The rate distortion function $R(D)$ is defined as

$$R(D) := \min\{R : \exists(f,g) \text{ with rate}(f) \leq R \text{ and fidelity } D \geq 0\},$$

where $f : \mathcal{X} \to \mathbb{N}, g : \mathbb{N} \to \hat{\mathcal{X}}$ are the encoding/decoding functions and the estimate $\hat{X}^n = g(f(X^n))$ has fidelity D , if $\sum_{t=1}^{n} Ed(x_t, \hat{x}_t) \leq nD$.

Rate Distortion Theorem: For every $D \geq 0$ and all distortion measures $d(\mathcal{X} \times \hat{\mathcal{X}}) \to \mathbb{R}^+$

$$R(D) = A(D) := \min_{W:\mathcal{X}\to\hat{\mathcal{X}}, Ed(X,\hat{X})\leq D} I(X, \hat{X}),$$

where X is the generic random variable of the source and $\hat{X}$ is the output variable of the channel W.

1 Let R(D) be the rate distortion function of a DMS with a finite distortion measure.

Show: For any $D \geq 0$, any $\delta > 0$, and any sufficient large blocklenght L , there exists a source code with $M < exp\{L(R(D)+\delta)\}$ code words such that

$$\bar{D}_l := \frac{1}{L} \sum_x P_L(x) \sum_{t=1}^{L} d(x_t, \hat{x}_t) \leq D + \delta,$$

i.e. the average distortion per letter is bounded by D and δ.

Let $W : \mathcal{X} \to \mathcal{Y}$ be a DMC with transmission matrix $W = (w(y|x))_{x\in\mathcal{X}, y\in\mathcal{Y}}$. A sequence $y^n \in \mathcal{Y}^n$ is called conditionally typical (or (n, δ) -generated), if for all $x \in \mathcal{X}, y \in \mathcal{Y}$

$$| < (x^n, y^n)|(x,y) > - < x^n|x > \cdot w(y,x)| \leq n\delta.$$

The set of all conditionally typical sequences is denoted as $\mathcal{T}^n_{Y|X,\delta}(x^n)$.

2 Consider the hypergraph

$$\mathcal{H} = (\mathcal{V}, \mathcal{E}) = (\mathcal{T}^n_{X,\delta}, (\mathcal{T}^n_{X|\hat{X},\hat{\delta}}(u))_{u\in\mathcal{T}^n_{\hat{X}}})$$

Show:

$$\deg(v) \geq exp\{nH(\hat{X}|X) - ng(\hat{\delta}, \delta)\} \text{ for all } v \in \mathcal{V},$$

where g is a function such that $\lim_{n\to\infty} g(\hat{\delta}, \delta) \to 0$.

Let $(\mathcal{V}, \mathcal{E})$ be an hypergraph and $(\mathcal{V}^n, \mathcal{E}^n)$ be the associated product hypergraph. Define $c(n)$ as the minumum number of vertices needed to cover the product hypergraph by a covering $\mathcal{C}$, i.e.

$$c(n) := min\{|\mathcal{C}| : \mathcal{C} \text{ covers } \mathcal{V}^n\}$$

3 Show: If $f(n)$ is subadditiv, i.e. for all $n, m \in \mathbb{N} : f(n+m) \leq f(n) + f(m)$, and bounded, then the limit

$$C := \lim_{n\to\infty} \frac{f(n)}{n}$$

exists. Remember: $c(n)$ is submultiplicativ, i.e. $\log c(n)$ is subadditiv!

4 Let $K := \max_{P\in\mathcal{P}(\mathcal{E})} \min_{v\in\mathcal{V}} \sum_{E\in\mathcal{E}} 1_E(v)P(E)$.

Show: $c(1) \geq \exp(K^{-1})$.