Overview seminar on Hodge locus

The main purpose of the seminar would be to understand recent results on the Hodge locus as proven in the paper [2], and some of the ingredients used in the proof.

The study of the Hodge locus starts with the classical Noether theorem which states that the Picard group of a very general degree $d \geq 4$ surface in $\mathbb{P}^3$ is of rank 1. If U_d parametrizes smooth degree d surfaces in $\mathbb{P}^3$ (open in the projectivization of the space of degree d homogeneous polynomial in 4 variables), the Noether-Lefschetz locus $\mathrm{NL}_d \subset U_d$ for degree d surfaces is the subset of those points in U_d such that the corresponding surface has Picard rank greater than 1. One can show that this subspace is a union of countably many analytic subvarieties of U_d.

More generally, given a smooth family of projective varieties $f : \mathcal{X} \to S$ over a smooth irreducible quasi-projective variety, one defines the Hodge locus for the family to be the subset of $s \in S$ such that the fibre $\mathcal{X} \to S$ acquires "extra" Hodge classes with respect to the general fibre. Again, the Hodge locus is a union of countably many analytic subvarieties of S. A famous theorem of Cattani, Deligne and Kaplan [7] states that each irreducible component of the Hodge locus is in fact an algebraic subvariety.

The understanding of the Hodge locus has greatly improved in the past years. The paper [2] presents and partially proves the so-called Zilber-Pink conjecture, which would predict rather precisely the general behaviour of Hodge loci. To explain it, one distinguishes between typical and atypical components. Roughly speaking, one can predict the dimension of a component Z of the Hodge locus; when the dimension of Z equals the expected dimension, then Z is said typical, else, Z is said atypical (to make this precise one needs to move from S to the period domain associated to the family $f : \mathcal{X} \to S$). We split the Hodge locus into the union of the atypical and typical Hodge locus, union of atypical and typical components respectively. The Zilber-Pink conjecture predicts the following dichotomy:

Conjecture. *The atypical Hodge locus is algebraic (i.e., union of finitely many components). The typical Hodge locus is either empty or dense (for the analytic topology).*

Among the main results of [2], it is (essentially) shown that the positive dimensional atypical Hodge locus is algebraic; however, it is hard to control atypical points. Moreover, they show that the typical Hodge locus is either empty or analytically dense. In fact, it is proven that it is empty for "most" families, specifically, when the level of the associated variation of Hodge structures is at least 3. This level is a measure of the complexity of the variation of Hodge structures; the case of level 1 is the most well-studied, arising from families of K3 surfaces or abelian varieties, and the corresponding period domains give Shimura varieties. As application, some results on the Hodge locus for the universal family of degree d hypersurfaces in $\mathbb{P}^n$ are given (a more detailed exposition of this special case can be found in [4]).

Arithmetic aspects. What discussed so far involves Hodge theory and it is done purely over the complex numbers. However, there are several related questions of arithmetic flavour which we may study. Just to give an example, it is not at all clear from the Noether theorem whether there exists a surface of degree d in $\mathbb{P}^3$ defined over $\bar{\mathbb{Q}}$ with Picard rank 1; in fact, a priori, the complement of the Noether-Lefschetz locus may contain no $\bar{\mathbb{Q}}$-point at all. One might also ask about the fields of definition of the components of the Hodge locus. See [13, Section 1.3] and references therein.

Moreover, when the family $f\colon \mathcal{X} \to S$ is defined over some field K finitely generated over $\mathbb{Q}$, instead of Hodge theory one can consider ℓ-adic étale cohomology and the associated $\mathrm{Gal}(\bar{K}/K)$-representations, and define the "ℓ-Tate locus" of a family $f\colon \mathcal{X} \to S$ in analogy with the the Hodge locus. The algebraicity of the components of this ℓ-Tate locus has been studied by Kreutz [12]. The results of [2] have some other strong arithmetic consequence as well, see [3].

Outline. We could maybe start discussing the classical Noether theorem; the article [6] gives a historical overview as well as explaining successive developments. We would then learn about variations of Hodge structures and period domains. References can be Voisin's book [14], or the book [9], and maybe Griffiths' papers [10, 11]. The language of Mumford-Tate groups should be introduced, and we could study Deligne's theorem of the fixed part [8] (relating the monodromy group with the Mumford-Tate group), a crucial result in Hodge theory. A nice introductory reference on the Hodge locus is the article of Voisin [13], touching several aspects. It is interesting (and not hard) to see how the Hodge conjecture would imply that the components of the Hodge locus are algebraic; from this perspective one might see the theorem of [7] as one of the strongest available piece of evidence supporting the validity of the Hodge conjecture. The proof of the result of Cattani, Deligne, Kaplan seems quite difficult, using Schmid's nilpotent orbit theorem, and we should probably leave it aside.

Perhaps the most important ingredient used in [2] is a Ax-Schanuel theorem, related to the topic of functional transcendence, as proven in [1]. Schanuel conjecture regards the transcendence degree of certain complex numbers over $\mathbb{Q}$; these Ax-Schanuel theorems prove so-called geometric versions of Schanuel conjecture, and several versions appeared in the literature. Most of the recent versions use o-minimality, a key source of developments in the study of Hodge loci, which led to a new proof of the theorem of Cattani-Deligne-Kaplan, to the proof of the André-Oort conjecture for Shimura varieties, and to other important algebraicity results for period maps. We could decide to learn some of this, but a proper treatment would probably require a large part of the seminar. Otherwise, in [1] a proof of the needed Ax-Schanuel theorem is given via the recent [5], which avoids o-minimality and uses instead foliations. It should be very interesting in any case to understand how these results on functional transcendence are related with the study of the Hodge locus. The article [1] also gives an application to Grothendieck period conjecture over function fields.

References

[1] B. Bakker & J. Tsimerman *Functional transcendence of Periods and the Geometric André–Grothendieck Period Conjecture.* arXiv:2208.05182 (2022)

[2] G. Baldi, B. Klingler & E. Ullmo *On the distribution of the Hodge locus.* arXiv:2107.08838 (2021), to appear in Inventiones Math.

[3] G. Baldi, B. Klingler & E. Ullmo *On the Geometric Zilber-Pink Theorem and the Lawrence-Venkatesh method.* Expo. Math., vol. 41, no.3, 718-722 (2023)

[4] G. Baldi, B. Klingler & E. Ullmo *Non-density of the exceptional components of the Noether-Lefschetz locus.* arXiv:2312.11246 (2023)

[5] D. Blásquez-Sans, G. Casales, J. Freitag & J. Nagloo *A differential approach to the Ax-Schanuel, I.* arXiv:2102.03384 (2021)

[6] J. Brevik & S. Nollet *Developments in Noether-Lefschetz theory.* in *Hodge theory, complex geometry, and representation theory, Fort Worth, TX, USA, 2012*, pp. 21–50. Providence, American Mathematical Society (2014)

[7] E. Cattani, P. Deligne & A. Kaplan *On the locus of Hodge classes.* JAMS, vol. 8, no. 2, pp. 483–506 (1995)

[8] P. Deligne *Théorie de Hodge, II.* Publ. Math. de l'IHES, vol. 40, pp. 5–57 (1971)

[9] M. Green, P. A. Griffiths & M. Kerr *Mumford-Tate Groups and Domains: Their Geometry and Arithmetic.* Princeton University Press, 2012

[10] P. A. Griffiths *On the periods of certain rational integrals, I, II.* Annals of Math., vol. 2, pp. 460–495, 496–541 (1969)

[11] P. A. Griffiths *Periods of integrals on algebraic manifolds, III.* Publ. Math. de l'IHES, vol. 38, pp. 125–180 (1970)

[12] T. Kreutz *ℓ-Galois special subvarieties and the Mumford-Tate conjecture.* arXiv:2111.01126 (2021)

[13] C. Voisin *Hodge loci.* in *Handbook of Moduli* (Eds G. Farkas and I. Morrison) Advanced Lectures in Mathematics 25, Volume III, International Press, pp. 507–546.

[14] C. Voisin *Hodge Theory and Complex Algebraic Geometry, I, II.* Cambridge University Press, 2002