Übungen zur Informationstheorie und Komplexität

Blatt 3 – SS '98

1. (a) *(Bayes's Rule)* Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B \in \mathcal{F}$. For a countable set I let $A_i \in \mathcal{F}$ $(\forall\, i \in I)$ with $\dot{\bigcup}_{i\in I} A_i = \Omega$. Show:

$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{i\in I} P(A_i)P(B|A_i)} \qquad \forall\, j \in I.$$

   (b) *(Laplace's Law of Succession)* Given urns $U_0, \ldots, U_N$ with urn U_k containing k red and $N-k$ white balls. Choose an urn at random and draw n of its balls replacing each ball immediately. Suppose that all n balls turn out to be red! Drawing now another ball from the same urn we ask for its probability to be red. Show:

$$P(\text{ next ball red} \mid \text{all } n \text{ balls red}) \underset{N\to\infty}{\longrightarrow} \frac{n+1}{n+2}\,.$$

   Hints:

   - Calculate $P(\,U_k \text{ has been chosen} \mid \text{all } n \text{ balls red})$.
   - It is often possible to replace '$\lim_{N\to\infty} \sum_{k=0}^{N} \cdots$' with an integral.

2. *(Kraft's Inequality)* Let $\mathcal{X}$ a finite set and $L\colon \mathcal{X} \to \mathbb{N}_1$ a map. Show that there is a prefix code $c\colon \mathcal{X} \to \{0,1\}^*$ (no codeword is the beginning of another) with the lenghts of the codewords given by L if and only if

$$\sum_{x\in\mathcal{X}} 2^{-L(x)} \leq 1\,.$$

   Hint: What's the meaning of the prefix property if codewords are represented as vertices (nodes) of a binary tree?

3. (a) *(Binary Entropy)* Show that the function $h\colon [0,1] \to \mathbb{R}$ with

$$h(p) \triangleq -p\,\log_2 p - (1-p)\log_2(1-p)$$

   is concave and takes its maximum value at $p = \frac{1}{2}$.

   (b) *(Entropy)* For all probability distributions P on a finite set $\mathcal{X}$ let

$$H(P) \triangleq -\sum_{x\in\mathcal{X}} P(x)\log_2 P(x)\,.$$

   Show that H takes its maximum value at the uniform distribution on $\mathcal{X}$.

Übung: Di 12^{30}-14, V2-216.