Applications of Baranyai's Theorem in Information Theory

Ulrich Tamm

I. Baranyai's Theorem

In order to motivate Baranyai's Theorem, let us present an example. Think of the director of a round– robin tournament (for instance, a football pool or a chess tournament) who has to schedule the rounds of the tournament in such a way that

i) each team plays against each other team exactly once,

ii) in every round all the teams are occupied, i. e. they have an opponent.

Of course, the second requirement can only be fulfilled, if there is an even number of teams or players, which we assume from now on. The following table for a tournament with 8 players obviously does the job. Here (i,j) means that Player i is scheduled against Player j. The games of the respective rounds are listed in the different rows.

Round 1 : (1,8) (2,7) (3,6) (4,5)
Round 2 : (8,5) (6,4) (7,3) (1,2)
Round 3 : (2,8) (3,1) (4,7) (5,6)
Round 4 : (8,6) (7,5) (1,4) (2,3)
Round 5 : (3,8) (4,2) (5,1) (6,7)
Round 6 : (8,7) (1,6) (2,5) (3,4)
Round 7 : (4,8) (5,3) (6,2) (7,1)

The conditions i) and ii) can be formulated in a mathematical language as follows: All possible $\binom{2n}{2}$ pairs have to be distributed among $2n-1$ rows (the rounds) such that every row partitions the set $\{1,\ldots,n\}$ into pairs.

Since the pairs can be regarded as the edges of the complete graph on $2n$ vertices, we also call this a *factorization* of the complete graph. A factor of a graph $(\mathcal{V},\mathcal{E})$ corresponds to one round in the tournament table (i. e., each vertex occurs in exactly one edge of the factor) and a graph is factorizable if the set of edges $\mathcal{E}$ is the union of disjoint factors.

Let us briefly analyze the above tournament table, where $2n=8$. In every round $l=1,\ldots,2n-1$ there are $n-1$ pairs (i,j) with $i+j=l+1 \bmod 2n-1$ and one pair $(2n, \frac{l+1}{2} \bmod 2n)$.

Such a construction is possible for all even numbers. The underlying idea is quite simple. However, the organizers of chess tournaments in the 19th century found the tables by heuristical methods. If one does so, one may run into difficulties to group the pairs of the final rounds.

University of Bielefeld, Dept. of Mathematics, P.O. Box 10 01 31, 33501 Bielefeld, Germany. E-mail: tamm@mathematik.uni-bielefeld.de

The chess tournament in Nürnberg 1883 almost couldn't begin in time, since the author of the tournament table, Schallopp, left the schedule at home. With the above scheme in mind it would have been no difficulty to find the pairs of the next round.

By the way, the table we presented is much more regular than required under i) and ii). These conditions only mean that the unordered pairs are factorized into rounds. In the above tournament table the pairs are ordered in such a way that the players alternately take the first and the second position in a pair. This is important, since by this property a chess tournament can be organized in such a way that each player alternately plays with black and white pieces, and in a football pool the teams alternately play at home or at the opponents place.

Now we will turn to disciplines in which more than two players participate in a single game. Let us denote this number of participants by k For example, in the German card game "Skat" we have $k=3$ participants and in "Bridge" there are $k=4$ participants in a single party.

According to conditions i) and ii) from a table of a round–robin tournament with kn players we would require

i') each k–tuple of players occurs exactly once,

ii') in every round all players are occupied.

This means that such a tournament can be played in $\binom{kn-1}{k-1}$ rounds, where each round corresponds to a partition of the set $\{1,\ldots,kn\}$ into k–tuples.

We can again interprete this in a combinatorial setting. The appropriate concept is a *hypergraph* $\mathcal{H}=(\mathcal{V},\mathcal{E})$. This is just a basic set $\mathcal{V}$ with a collection $\mathcal{E}$ of subsets of $\mathcal{V}$. A hypergraph is said to be *k–uniform*, if all the subsets or edges in $\mathcal{E}$ have the same size k, e. g. if all the edges consist of two elements we just have a graph. Further we say that the k–uniform hypergraph is *complete* if $\mathcal{E}$ contains all possible subsets of size k. A tournament table fulfilling conditions i') and ii') then corresponds to a *factorization* (defined analogously as for the graph) of the complete uniform hypergraph $\mathcal{V},\mathcal{E}$, where $\mathcal{V}=\{1,\ldots,kn\}$ and $\mathcal{E}$ consists of all possible k–tuples.

Baranyai's Theorem now states that a factorization of the complete k–uniform hypergraph on the vertex set $\mathcal{V}=\{1,\ldots,kn\}$ exists for all $k=1,2,3,\ldots$. So it is always possible to find a tournament table fulfilling the conditions i') and ii').

Actually, Baranyai [1], proved a more general result, from which it can be derived, for instance, that the k–elementary subsets of an N–elementary set $\mathcal{V}$ can be distributed in the most regular way, even if k is not a divisor of N. We shall use this fact in the following sections.

THEOREM (Baranyai, 1973): Let $a_1, \dots, a_t$ be natural numbers such that $\sum_{j=1}^{t} a_j = \binom{N}{k}$. Then the k–elementary subsets of an N–elementary set can be partitioned into disjoint families $S_1, \dots, S_t$ with $|S_j| = a_j$ and each $i \in \{1, 2, \dots N\}$ is included in exactly $\lceil \frac{a_j \cdot k}{N} \rceil$ or $\lfloor \frac{a_j \cdot k}{N} \rfloor$ elements of S_j.

With the parameters $N = k \cdot n$, $t = \binom{kn-1}{k-1}$ and $a_j = N$ for all j it is clear that a factorization of the complete k–uniform hypergraph is always possible.

We are now going to present three applications of Baranyai's Theorem in Information Theory.

II. Interactive Communication

Two persons, $P_\mathcal{V}$ and$P_\mathcal{E}$ are communicating over a noiseless, binary channel. Person $P_\mathcal{V}$ knows a vertex $v \in \mathcal{V}$ and Person $P_\mathcal{E}$ knows an edge $E \in \mathcal{E}$ of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. We denote by $C_l(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$ the number of bits that have to be exchanged such that finally $P_\mathcal{V}$ knows the edge E and $P_\mathcal{E}$ knows the vertex v. The index l denotes the number of messages allowed during the transmission. Interactive communication was introduced by Orlitsky [2] in a slightly different model where only $P_\mathcal{E}$ has to be informed about the vertex v. He studied the possible gaps between C_l and C_{l-1}. Ahlswede, Cai, and Zhang [3] introduced the complexity in case that both persons have to be informed and determined $C_l(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E})$ for some special hypergraphs. Especially for the complete k–uniform hypergraph they could show that

$$C_l(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E}) = \lceil \log k \rceil + \lceil \log \binom{|\mathcal{V}|-1}{k-1} + \epsilon \rceil, \epsilon \in \{0,1\}$$

The decisive parameter in the bounds on interactive communication often is a colouring number of the underlying hypergraph. Here this turns out to be $ind(\mathcal{V}, \mathcal{E})$, which is the number of different colours needed to colour the edges in such a way that no two overlapping edges (which share a common vertex) have the same colour. With Baranyai's Theorem it can be seen that $ind(\mathcal{V}, \mathcal{E})$ differs by at most one bit from $\binom{|\mathcal{V}|-1}{k-1}$ (indeed, in the case "k divides $|\mathcal{V}|$" which we only considered in the previous section $ind(\mathcal{V}, \mathcal{E}) = \binom{|\mathcal{V}|-1}{k-1}$). The reason is that the sets in one factor cannot overlap by definition and hence all of them can be assigned the same colour.

III. WUM–Codes

We are given a storage medium on which information is encoded binary. For technical reasons, updating of information is only possible by changing either 0's to 1's or 1's to 0's. To be more exactly, in odd rounds of updating it is only possible to change a 0 to a 1 and in even rounds of updating it is only possible to change 1's to 0's. The question is: How many messages can be stored using n bits? Denoting by $M(n)$ this number of messages, we obtain the rate as usual as $R = \frac{1}{n} \log M(n)$. It is easily seen that a rate $R = \frac{1}{2}$ is achievable for even n. Namely, in this case we could divide the n bits of storage space into two blocks, B_0 and B_1 say, of size $\frac{n}{2}$ each, where B_0 is the block on which information is stored in odd rounds and B_1 is the block on which information is stored in even rounds. So, in odd rounds all the zeroes in B_1 are switched to 1 in order to prepare B_1 for storing information in the next round. B_0 has been prepared in a similar way the round before and hence consists of $\frac{n}{2}$ 0's, which may all be changed to 1 or left as a 0 such that $2^{n/2}$ different messages may be stored.

Willems and Vinck [4] demonstrated that it is possible to achieve better bounds by using the following code allowing to encode 6 messages with 5 bits (hence $R = \frac{1}{5} \log 6 = 0.517$. In odd rounds only words of weights $3, 4$ or 5 are written and in even rounds only words of small weights $0, 1$ or 2 are used. We only present the codebook for the odd rounds, the one for the even rounds is obtained by taking complements of the vectors in the scheme below.

m_1	m_2	m_3	m_4	m_5	m_6
01111	10111	11011	11101	11110	11111
10011	01101	01110	01011	00111	
11100	11010	10101	10110	11001	

The idea is to use more than one codeword for the same message m_i. Whatever will be written on the device (this must be a word with at most two 1's), it is always possible to use at least one of the codewords listed below m_i if m_i is the message to be stored.

If we take a closer look at this code it turns out that the positions of the zeroes just yield a factorization of the complete graph on six vertices. Namely, the codewords of the first 5 messages correspond to the following tournament table (with Player 6 taking the role of a joker):

m_1: (1,(6)) (2,3) (4,5), m_2: (2,(6)) (1,4) (3,5),
m_3: (3,(6)) (1,5) (2,4), m_4: (4,(6)) (1,3) (2,5),
m_5: (5,(6)) (1,2) (3,4).

A similar construction based on Baranyai's Theorem (for k=3) was used by Simonyi [5] in order to obtain a WUM-code of rate $\frac{1}{11} \log 58 = 0.5325$. The best known WUM-code is due to Koschnick [6], it achieves rate $\frac{1}{17} \log 767 = 0.5637$. There is still a considerable gap to the upper bound $\log \frac{1+\sqrt{5}}{2} = 0.6942$ due to Borden [7].

IV. Superimposed Codes

We call a set system $\mathcal{A}$ *r–cover–free*, if no set is contained in the union of r other sets, i. e., for all

$A_0, A_1, \ldots, A_r \in \mathcal{A}$ it is

$$A_0 \not\subseteq A_1 \cup A_2 \cup \ldots \cup A_r$$

Now each set $A \subseteq \{1, \ldots n\}$ corresponds to a $\{0,1\}$–vector $(x_1, \ldots x_n)$ by the bijection $x_i = 1$ exactly if $i \in A$. The $\{0,1\}$–vectors corresponding to the sets of an r–cover–free family are said to form a *superimposed* (or *disjunctive*) code.

Superimposed codes had been introduced by Kautz and Singleton [8], who also gave some applications, e. g., in data communication, memories, and retrieval of files. The parameters of a superimposed code are

n	the length of the single codewords,
r	the covering number,
$T(r,n)$	the maximum cardinality of an n, r–code,

The best known bounds on the size of such codes are due to Dyachkov and Rykov [9], who demonstrated that there are constants c_1 and c_2 such that

$$c_1 \frac{1}{r^2} \leq \frac{1}{n} \log T(r,n) \leq c_2 \frac{\log r}{r^2}$$

Their proof of the upper bound is rather involved and recently Ruszinkó [10] gave a new derivation of the upper bound using Baranyai's Theorem.

V. Constructions

Observe that Baranyai's Theorem is a statement about the existence of a certain combinatorial configuration. Indeed, Baranyai's proof, which makes use of flows in networks, does not give an idea how a factorization of the complete k–uniform hypergraph may look like. For the special case $k = 3$, Peltesohn [11] already back in 1936 could prove the existence of such a factorization. Her proof is based on an induction and one needs the factorizations for all smaller 2– and 3– uniform hypergraphs, which means that the complexity of finding a factorization for a special hypergraph is very high. Beth [12] gave a procedure to quickly find a factorization of the complete 3– uniform hypergraphs on $6n$ vertices in the case that $q := 6n - 1$ is a prime number.

As sets of a basic factor he took the orbits (consisting of three elements each) of the mapping $\pi(x) := -\frac{1+x}{x}$ acting on $\mathrm{PG}(1,q) = \{1, \ldots, q\} \cup \{\infty\}$, where addition and multiplication is modulo q. All the other factors are obtained by manipulation of this basic factor. To be more exactly, let $\mathcal{A}$ be a partition of $\mathrm{PG}(1,q)$ into the orbits of π, let ω be a primitive element of $\mathrm{GF}(q)$ and let $\mathcal{B} := \{\omega^j : j = 1, \ldots, \frac{q-1}{2}\}$, then

$$\{x \cdot \mathcal{A} + g : x \in \mathcal{B}, g \in \mathrm{GF}(q)\}$$

is a factorization of the complete 3–uniform hypergraph.

For $k \geq 4$ no general constructions seem to be known.

References

[1] Z. Baranyai, "On the factorization of the complete uniform hypergraph", *Colloquia Mathematica Societatis Janos Bolyai*, Vol.10, Infinite and Finite Sets, 91–108, 1973.

[2] A. Orlitski, "Worst case interactive communication I: two messages are almost optimal", *IEEE Trans. Inform. Theory*, Vol. 36, 1111–1127, 1990.

[3] R. Ahlswede, N. Cai, and Z. Zhang, "On interactive communication", to appear in *IEEE Trans. Inform. Theory*, Preprint 93-066, SFB 343, University of Bielefeld.

[4] F. M. J. Willems and A. J. Vinck, "Repeated recording for an optical disc", Proc. 7th Symp. Information Theory in the Benelux, May 1986, Delft University Press, 49–53, 1986.

[5] G. Simonyi, "On write–unidirectional memory codes", *IEEE Trans. Inform. Theory*, Vol. 35, 663–669, 1989.

[6] K.–U. Koschnick, "Good code design with combinatorial approximation algorithms", Preprint 91–017, SFB 343, University of Bielefeld, 1991.

[7] J. M. Borden, "Coding for write–unidirectional memories", 1986.

[8] W. H. Kautz and R. C. Singleton' "Nonrandom binary superimposed codes", *IEEE Trans Inform. Theory*, Vol. 10., No. 4, 363–377, 1964.

[9] A. G. Dyachkov and V. V. Rykov, "Bounds on the length of disjunctive codes", *Problemy Peredachi Informatsii*, Vol. 18, No. 3, 7–13, 1982.

[10] M. Ruszinkó, "On the upper bound of the size of the r-cover free families", *J. Combin. Theory Ser. A*, Vol. 66, No. 2, 302–310, 1994.

[11] R. Peltesohn, "Das Turnierproblem für Spiele zu je dreien", Doctoral Thesis, Berlin, 1936.

[12] T. Beth, "On resolutions of Steiner systems", Mitteilungen aus dem Mathematischen Seminar Giessen, Giessen 1979.