Section 5 we present an algorithm for the last 15 questions. This algorithm was developed by Guzicki [14] in order to solve Ulam's problem in the case of two lies. It is easy to prove that we can also use this algorithm if three lies are allowed. The main theorem of the paper is presented in Section 6 where we combine all results of the previous parts. Because of lack of space some proofs and some formulations of questions are omitted. They can be found in [12]. In each case all ideas and constructions can be found in this paper.

The important auxiliary function (the volume) was defined by Berlekamp [6], who considered an equivalent problem in 1964, the problem of transmitting messages over a noisy binary channel with noiseless feedback. In this model, the sender wants to transmit a message over a noisy binary channel. Let $\mathcal{X} = \{1, \ldots, N\}$ denote the set of possible messages and let $\mathcal{Y} = \{0, 1\}$ be the binary coding alphabet. We have a passive feedback, that is, the sender always knows what has been received. The codewords are elements of $\mathcal{Y}^n$, and a codeword has the following form: $(c_1(x, y^0), c_2(x, y^1) \ldots, c_n(x, y^{i-1}))$, where $c_i : \mathcal{X} \times \mathcal{Y}^{i-1} \to \mathcal{Y}$ is a function defined for the i-th code letter which depends on the message we want to transmit and on $i - 1$ bits that have been received before. We suppose that the noise does not change more than $l \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ bits of a codeword. One may consider each transmission as the following quiet-question-noisy-answer-game: The sender and receiver play this game against a "Devil". The sender chooses a message $x \in \mathcal{X}$. The sender and receiver have a common partition strategy depending on the received bits. Thus, the sender chooses a subset $S \subset \mathcal{X}$ by following the partition strategy. He sends either "1", if $x \in S$ or "0", if $x \notin S$ over a noisy channel. A new subset $S \subset \mathcal{X}$ is chosen depending on the previously received bits. The aim is that the receiver tries to get the message after n transmissions, and the "Devil" wants to avoid this by changing at most l bits.

Berlekamp [6] considered the asymptotic problem ($\frac{l}{n} \to p, n \to \infty$). Obviously, Berlekamp's quiet-question-noisy-answer-game is equivalent to Ulam's game. Many search problems, which are equivalent to a coding problem, can be found in [1] and [4]. A overview and some results concerning channels with noiseless feedback is given in [30]. In 1992 Spencer [27] introduced a game on a board, which is equivalent to Ulam's game. The authors give an algorithm based on Berlekamp's volume in [18]. Some other variants of Ulam's game can be found in [16], [8], [17], [22], [3], [28], [13], [26], [2], [10] and [23].

2 Definitions and notations

We use the notations of Berlekamp [6]. In each round of the game Person 2 receives a negative vote for a subset $T \subset \mathcal{X}$. If Person 2 gets more than l negative votes for a number, then this number cannot be the searched one because Person 1 is allowed to lie at most l times. Therefore, we consider the sets $S_j \triangleq \{x \in \mathcal{X} : \text{Person 2 gets } (l-j) \text{ negative votes for } x\}$.

Definition 1 *The vector $\underline{v} = (|S_l|, |S_{l-1}|, \ldots, |S_0|) = (v_l, v_{l-1}, \ldots, v_0)$ is referred to as a state (of the game). $\underline{v}$ is called a k-state if k questions are left.*

Neither the states nor the dividing questions depend on the specific number, which is chosen by Person 1. They depend only on the cardinality of the sets S_j.

Definition 2 *Let $\underline{s}$ be an arbitrary state. The question whether "$x \in S$" ($S \subset \mathcal{X}$) is introduced as a vector $[\underline{u}] = [u_l, \ldots, u_0]$, where $u_i \triangleq |S_i \cap S|$. The state $\underline{x}$ is reduced to the states $\underline{y}(\triangleq YES^{\underline{s}})$ and $\underline{z}(\triangleq NO^{\underline{s}})$ by the question $[\underline{u}]$ if there exists a $\underline{v} \geq 0$ such that $\underline{x} = \underline{u} + \underline{v} \triangleq (u_l + v_l, u_{l-1} + v_{l-1}, \ldots, u_0 + v_0)$, $y_i = u_i + v_{i+1}$, $z_i = v_i + u_{i+1}$.*

Definition 3 *Let $\underline{v}$ be an arbitrary state and let $[\underline{x}]$ be a question. The question is called legal if $0 \leq x_i \leq v_i$ for all $i = 0, \ldots, l$.*

Definition 4 *(1) A 0-state $\underline{x}$ is called winning if $\sum_{i=0}^{l} x_i \leq 1$. Otherwise, $\underline{x}$ is called losing.*
(2) A k-state $\underline{x}$ is called winning if it can be reduced to two winning $(k-1)$-states. Otherwise, $\underline{x}$ is called losing.
(3) A winning k-state is called borderline winning if it is a losing $(k-1)$-state.

Proposition 5 *(1) A winning n-state is also a winning k-state if $k > n$.*
(2) All borderline winning 1-states have the form $(0, \ldots, 0, 2)$.
(3) Let $\underline{x}$ be a winning k-state and let $\underline{y}$ be some state, where $y_i \leq x_i$ holds for all $i \leq l$. Then $\underline{y}$ is also a winning k-state.

Definition 6 *For a given state* $\underline{x}$ *the function* $V_n(\underline{x}) \triangleq \sum_{i=0}^{l} x_i \sum_{j=0}^{i} \binom{n}{j}$ *is called the n-th volume of* $\underline{x}$.

Theorem (Berlekamp's Conservation of Volume [6]) *Let* $\underline{x}$ *be a state which can be reduced to the states* $\underline{y}$ *and* $\underline{z}$. *Then* $V_n(\underline{x}) = V_{n-1}(\underline{y}) + V_{n-1}(\underline{z})$.

Theorem (Berlekamp's Volume Bound [6]) *Let* $\underline{x}$ *be a winning n-state. Then* $V_n(\underline{x}) \leq 2^n$.

We denote by $L_l(N)$ the minimal number of questions, which Person 2 needs to find the searched number by using an optimal strategy.

Corollary 7 (The Hamming Bound) *Let* $|\mathcal{X}| = N$ *and* $l \in \mathbb{N}$. *If* $L_l(N) = n$, *then* $N \leq \frac{2^n}{\sum_{j=0}^{l} \binom{n}{j}}$.

Definition 8 *Let* $\underline{s}$ *be an arbitrary state. The number*

$$ch(\underline{s}) \triangleq \min\{k : V_k(\underline{s}) \leq 2^k\}$$

is called character of $\underline{s}$.

Hereafter, we restrict our considerations to the case $l = 3$. We set $F, G : \mathbb{N} \to \mathbb{N}$ with $G(k) \triangleq \sum_{i=0}^{3} \binom{k}{i} = \frac{k^3+5k+6}{6}$, $F(k) \triangleq \sum_{i=0}^{2} \binom{k}{i} = \frac{k^2+k+2}{2}$. It is easy to see that $V_k(\underline{v}) = v_3 G(k) + v_2 F(k) + v_1(k+1) + v_0$.

3 The first three crucial questions

Let $\underline{v}$ be the state of the game after the first three questions. We show that the volume of $\underline{v}$ is crucial if N is big enough in Section 5. It turns out that we can solve the game with $ch(\underline{v})$ or $ch(\underline{v}) + 1$ questions by starting from the state $\underline{v}$. We construct an optimal strategy step by step and call it mini-strategy. We analyse the best queries and the worst answers. In every step of the game, the Questioner reduces the current state to the YES- and NO-states. The Responder chooses the worst state. The aim of the Questioner is to reduce the character by one in every step.

Let $\underline{v}$ be a state and $\underline{x}$ be the next question. Then the question $[\underline{z}] = [v_3 - x_3, v_2 - x_2, v_1 - x_1, v_0 - x_0]$ is called the complement of the question $\underline{x}$ because it exchanges the YES- and NO-states. Let $\underline{v}$ be a $k+1$-state.

Definition 9 *(1) Let* $\underline{s}$ *be a state and* $k \triangleq ch(\underline{s})$. *The state is called balanced if there exists a query* $[\underline{u}]$ *such that* $|V_{k-1}(YES^{\underline{s}}) - V_{k-1}(NO^{\underline{s}})| \leq 1$.

(2) The state $\underline{s}$ *is called nice if the Questioner wins in* $ch(\underline{s})$ *questions starting from this state.*

In the next three propositions we present the first three questions of the mini-strategy of the game.

Definition 10 *Let $\underline{s} = (N, 0, 0, 0)$ be an initial state and $k = ch(\underline{s})$. We denote the first question by $[\underline{x}]$, the state after the first question by $\underline{t}$, the second question by $[\underline{y}]$, the state after the second question by $\underline{u}$, the third question by $[\underline{z}]$ and the state after the third question by $\underline{v}$.*

(1) The function $a_1(N) \triangleq \frac{\min|V_{k-1}(YES^{\underline{s}})-V_{k-1}(NO^{\underline{s}})|}{\binom{k-1}{3}}$ is called loss in the first question.
(2) The function $a_2(N, \underline{t}) \triangleq \frac{\min|V_{k-2}(YES^{\underline{t}})-V_{k-2}(NO^{\underline{t}})|}{\binom{k-2}{2}}$ is called loss in the second question.
(3) The function $a_3(N, \underline{u}) \triangleq \frac{\min|V_{k-3}(YES^{\underline{u}})-V_{k-3}(NO^{\underline{u}})|}{k-3}$ is called loss in the third question.

In all functions we take the minimum over all legal questions. A question is called locally optimal if it assumes the minimum of the loss function.

Now we will present constructions of the first three questions so that they are locally optimal. In the end it turns out that these questions are optimal as well. We write e for an even component of a state and o for an odd one.

Proposition 11 *Let $\underline{s} = (s_3, 0, 0, 0)$ be a k-state. The following questions are legal and locally optimal:*

(1) $[\frac{s_3}{2}, 0, 0, 0]$, if s_3 is even,
(2) $[\frac{s_3+1}{2}, 0, 0, 0]$, if s_3 is odd.

Furthermore, $a_1(N) = \begin{cases} 0 \text{ , if } N \text{ is even,} \\ 1 \text{ , otherwise.} \end{cases}$.

PROOF. Let us denote a query by $[\underline{x}]$. We have to minimize $|V_{k_1}(YES) - V_{k_1}(NO)|$ that can be expressed by $|(2x_3 - s_3)\binom{k_1}{3}|$. Obviously, $x_3 = \frac{s_3}{2}$, if s_3 is even and $x_3 = \frac{s_3+1}{2}$, if s_3 is odd. □

We will use the questions stated in Proposition 11 as the first question of the mini-strategy. Depending on N mod 4 we get the four different new states: $(e, e, 0, 0), (e, o, 0, 0), (o, e, 0, 0)$ and $(o, o, 0, 0)$. We have to minimize $|V_{k-2}(YES) - V_{k-2}(NO)|$ that can be expressed by $|\binom{k-2}{2}((2y_3 - t_3)\frac{k-4}{3} + 2y_2 - t_2)|$ in order to get locally optimal questions. We will construct them as follows. We assume that k mod 6, t_3 mod 2 and t_2 mod 2 are given. Now it is possible to calculate $\min_{y_3,y_2\in\mathbb{N}} |(2y_3 - t_3)\frac{k-4}{3} + 2y_2 - t_2)|$. We set $y_3 = \lceil \frac{t_3}{2} \rceil$. If a $y_2 \in \mathbb{N}$ exists such that the minimum is assumed, we set $[\underline{y}] = [y_3, y_2, 0, 0]$ (*). Otherwise let increase y_3 by 2 and so on. The result of this calculation can be found in [12].

Proposition 12 *Let $\underline{s} = (N, 0, 0, 0)$ be an initial state, $k = ch(\underline{s})$, $[\underline{x}]$ be the first question (stated in Proposition 11) and $\underline{t} = (t_3, t_2, 0, 0)$ be a resulting state. If $t_3 \geq 3$ and $t_2 \geq k - 4$, the constructed questions $[\underline{y}]$ (c.f. *) are locally optimal and legal. Furthermore, $a_2(N, \underline{t}) \in \{0, \frac{1}{3}, 1\}$ and the values of the function can be found in Table 1.*

PROOF. It is easy to see that this proposition holds, if we calculate all questions. This is done in [12]. □

We will use the question $[\underline{y}]$ as the second question of the mini-strategy. Depending on N mod 8 we get the eight different new states. We have to minimize $|V_{k-3}(YES) - V_{k-3}(NO)|$ that can be expressed by $|(k-3)(\frac{k-4}{2}((2z_3 - u_3)\frac{k-5}{3} + 2z_2 - u_2) + 2z_1 - u_1)|$ in order to get locally optimal questions. We will construct them as follows. We assume that k mod 12, u_3 mod 2, u_2 mod 2 and u_3 mod 2 are given. Now it is possible to calculate $\min_{z_3,z_2,z_1 \in \mathbb{N}} |(\frac{k-4}{2}((2z_3 - u_3)\frac{k-5}{3} + 2z_2 - u_2) + 2z_1 - u_1)|$. We set $z_3 = \lceil \frac{t_3}{2} \rceil$. If $z_2, z_1 \in \mathbb{N}$ exist such that the minimum is assumed, then we set $z_2 = \left\lceil \frac{3u_2 - (2z_3 - u_3)(k-5)}{6} \right\rceil$. If $z_1 \in \mathbb{N}$ exists such that the minimum is assumed, we set $[\underline{z}] = [z_3, z_2, z_1, 0]$ (**). Otherwise we increase z_2 by 2 etc. If such $z_1, z_2 \in \mathbb{N}$ do not exist we increase z_3 by 2 etc. The result of this calculation can be found in [12].

Proposition 13 *Let $\underline{s} = (N, 0, 0, 0)$ be an initial state, $k = ch(\underline{s})$, $[\underline{x}]$ be the first question, $\underline{y}$ be the second question and $\underline{u} = (u_3, u_2, u_1, 0)$ be a resulting state. If $u_3 \geq 4$, $u_2 \geq k - 3$ and $u_2 \geq k - 3$, the constructed questions $[\underline{z}]$ (c.f. **) are locally optimal and legal. Furthermore, $a_3(N, \underline{u}) \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1\}$ and the values of the function can be found in Table 2.*

PROOF. It is easy to see that this proposition holds, if we calculate all questions. This is done in [12]. □

We will use the question $[\underline{z}]$ as the third question of the mini-strategy. Let $\underline{s}$ be the initial state and let $k = ch(\underline{s})$. If we calculate the first three questions of the mini-strategy (and its complement) it turns out that

$$\frac{N-1}{2} \leq x_3 \leq \frac{N+1}{2},$$

$$\frac{t_3-3}{2} \le y_3 \le \frac{t_3+3}{2},$$
$$\frac{t_2-k+4}{2} \le y_2 \le \frac{t_2+k-4}{2},$$
$$\frac{u_3-3}{2} \le z_3 \le \frac{u_3+3}{2},$$
$$\frac{u_2-k+3}{2} \le z_2 \le \frac{u_2+k-3}{2},$$
$$\frac{u_1-k+4}{2} \le z_1 \le \frac{u_1+k-4}{2}.$$

Definition 14 *The first three questions are called optimal if they can be used to get a minimal $(k-3)$-th volume of the resulting state.*

Proposition 15 *Let $\underline{s} = (N, 0, 0, 0)$ be a state and let $k = ch(\underline{s})$. If $N > 265$ and we follow the mini-strategy, then the assumptions of Proposition 11, Proposition 12 and Proposition 13 are satisfied.*

PROOF. If $N > 265$, then $ch(\underline{s}) > 18$. First we show:

$$\text{If } k > 18, \text{ then } N \ge 6k - 7. \tag{1}$$

We assume that $k > 18$, but $N < 6k-7$. Then $NG(k) \le 2^k$ and $NG(k-1) > 2^{k-1}$. However, $NG(k-1) < (6k-7)G(k-1) \le 2^{k-1}$, if $k > 18$.

(1) We show that the assumptions of Proposition 12 are satisfied if $k > 18$. Let $\underline{t}$ be the state after the first question. After this question, $t_3 \ge \frac{N-1}{2}$ and $t_2 \ge \frac{N-1}{2}$. It follows that $t_3 \ge 132$ and $t_2 \ge \frac{N-1}{2} \ge \frac{6k-8}{2} \ge 3k-4 > k-4$.

(2) We show that the assumptions of Proposition 13 are satisfied if $k > 18$. Let $\underline{u}$ be the state after the second question. After this question $u_3 \ge \frac{N-7}{4}$, $u_2 \ge \frac{N-k+3}{2}$ and $u_1 \ge \frac{N-2k+1}{4}$. It follows that $u_3 \ge 65$ because $k > 18$, $u_2 \ge \frac{N-k}{2} \ge \frac{5k-7}{2} > k-3$ and $u_1 \ge \frac{N-2k+7}{4} \ge \frac{4k}{2} \ge 2k > k-4$. □

Lemma 16 *Let $\underline{s} = (s_3, 0, 0, 0)$, $\underline{t} = (t_3, t_2, 0)$ and $\underline{u} = (u_3, u_2, u_1, 0)$ be states, $l \in \mathbb{N}$.*

(1) There exists a question which reduces $\underline{s}$ to $\underline{t}$ such that

$$V_l(\underline{s}) \ge 2V_{l-1}(\underline{t}) - \binom{l-1}{3}.$$

(2) There exists a question which reduces $\underline{t}$ to $\underline{u}$ such that

$$V_l(\underline{t}) \ge 2V_{l-1}(\underline{u}) - \binom{l-1}{2}.$$

(3) There exists a question which reduces $\underline{u}$ to $\underline{v}$ such that

$$V_l(\underline{u}) \geq 2V_{l-1}(\underline{v}) - (l-1).$$

PROOF.

(1) Choose the question $[\left\lceil \frac{s}{2} \right\rceil, 0, 0, 0]$. The proof follows by Berlekamp's Conservation of Volume.
(2) Choose the question $[\left\lceil \frac{t}{2} \right\rceil, \left\lfloor \frac{t}{2} \right\rfloor, 0, 0]$. The proof follows by Berlekamp's Conservation of Volume.
(3) Choose the question $[\left\lceil \frac{u}{2} \right\rceil, \left\lfloor \frac{u}{2} \right\rfloor, \left\lfloor \frac{u}{2} \right\rfloor, 0]$. The proof follows by Berlekamp's Conservation of Volume.

Proposition 17 *Let $\underline{s} = (s_3, 0, 0, 0)$ be a state with $ch(\underline{s}) = k \geq 17$ and let $\underline{v}$ be the state after the third question. There exists a strategy such that $ch(\underline{v}) \leq k-2$.*

PROOF. Let $k \geq 17$. Because of Lemma 16 and Berlekamp's Conservation of Volume, we have $V_{k-2}(\underline{v}) \leq \frac{\frac{\frac{V_{k+1}(\underline{s})+\binom{k}{3}}{2}+\binom{k-1}{2}}{2}+k-2}{2}$. It holds that $V_{k+1}(\underline{s}) = V_k(\underline{s}) + s_3 \sum_{i=0}^{2} \binom{k}{i} \leq 2^k + \frac{2^k \sum_{i=0}^{2}\binom{k}{i}}{\sum_{i=0}^{3}\binom{k}{i}}$. Thus, $V_{k-2}(\underline{s}) \leq 2^{k-3} + \frac{2^{k-3}\sum_{i=0}^{2}\binom{k}{i}}{\sum_{i=0}^{3}\binom{k}{i}} + \frac{\binom{k}{3}}{8} + \frac{\binom{k-1}{2}}{4} + \frac{k-2}{2}$ and $\frac{2^{k-3}\sum_{i=0}^{2}\binom{k}{i}}{\sum_{i=0}^{3}\binom{k}{i}} + \frac{\binom{k}{3}}{8} + \frac{\binom{k-1}{2}}{4} + \frac{k-2}{2} \leq 2^{k-3}$ if $k \geq 17$. We get $V_{k-2}(\underline{v}) \leq 2^{k-2}$. $\square$

Proposition 18 *Let $\underline{s} = (s_3, 0, 0, 0)$ be a state and $ch(\underline{s}) = k \geq 7$. Let Person 2 use the mini-strategy. The best counter-strategy for Person 1 is to choose the maximal value in every step.*

Proposition 19 *Let $\underline{s} = (s_3, 0, 0, 0)$ be a state with $ch(\underline{s}) = k \geq 7$. Person 2 gets the smallest possible character after the third question, if he uses local optimal questions.*

PROOF. Let $\underline{v}$ be the state of the game after the third question and $e = ch(\underline{v})$. If $e = k-3$, then Person 2 cannot do better. Thus we assume that Person 2 uses local optimal questions and $e \geq k-2$. By Proposition 11, Table 1 and 2, we know how much we lose in each question. Let $\underline{v}$, $\underline{u}$, $\underline{t}$, $\underline{s}$ and k, be the worst state after the first three questions, the worst state after the first two questions, the worst state after the first question, the initial state, and the character of $\underline{s}$, , respectively. We show that there is no other strategy such that $V_{k-3}(\underline{v})$ is smaller by using local optimal questions. We set $a_1 = |V_{k-1}(YES^{\underline{s}} - V_{k-1}(NO^{\underline{s}})|$, $a_2 = |V_{k-2}(YES^{\underline{t}} - V_{k-2}(NO^{\underline{t}})|$ and

$a_3 = |V_{k-3}(YES^{\underline{u}} - V_{k-3}(NO^{\underline{u}})|$. By Berlekamp's Conservation of Volume we get $V_{k-3}(\underline{v}) = \frac{V_k(\underline{s})}{8} + \underbrace{\frac{a_1\binom{k-1}{3}}{8} + \frac{a_2\binom{k-2}{2}}{4} + \frac{a_3(k-3)}{2}}_{=R}$.

(1) We prove: In order to get a minimal character of the state after the third question it is an optimal strategy for Person 2 to minimize $|V_{k-1}(YES) - V_{k-1}(NO)|$ in the first step.
- s_3 is even: In this case $R \leq \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$, if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $R \geq \frac{\binom{k-1}{3}}{4}$. But $\frac{\binom{k-1}{3}}{4} > \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$ if $k > 5$.
- s_3 is odd: In this case $R \leq \frac{\binom{k-1}{3}}{8} + \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$, if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $R \geq \frac{3\binom{k-1}{3}}{8}$. But $\frac{\binom{k-1}{3}}{4} > \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$ if $k > 5$.

Therefore, we know that we have to choose a local optimal question. Let $S = \frac{a_2\binom{k-2}{2}}{4} + \frac{a_3(k-3)}{2}$.

(2) We prove: In order to get a minimal character of the state after the third question it is an optimal strategy for Person 2 to minimize $|V_{k-2}(YES) - V_{k-2}(NO)|$ in the second step. Let $D = (2y_3 - t_3)\frac{k-4}{3} + 2y_2 - t_2$.
- D is an odd number: In this case $S \leq \frac{\binom{k-2}{2}}{4} + \frac{k-3}{2}$, if Person 2 asks a local optimal question. If he does not ask such a question in the second step, then Person 1 can choose the answers in such a way that $S \geq \frac{3\binom{k-2}{2}}{4}$. But $\binom{k-2}{2} \geq k-3$, if $k > 2$.
- D is an even number: In this case $S \leq \frac{k-3}{2}$, if Person 2 asks a local optimal question. If he does not ask such a question in the second step, then Person 1 can choose the answers in such a way that $S \geq \frac{\binom{k-2}{2}}{2}$. But $\binom{k-2}{2} \geq k-3$ if $k > 2$.
- $D = \frac{z}{3}$, where z is an odd number which is not divisible by 3: In this case $S \leq \frac{\binom{k-2}{2}}{12} + \frac{k-3}{2}$, if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $S \geq \frac{\binom{k-2}{2}}{4}$. But $\frac{\binom{k-2}{2}}{4} > \frac{\binom{k-2}{2}}{12} + \frac{k-3}{2}$, if $k > 6$.
- $D = \frac{z}{3}$, where z is an even number not divisible by 3: In this case $S \leq \frac{k-3}{2}$ if Person 2 asks a local optimal question. If he does not ask such a question in the first step, then Person 1 can choose the answers in such a way that $S \geq \frac{\binom{k-2}{2}}{3}$. But $\frac{\binom{k-2}{2}}{3} > \frac{k-3}{3}$ if $k > 6$.

(3) We prove: In order to get a minimal character of the state after the third

question it is an optimal strategy for Person 2 to minimize $|V_{k-3}(YES) - V_{k-3}(NO)|$ in the third step. This is obvious.

□

4 Typical questions

Definition 20 *Let $\underline{v} = (v_3, v_2, v_1, v_0)$ be a state with $ch(\underline{v}) = k$. The state is called typical if*

(1) $3v_3 < v_2 + \frac{3}{2}k$,
(2) $v_2 < v_1 + \frac{15}{8}k$,
(3) $v_0 > k$.

Proposition 21 *Let $k = ch((N, 0, 0, 0)) > 18$ and let Person 2 follow the mini-strategy. Then the state is typical after the first three questions.*

PROOF. Let $(N, 0, 0, 0)$, $(t_3, t_2, 0, 0)$, $(u_3, u_2, u_1, 0)$ and (v_3, v_2, v_1, v_0) be the states before the first, second, third and fourth question respectively, when the mini-strategy is used. Let $[\underline{x}]$, $[\underline{y}]$ and $[\underline{z}]$ be the first three questions and $c = ch(\underline{v})$. Since $c + 2 \leq k \leq c + 3$, we have $c > 15$.

(1) It is easy to show that the maximal value for v_3 and the minimal value for v_2 satisfy $3v_3 < v_2 + \frac{3}{2}c$ if $c > 11$.
(2) It is easy to show that the maximal value for v_2 and the minimal value for v_1 satisfy $v_2 \leq v_1 + \frac{15}{8}k$ if $c > 12$.
(3) It is easy to show that the minimal value for v_0 is greater than k if $c > 18$.

□

Proposition 22 *Let $\underline{v}$ be a typical state and $ch(\underline{v}) = k + 1 \geq 8$. Then the state is balanced and can be reduced to two typical states if $v_0 > \binom{k}{3} + 2k$.*

PROOF. We have to consider eight cases. We show that we can always find a question $[\underline{x}]$ which reduces the state $\underline{v}$ to two typical states: $YES^{\underline{v}} = \underline{y} = (x_3, x_2 + v_3 - x_3, x_1 + v_2 - x_2, x_0 + v_1 - x_1)$ and $NO^{\underline{v}} = \underline{n} = (v_3 - x_3, v_2 - x_2 + x_3, v_1 - x_1 + x_2, v_0 - x_0 + x_1)$. We use the fact that $\underline{v}$ is typical. This means that we use the inequalities: $3v_3 < v_2 + \frac{3}{2}(k + 1)$, $v_2 < v_1 + 2(k + 1)$, $v_0 > \binom{k}{3} + 2k$, $v_0 > 6$. In all these cases, it is easy to verify that all questions are legal. It holds that $ch(YES^{\underline{v}}) = ch(NO^{\underline{v}}) = k$. The proof that the states are typical can be found in [12] in detail.

(1) v_3 is even, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil$.

(2) v_3 is odd, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{3}}{2} \right\rfloor$.

(3) v_3 is even, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{2}}{2} \right\rfloor$.

(4) v_3 is odd, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}}{2} \right\rfloor$.

(5) v_3 is even, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1+1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{k}{2} \right\rfloor$.

(6) v_3 is odd, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1-1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{3}-k}{2} \right\rfloor$.

(7) v_3 is even, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1-1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{2}-k}{2} \right\rfloor$.

(8) v_3 is odd, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1-1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}-k}{2} \right\rfloor$. $\square$

Proposition 23 *Let $\underline{v}$ be a typical state and $ch(\underline{v}) = k+1 \geq 8$. Then the state is balanced and can be reduced to two typical states if $v_1 > \binom{k}{2} + 2k$.*

PROOF. We have to consider eight cases. We show that we can always find a question $[\underline{x}]$ which reduces the state $\underline{v}$ to two typical states: $YES^{\underline{v}} = \underline{y} = (x_3, x_2 + v_3 - x_3, x_1 + v_2 - x_2, x_0 + v_1 - x_1)$ and $NO^{\underline{v}} = \underline{n} = (v_3 - x_3, v_2 - x_2 + x_3, v_1 - x_1 + x_2, v_0 - x_0 + x_1)$. In all these cases, it is easy to verify that all questions are legal. It holds that $ch(YES^{\underline{v}}) = ch(NO^{\underline{v}}) = k$. The proof that the states are typical can be found in [12] in detail.

(1) v_3 is even, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil$.

(2) v_3 is odd, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{(k-1)(k-2)}{12} \right\rfloor$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{3}+(2x_1-v_1)k}{2} \right\rfloor$.

(3) v_3 is even, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1}{2} - \left\lceil \frac{(k-1)}{4} \right\rceil$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(4) v_3 is odd, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1}{2} - \left\lceil \frac{\binom{k}{3}-\binom{k}{2}}{2k} \right\rceil$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(5) v_3 is even, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1+1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{k}{2} \right\rfloor$.

(6) v_3 is odd, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lceil \frac{\binom{k}{3}-k}{2k} \right\rceil$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{3}+(2x_1-v_1)k}{2} \right\rfloor$.

(7) v_3 is even, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lceil \frac{\binom{k}{2}-k}{2k} \right\rceil$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(8) v_3 is odd, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lceil \frac{\binom{k}{3}-\binom{k}{2}-k}{2k} \right\rceil$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil - \left\lfloor \frac{\binom{k}{3}-\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

□

Proposition 24 *Let $\underline{v}$ be a typical state and $ch(\underline{v}) = k+1 \geq 8$. Then the state is balanced and can be reduced to two typical states if $v_2 > 4k$.*

PROOF. We have to consider eight cases. We show that we can always find a question $[\underline{x}]$ which reduces the state $\underline{v}$ to two typical states: $YES^{\underline{v}} = \underline{y} = (x_3, x_2+v_3-x_3, x_1+v_2-x_2, x_0+v_1-x_1)$ and $NO^{\underline{v}} = \underline{n} = (v_3-x_3, v_2-x_2+x_3, v_1-x_1+x_2, v_0-x_0+x_1)$. If $v_2 > 4k$, then $v_1 > 2k$. In all these cases, it is easy to verify that all questions are legal. It holds that $ch(YES^{\underline{v}}) = ch(NO^{\underline{v}}) = k$.

(1) v_3 is even, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil$.

(2) v_3 is odd, v_2 is even and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2} - \left\lfloor \frac{k-2}{6} \right\rfloor$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}}{2k} \right\rfloor$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(3) v_3 is even, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{\binom{k}{2}}{2k} \right\rfloor$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(4) v_3 is odd, v_2 is odd and v_1 is even. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2} - \left\lfloor \frac{k-5}{6} \right\rfloor$, $x_1 = \frac{v_1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}}{2k} \right\rfloor$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(5) v_3 is even, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2}{2}$, $x_1 = \frac{v_1+1}{2}$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor$.

(6) v_3 is odd, v_2 is even and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2}{2} - \left\lfloor \frac{k-2}{6} \right\rfloor$, $x_1 = \frac{v_1-1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}-k}{2k} \right\rfloor$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(7) v_3 is even, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3}{2}$, $x_2 = \frac{v_2+1}{2}$, $x_1 = \frac{v_1-1}{2} - \left\lfloor \frac{\binom{k}{2}-k}{2k} \right\rfloor$ and $x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor$.

(8) v_3 is odd, v_2 is odd and v_1 is odd. We set $x_3 = \frac{v_3+1}{2}$, $x_2 = \frac{v_2-1}{2} - \left\lfloor \frac{k-5}{6} \right\rfloor$,

$$x_1 = \frac{v_1-1}{2} - \left\lfloor \frac{\binom{k}{3}+(2x_1-v_1)\binom{k}{2}-k}{2k} \right\rfloor \text{ and } x_0 = \left\lceil \frac{v_0}{2} \right\rceil + \left\lfloor \frac{\binom{k}{3}+(2x_2-v_2)\binom{k}{2}+(2x_1-v_1)k}{2} \right\rfloor.$$

□

Proposition 25 *Let $\underline{v}$ be a typical state with $ch(\underline{v}) = k+1 > 15$. Then $v_2 > 4k$ or $v_1 > \binom{k}{2} + 2k$ or $v_0 > \binom{k}{3} + 2k$.*

PROOF. We assume the contrary. Let $v_2 \leq 4k$, $v_1 \leq \binom{k}{2} + 2k$ and $v_0 \leq \binom{k}{3} + 2k$. It holds that $3v_3 < v_2 + \frac{3}{2}k$, thus $v_3 < 2k$. Therefore $V_k(\underline{v}) = v_3G(k)+v_2F(k)+v_1(k+1)+v_0 < 2kG(k)+4kF(k)+(\binom{k}{2}+2k)(k+1)+\binom{k}{3}+2k = \frac{k}{6}(2k^3+16k^2+31k+59)$. If $k > 15$, then $V_k(\underline{v}) < \frac{k}{6}(2k^3+16k^2+31k+59) < 2^k$. This is a contradiction to the hypothesis that $ch(\underline{v}) = k+1$. □

5 The last questions

In this section we show that all typical states with a character greater than or equal 15 are nice. We know from the previous section that we can reduce a typical state to two balanced typical states if its character is greater than 15. Thus we have to show that all typical states with character 15 are nice (starting from this state Person 2 needs 15 questions to win the game). In order to show this we use Guzicki's Algorithm. This algorithm, originally described for the case of two lies, can be generalized about the case of three lies without difficulty.

Lemma 26 *Let $\underline{v}$ be a k-winning state, $ch(\underline{v}) \geq 1$ and let $[\underline{x}]$ be the first question of the k-winning strategy that is chosen in such a way that $V_{k-1}(YES^{\underline{v}}) \geq V_{k-1}(NO^{\underline{v}})$.*

(1) If $V_k(\underline{v}) < 2^k$, then $\underline{v}' = (v_3, v_2, v_1, v_0+1)$ is also a k-winning state and there exists a k-winning strategy with the first question $[\underline{x}]$.
(2) If $V_k(\underline{v}) = 2^k$, then $\underline{v}' = (v_3, v_2, v_1, v_0+1)$ is a $(k+1)$-winning state and there exists a $(k+1)$-winning strategy with the first question $[\underline{x}]$.

PROOF. We prove this Lemma by induction in k. □

Corollary 27 *Let $\underline{v}$ be a nice state. Then $\underline{v} = (v_3, v_2, v_1, v_0+1)$ is also a nice state.*

PROOF. If $ch(\underline{v}) = 0$ the statement can be easily checked. In the other cases it follows directly by Lemma 26. $\square$

Definition 28 *Let $v_3, v_2, v_1 \in \mathbb{N}$. Then*

$$M(v_3, v_2, v_1) \triangleq \min\{v_0 : \text{ the state } (v_3, v_2, v_1, v_0) \text{ is nice}\}.$$

Lemma 29 *Let $M(v_3, v_2, v_1) = c$, $ch((v_3.v_2, v_1, c)) = k$ and $c \neq 0$, then*

$$V_{k-1}((v_3, v_2, v_1, c)) = 2^{k-1} + 1.$$

PROOF. Let $M(v_3, v_2, v_1) = c$, $ch((v_3.v_2, v_1, c)) = k$ and $c \neq 0$. We assume that $V_{k-1}(v_3, v_2, v_1, c) > 2^{k-1} + 1$. Therefore $V_{k-1}(v_3, v_2, v_1, c-1) > 2^{k-1}$ and $ch(v_3, v_2, v_1, c-1) = k$. Thus the state $(v_3, v_2, v_1, c-1)$ is nice. This contradicts the equation $M(v_3, v_2, v_1) = c$. $\square$

Definition 30 *We set $C(v_3, v_2, v_1, x_3, x_2, x_1) \triangleq \min\{c :$ there exists an integer z and a question $[x_3, x_2, x_1, z]$ which reduces the state (v_3, v_2, v_1, c) to two $(k-1)$-winning states $(k = ch(v_3, v_2, v_1, c))\}$. If $(x_3, x_2, x_1) \in \{(0, 0, 0), (v_3, v_2, v_1)\}$ we set $C(v_3, v_2, v_1, x_3, x_2, x_1) = \infty$.*

Calculation of $M(v_3, v_2, v_1)$: Set $M(0, 0, 0) = 1$, $M(0, 0, 1) = 0$, $M(0, 1, 0) = 0$, $M(1, 0, 0) = 0$, $M(v_3, v_2, v_1) = \min\{C(v_3, v_2, v_1, x_3, x_2, x_1) : 0 \leq x_3 \leq \left\lceil \frac{v_3}{2} \right\rceil, 0 \leq x_2 \leq v_2,\ 0 \leq x_1 \leq v_1$.

Because of the complement of the questions, it is sufficient to restrict x_3 to $\leq \left\lceil \frac{v_3}{2} \right\rceil$, .

Calculation of $C(v_3, v_2, v_1, x_3, x_2, x_1)$: Consider the state $\underline{v} = (v_3, v_2, v_1, 0)$ and the question $[x_3, x_2, x_1, 0]$, $YES^{\underline{v}} = (y_3, y_2, y_1, y_0)$, $NO^{\underline{v}} = (n_3, n_2, n_1, n_0)$, $c_1 = M(y_3, y_2, y_1)$, $c_2 = M(n_3, n_2, n_1)$, $k_1 = ch(YES^{\underline{v}})$, $k_2 = ch(NO^{\underline{v}})$, $k_3 = ch(y_3, y_2, y_1, c_1)$, $k_4 = ch(n_3, n_2, n_1, c_2)$.

Proposition 31 *Let $k = \max\{k_1, k_2, k_3, k_4\}$. Then $C(v_3, v_2, v_1, x_3, x_2, x_1) = \min\{c : ch(v_3, v_2, v_1, c) > k\}$, if $(x_3, x_2, x_1) \notin \{(0, 0, 0), (v_3, v_2, v_1)\}$.*

PROOF. Let $C(v_3, v_2, v_1, x_3, x_2, x_1) = c$. First we show that $ch(v_3, v_2, v_1, c) > k$. Suppose that $ch(v_3, v_2, v_1, c) \leq k$. Then (see the definition of C) there exists a question $[(x_3, x_2, x_1, z)]$ that reduces (v_3, v_2, v_1, c) to the $(k-1)$-winning states $YES' = (y_3, y_2, y_1, y_0 + z)$ and $NO' = (n_3, n_2, n_1, n_0 + c - z)$, where $y_3, y_2, y_1, y_0, n_3, n_2, n_1, n_0$ are defined above.

- Case 1: $k = k_1$. In this case $ch(YES^{\underline{v}}) = k$. It follows that $V_{k-1}(YES') \geq V_{k-1}(YES^{\underline{v}}) > 2^{k-1}$. This is a contradiction.

- Case 2: $k = k_2$. In this case $ch(NO^{\underline{v}}) = k$. It follows that $V_{k-1}(NO') \geq V_{k-1}(NO^{\underline{v}}) > 2^{k-1}$. This is a contradiction.
- Case 3: $k = k_3$. In this case $ch(y_3, y_2, y_1, c_1) = k$, (y_3, y_2, y_1, c_1) is nice because of the definition of M and for every $c' < c_1$ it holds that (y_3, y_2, y_1, c') is not nice. It follows from Lemma 29 that we can distinguish between two sub-cases:

(1) Case $c_1 > 0$. In this sub-case $V_{k-1}(y_3, y_2, y_1, c_1 - 1) = 2^{k-1}$. $(y_3, y_2, y_1, 0)$ is $(k-1)$-winning and from Lemma 26 it follows that $(y_3, y_2, y_1, c_1 - 1)$ is nice. This is a contradiction to c_1 is minimal.

(2) Case $c_1 = 0$. In this sub-case $V_{k-1}(y_3, y_2, y_1, 0) > 2^{k-1}$. YES' is not $(k-1)$-winning. This is a contradiction.

- Case 4: $k = k_4$. In this case $ch(y_3, y_2, y_1, c_2) = k$, (n_3, n_2, n_1, c_2) is nice and for every $c' < c_2$ it holds that (n_3, n_2, n_1, c') is not nice. It follows from Lemma 29 that we can distinguish between two sub-cases:

(1) Case $c_2 > 0$. In this sub-case we get $V_{k-1}(n_3, n_2, n_1, c_2 - 1) = 2^{k-1}$. $(n_3, n_2, n_1, 0)$ is $(k-1)$-winning and from Lemma 26 it follows that $(n_3, n_2, n_1, c_2 - 1)$ is nice. This is a contradiction to c_2 is minimal.

(2) Case $c_2 = 0$. In this sub-case $V_{k-1}(n_3, n_2, n_1, c_2) > 2^{k-1}$. Thus NO' is not $(k-1)$-winning. This is a contradiction.

Now we have proved that $C(v_3, v_2, v_1, x_3, x_2, x_1) \geq \min\{c : ch(v_3, v_2, v_1, c) > k\}$. Let $c = \min\{c' : ch(v_3, v_2, v_1, c') > k\}$. We show that the question $[x_3, x_2, x_1, 0]$ reduces (v_3, v_2, v_1, c) to two k-winning states.

- Case $c = 0$: Because of the definition of k_1 and k_2 we get $ch(YES^{\underline{v}}) \leq k$ and $ch(NO^{\underline{v}}) \leq k$. Let us consider the $YES^{\underline{v}}$ state. If $y_0 \geq c_1$, then $YES^{\underline{v}}$ is k-winning due to the definition of function M. If $y_0 < c_1$, then $k_1 \leq k_3 \leq k$. Thus the state is k-winning. Let us consider the $NO^{\underline{v}}$ state. If $n_0 \geq c_2$, then NO is k-winning because of the definition of function M. If $n_0 < c_2$, then $k_2 \leq k_4 \leq k$. Thus the state is k-winning.
- Case $c > 0$: In this case $V_k(v_3, v_2, v_1, c) > 2^k$ and $V_k(v_3, v_2, v_1, c - 1) \leq 2^k$ because c is minimal. Therefore we get $V_k(v_3, v_2, v_1, c) = 2^k + 1$ and $V_k(v_3, v_2, v_1, 0) \leq V_k(v_3, v_2, v_1, c-1) = 2^k$. Let $[x_3, x_2, x_1, 0]$ be the question, then $YES' = YES^{\underline{v}} = (y_3, y_2, y_1, y_0)$ and $NO' = NO^{\underline{v}} = (n_3, n_2, n_1, n_0 + c)$. Obviously $ch(YES') = ch(YES^{\underline{v}}) \leq k$. In order to show that $ch(NO') \leq k$, it suffices to show that $V_k(NO') < V_k(v_3, v_2, v_1, c)$. This condition is satisfied because $x_1(k+1) \geq x_1$, $x_2F(k) \geq x_2(k+1)$ and $x_3G(k) \geq x_3F(k)$, if $k > 0$ and $(x_3, x_2, x_1) \neq (0, 0, 0)$ (i.e. one of the three terms is greater). It remains to show that the states YES' and NO' are $k - winning$. YES' is k-winning because $YES^{\underline{v}}$ is k-winning. Let us consider NO'. If $n_0 + c \geq c_2$, then it is k-winning by the definition of function M. If $n_0 + c < c_2$, then it is k_4-winning and thus k-winning.

□

Now we can write a computer program and by using Proposition 25, we obtain the following result:

Proposition 32 *Let $\underline{v}$ be a typical state and $k = ch(\underline{v})$, $\underline{v}$ is nice if $k \geq 15$.*

PROOF. Let $\underline{v} = (v_3, v_2, v_1, v_0)$ be a typical state with $k = ch(\underline{v}) \geq 15$. By exhaustive computer search (Guzicki's Algorithm) we get $M(v_3, v_2, v_1) \leq v_0$. $\square$

Proposition 33 *Let $\underline{v}$ be a state. If $ch(\underline{v}) \geq 15$, Person 2 needs at most $ch(\underline{v}) + 1$ questions to win the game.*

PROOF. This follows by Proposition 32 and Proposition 17. $\square$

Guzicki's Algorithm gives an optimal winning strategy for each state as well. Let (v_3, v_2, v_1, v_0) be a state, $k = ch(v_3, v_2, v_1, v_0)$, $v_0' = M(v_3, v_2, v_1)$ and $k' = ch(v_3, v_2, v_1, v_0')$. If $v_0 \leq v_0'$, then (v_3, v_2, v_1, v_0) is a k'-winning state. We can reduce it to two $(k'-1)$-winning states by the question $[x_3, x_2, x_1, 0]$, where x_3, x_2, x_1 minimizes the function C. If $v_0 > v_0'$, then (v_3, v_2, v_1, v_0) is a k-winning state. The question $[x_3, x_2, x_1, x_0]$, which reduces the state to two $(k-1)$-winning states, can be calculated by Lemma 26.

6 The main result

Let us now consider all given properties. If $N > 265$, then the character of the initial state is greater than 18. In this case we can use the optimal mini-strategy for the first three questions and get a nice state. The result is given in Theorem 40. If $N \leq 265$, we need some more considerations.

Lemma 34 *Let $M(v_3, v_2, v_1) \neq 0$ and $k = ch(v_3, v_2, v_1, 0)$. Then there exists a $j > 0$ such that $M(v_3, v_2, v_1) = \min\{d : ch(v_3, v_2, v_1, d) > k + (j-1)\}$.*

PROOF. Let $M(v_3, v_2, v_1) = c$. Thus $k < ch(v_3, v_2, v_1, c)$. There exists a j such that $k + j = ch(v_3, v_2, v_1, c)$. It holds $ch(v_3, v_2, v_1, c-1) = k + j - 1$ because c is minimal. The statement follows directly. $\square$

Lemma 35 *Let $(v_3, v_2, v_1, 0)$ be a borderline k-winning, but not a nice state and let $v_0 = M(v_3, v_2, v_1)$. Then (v_3, v_2, v_1, v_0) is a nice k-winning state.*

PROOF. There exists an integer v_0' such that (v_3, v_2, v_1, v_0') is a k-winning state and $ch(v_3, v_2, v_1, v_0') = k$ because of Lemma 26. $v_0 = v_0'$ because M is minimal. □

Proposition 36 *Let $\underline{v} = (v_3, v_2, v_1, 0)$ be a state with $k = ch(v_3, v_2, v_1, 0)$ and $m > 0$. Then the following holds*

(1) $\underline{v}$ is a borderline winning k-state $\iff M(v_3, v_2, v_1) = 0$.
(2) $\underline{v}$ is a borderline winning $k + m$-state

$$\iff M(v_3, v_2, v_1) = \min\{c : ch(v_3, v_2, v_1, c) > k + (m-1)\}.$$

PROOF.

(1) This statement is true because of the definition of M.
(2)
- Let $\underline{v}$ be a borderline winning $(k+m)$-state. First we will show that $M(v_3, v_2, v_1) \geq \min\{c : ch(v_3, v_2, v_1, c) > k + (m-1)\}$.
  If $M(v_3, v_2, v_1) < \min\{k : ch(v_3, v_2, v_1, k) > k + (m-1)\}$, then we know that $(v_3, v_2, v_1, M(v_3, v_2, v_1))$ is a winning $(k+m-1)$-state. It follows that $(v_3, v_2, v_1, 0)$ is also a winning $(k+m-1)$-state. This is a contradiction. It remains to show that $M(v_3, v_2, v_1) \leq \min\{c : ch(v_3, v_2, v_1, c) > k + (m-1)\}$. This follows by Lemma 35.
- Let $m = M(v_3, v_2, v_1) = \min\{c : ch(v_3, v_2, v_1, c) > k + (m-1)\}$. It follows that (v_3, v_2, v_1, m) is a borderline winning $(k+m)$-state. Thus $(v_3, v_2, v_1, 0)$ is a winning $(k+m)$-state. It is impossible to win it with less than $k+m$ questions because of Lemma 35 and M being minimal.

□

Proposition 37 *Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$. Person 2 needs k questions to win the game if $N \in M_0 (=$*

$$\{1, 2, 14, \ldots, 16, 22, \ldots, 28, 35, \ldots, 50, 57, \ldots, 88, 95, \ldots, 154, 158, \ldots, 264\}).$$

PROOF. Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$ according to Proposition 36. Person 2 needs k questions to win the game if $M(N, 0, 0) = 0$. By exhaustive computer search we get all numbers with this property less or equal 265. □

Proposition 38 *Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$. Person 2 needs $k + 2$ questions to win the game if $N \in M_2 = \{3, 5\}$.*

PROOF. Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$ according to Proposition 36. Person 2 needs $k + 2$ questions to win the game if $M(N, 0, 0) = \min\{c :$

$ch(v_3, v_2, v_1, c) > k+1\}$. By exhaustive computer search we get all numbers with this property less or equal 265. $\square$

Proposition 39 *Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$. Person 2 needs $k+1$ questions to win the game if $N \in M_1 = \{1, \ldots, 165\} \backslash (M_0 \cup M_2)$, where M_0 is defined in Proposition 37 and M_1 is defined in Proposition 39.*

PROOF. Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$ according to Proposition 36. Person 2 needs $k+2$ questions to win the game if $M(N, 0, 0) = \min\{c : ch(v_3, v_2, v_1, c) > k\}$. By exhaustive computer search we get all numbers with this property less or equal 265. $\square$

Let us combine all results. In section 3 we show how to get local optimal questions. The second local optimal question and the third local optimal question is not unequivocal. We know that the first three questions have to be optimal (Definition 14). The third question is obviously optimal. We have to show that the second question is also optimal. This holds either if the loss in the third question is equal 0 or if there is no local optimal question such that the loss in the third question is equal 0.

Theorem 40 *Let $\underline{s} = (N, 0, 0, 0)$ and $k = ch(\underline{s})$.*
Case $N < 266$:

(1) *Person 2 needs $k+2$ questions to win the game if $N \in M_2 = \{3, 5\}$.*
(2) *Person 2 needs k questions to win the game if $N \in M_0 = \{1, 2, 14, \ldots, 16, 22, \ldots, 28, 35, \ldots, 50, 57, \ldots, 88, 95, \ldots, 154, 158, \ldots, 264\}$.*
(3) *Person 2 needs $k+1$ questions to win the game if $N \notin M_0 \cup M_2$.*

Case $N \geq 266$:

(1) *Person 2 needs k questions to win the game if*

$$V_k(\underline{s}) + a_1(N)\binom{k-1}{3} + 2a_2(N, \underline{t})\binom{k-2}{2} + 4a_3(N, \underline{u})[k-3] \leq 2^k,$$

where $\underline{t}$ is the state after the second question and $\underline{s}$ is the state after the third question of the mini-strategy.
(2) *Person 2 needs $k+1$ questions to win the game, otherwise.*

PROOF. Let $N \geq 266$, $\underline{s}$ be the initial state, $\underline{t}$ be the state after the first question $[\underline{x}]$, $\underline{u}$ be the state after the second question $[\underline{y}]$ and $\underline{v}$ be the state after the third question $[\underline{z}]$. The character of the state after three questions of the mini-strategy is the character of the initial state minus three or two. If this character is the same as the one of the initial state minus 3, then we can win the game with $ch(\underline{s})$ questions, otherwise we need $ch(\underline{s})+1$ questions. Thus we have to calculate if the character of the initial state is equal to the character of the state after three questions plus 3. By Proposition 11, Table 1 and 2 we know how much we lose in each question. Let $\underline{v}$ be the worst state after the first three questions, $\underline{s}$ the initial state, $k = ch(\underline{s})$ and $a_1(N)$, $a_2(N, \underline{t})$ and $a_3(N, \underline{u})$ the loss in the first, second and third question. By Berlekamp's Conservation of Volume we get $V_{k-3}(\underline{v}) = \frac{V_k(\underline{s})+a_1(N)\binom{k-1}{3}}{8} + \frac{a_2(N,\underline{t})\binom{k-2}{2}}{4} + \frac{a_3(N,\underline{u})(k-3)}{2}$. Thus $ch(\underline{s}) = ch(\underline{v}) + 3$ if $V_{k-3}(\underline{v}) \leq 2^{k-3}$. We have to check $V_k(\underline{s}) + a_1(N)\binom{k-1}{3} + 2a_2(N, \underline{t})\binom{k-2}{2} + 4a_3(N, \underline{u})(k-3) \leq 2^k$. Due to the complementary questions we only have to consider the cases in which the resulting YES-state is greater than or equal to the resulting NO-state. It remains to show that the second question is optimal. The proof is not complicated but very long, because it has to be checked for each question. The complete proof can be found in [12]. □

In [12] a theorem can be found which gives an exact formula if Player 2 needs either k questions or $k+1$ questions, depending on the conditions, concerning k and p if $N = 8m + p$. Let us consider the values of the function M. Let $\underline{v}$ be a typical state with $ch(\underline{v}) < 15$. Then it can happen that $M(v_3, v_2, v_1) = c > ch(v_3, v_2, v_1, c)$. In this case we can find some typical states. There are some states (v_3, v_2, v_1, j) in which $ch(\underline{v}) < j < M(v_3, v_2, v_1)$, the states are typical but not nice. Some examples of the states are given below.

$ch = 6:$ $(0, 1, 5, j)$ $6 < j < 14$ (in total 3 such sequences)

$ch = 7:$ $(0, 3, 0, j)$ $7 < j < 42$ (in total 12 such sequences)

$ch = 8:$ $(0, 5, 0, j)$ $8 < j < 72$ (in total 39 such sequences)

$ch = 9:$ $(1, 6, 0, j)$ $9 < j < 107$ (in total 119 such sequences)

$ch = 10:$ $(3, 0, 12, j)$ $10 < j < 365$ (in total 479 such sequences)

$ch = 11:$ $(6, 6, 1, j)$ $11 < j < 243$ (in total 1132 such sequences)

$ch = 12:$ $(7, 17, 0, j)$ $12 < j < 661$ (in total 1638 such sequences)

$ch = 13:$ $(9, 45, 26, j)$ $13 < j < 287$ (in total 1801 such sequences)

$ch = 14:$ $(22, 52, 39, j)$ $14 < j < 128$ (in total 921 such sequences)

In total there are 6144 sequences of typical and not nice states.

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Table 1
The loss in the second question

k − 2 mod 6 state	0	1	2	3	4	5
$(e, e, 0, 0)$	0	0	0	0	0	0
$(e, o, 0, 0)$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	1
$(o, e, 0, 0)$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	1
$(o, o, 0, 0)$	$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	0

Table 2
The loss in the third question

$k-3 \bmod 12$ state	0	1	2	3	4	5	6	7	8	9	10	11
$(e,e,e,0)$	0	0	0	0	0	0	0	0	0	0	0	0
$(e,e,o,0)$	0	1	0	$\frac{1}{3}$	0	1	0	1	0	$\frac{1}{3}$	0	1
$(e,o,e,0)$	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
$(e,o,o,0)$	$\frac{1}{6}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0
$(o,e,e,0)$	0	0	0	$\frac{1}{3}$	0	0	0	1	0	0	0	1
$(o,e,o,0)$	0	1	0	0	0	1	0	0	0	$\frac{1}{3}$	0	0
$(o,o,e,0)$	$\frac{1}{6}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{6}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$(o,o,o,0)$	$\frac{1}{6}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{6}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1