**Übungen zur Informationstheorie und Komplexität**

1. (a) (Log–Sum Inequality) For arbitrary non–negativ numbers $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ with $\sum\limits_{i=1}^{n} a_i \neq 0$ and $\sum\limits_{i=1}^{n} b_1 \neq 0$ it is

$$\sum_{i=1}^{n} a_i \cdot log_2 \frac{a_i}{b_i} \geq \left( \sum_{i=1}^{n} a_i \right) \cdot log_2 \frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i}$$

with equality exactly if $a_i \cdot \sum\limits_{j=1}^{n} b_j = b_i \cdot \sum\limits_{j=1}^{n} a_j$ for $1 \leq i \leq n$.

   (b) For PDs $P$ and $Q$ on the measurable space $(\mathcal{X}, \mathcal{A})$ and a partition $\mathcal{P} = \{A_1, \ldots, A_m\}$ of $\mathcal{X}$ let $f(\mathcal{P}) \triangleq \sum_{i=1}^{m} P(A_i) \log \frac{P(A_i)}{Q(A_i)}$. Show that if the partition $\mathcal{P}'$ is a refinement of $\mathcal{P}$ then $f(\mathcal{P}') \geq f(\mathcal{P})$.

   (c) Show that Dobrushin's definition of divergence is equivalent to the definition used in the lecture, i. e.,

$$\sup_{\mathcal{P}} f(\mathcal{P}) = \begin{cases} \int p_Q \log p_Q dQ & P << Q \\ \infty & \text{otherwise} \end{cases}$$

where $p_Q$ is the Radon-Nikodym derivative of $P$ with respect to $Q$.

2. (a) Let $l_2$ be the set of all sequences $a = (a_1, a_2, \ldots)$ of real numbers with $\sum_{i=1}^{\infty} a_i^2 < \infty$. For $a, b \in l_2$ let $\langle a, b \rangle \triangleq \sum_{i=1}^{\infty} a_i b_i$. Show that $l_2$ with componentwise addition and scalar multiplication is together with the inner product $\langle \cdot, \cdot \rangle$ a Hilbert space. Especially show that $\langle a, b \rangle < \infty$ for all $a, b \in l_2$.

   (b) Prove the Parallelogram identity for any Hilbert space, i. e.,

$$2(||x||^2 + ||y||^2) = ||x + y||^2 + ||x - y||^2.$$

   (c) Show that in a Hilbert space the projection of a point to a convex set is unique.

3. (a) Show that total variation is a norm.

   (b) Show that if $P_n \to P$ (in total variation) then $\lim\limits_{n \to \infty} D(P_n||Q) \geq D(P||Q)$.