Übungen zur Informationstheorie und Komplexität

Blatt 5

SS '98

1. Modify the proofs (and statements) of theorems 2.3 and 6.1 (on the back) slightly to correct them.

2. A *clique* of a graph is a complete subgraph of that graph. The *Ramsey number* $r(k,k)$ is the least integer such that for each graph G of size $r(k,k)$, either G or G's complement contains a clique of size k. Give a rigorous (!) proof for theorem 6.3 of Li/Vitányi's book, using the incompressibility method:

$$r(k,k) \geq k2^{\frac{k}{2}}(\frac{1}{e\sqrt{2}} - o(1)) \ .$$

   ($o(1)$ denotes some function φ with $\lim_{k\to\infty}\varphi(k) = 0$.)

3. Let $w(n)$ be the largest integer so that for each tournament T on $[n] = \{1,\ldots,n\}$ there exist disjoint sets A and B, each of cardinality $w(n)$, in $[n]$ such that $A \times B \subseteq T$. Prove $w(n) \leq 2\lceil\log n\rceil$ using the incompressibility method.

   (Hint: Add $2w(n)\lceil\log n\rceil$ bits to describe nodes, and save $w(n)^2$ bits on edges.)

Übung: Di 12^{30}-14, V2-216.

Some quotations from Li/Vitányi's book:

Theorem 2.3 Let $c \in \mathbb{N}_0$. For each fixed y, every finite set A of cardinality m has at least $m(1-2^{-c})+1$ elements x with $C(x|y) \geq \log m - c$.

Proof. The number of programs of lenght less than $\log m - c$ is

$$\sum_{i=0}^{\log m-c-1} 2^i = 2^{\log m - c} - 1.$$

Hence, there are at least $m - m2^{-c} + 1$ elements in A which have no program of lenght less than $\log m - c$.

(Transitive Tournament) A *tournament* is defined to be a complete directed graph. That is, for each pair of nodes i and j, exactly one of the edges (i,j) or (j,i) is in T. The nodes of a tournament can be viewed as players in a game tournament. If (i,j) is in T, we say player j dominates player i. We call T *transitive* if $(i,j), (j,k)$ in T implies (i,k) in T.

Let Γ_n be the set of all tournaments on $[n] = \{1,\ldots,n\}$. Given a tournament $T \in \Gamma_n$, fix a standard encoding $E\colon \Gamma_n \to \{0,1\}^{\binom{n}{2}}$, one bit for each edge. The bit for edge (i,j) is set to 1 if $i<j$ and 0 otherwise. There is a 1-1 correspondence between the members of Γ_n and the binary strings of lenght $\binom{n}{2}$.

Let $v(n)$ be the largest integer such that every tournament on $[n]$ contains a transitive subtournament on $v(n)$ nodes.

Theorem 6.1 $v(n) \leq 1 + \lfloor 2\log n \rfloor$.

Proof. Fix $T \in \Gamma_n$ such that

$$C(E(T)|n,p) \geq \binom{n}{2} \ ,$$

where p is a fixed program that on input n and $E'(T)$ (below) outputs $E(T)$. Let S be the transitive subtournament of T on $v(n)$ nodes. We try to compress $E(T)$, to an encoding $E'(T)$, as follows.

1. Prefix the list of nodes in S in lexicographical order of dominance to $E(T)$, each node using $\lceil \log n \rceil$ bits, adding $v(n)\lceil \log n \rceil$ bits.
2. Delete all redundant bits from the $E(T)$ part, representing the edges between nodes in S, saving $v(n)(v(n)-1)/2$ bits.

Then,

$$l(E'(T)) = l(E(T)) - \frac{v(n)}{2}(v(n) - 1 - 2\lceil \log n \rceil) \ .$$

Given n, the program p reconstructs $E(T)$ from $E'(T)$. Therefore,

$$C(E(T)|n,p) \leq l(E'(T)) \ .$$

These equations can only be satisfied with $v(n) \leq 1 + \lfloor 2\log n \rfloor$.