Condensed mathematics **Winter 2022/23**

We will study condensed mathematics after Clausen and Scholze, following mainly the lecture notes [Sch1].[1] See Scholze's MSRI talk for an introduction. Please contact the organizer for additional information on the talks and additional literature.

List of talks:

1) Condensed sets, [Sch1, Lecture I].
Recall also profinite sets (e.g. [Stacks, Tag 08ZW]) and the Stone-Cech compactification. Leave out Theorem 1.10, which appears in the next talk.

2) Condensed abelian groups, [Sch1, Lecture II].
Condensed ablelian groups form an abelian category with especially good properties. Explain also the cardinality-free definition of condensed sets and the relation with topological spaces (Appendix to Lecture II). See [Sch2, Prop 1.2] for a summary of the last point. See [Sch2, Prop 2.1] for a concrete description of the projective condensed abelian groups. See [Stacks, Tag 08YH] for more details on extremally disconnected spaces.

3) Cohomology, [Sch1, Lecture III].
Brief summary: The condensed cohomology of a compact Hausdorff space coincides with sheaf cohomology for coefficients $\mathbb{Z}$ and vanishes in higher degrees for coefficients $\mathbb{R}$. See for example [Stacks, Tag 01GU] for Cech-Cohomology and hypercoverings.

4) Locally compact abelian groups, [Sch1, Lecture IV].
Locally compact abelian groups can be viewed as condensed abelian groups, and all $R\,\underline{\mathrm{Hom}}$'s between these can be computed effectively.

5) Solid abelian groups I, [Sch1, Lecture V].
Solid abelian groups are condensed abelian groups which are complete in a

[1] The choice of topics is canonical up to a certain point. The lectures of [Sch1] can be divided into three parts:

I–IV: Condensed generalities

V–VIII: Solid modules and applications (local duality)

IX-XI: Globalization and coherent duality

Most seminars on condensed mathematics treat I–VIII in some form, but after that other continuations can be chosen, for example: the definition of liquid vector spaces and statement of the main result [Sch2, Theorem 6.5], whose proof is difficult but computer-verified ([LTE]); non-archimedean condensed functional analysis and folid locally analytic representations of p-adic Lie groups following [JC]; or aspects of the condensed approach to complex geometry following [CS]. We can discuss this at the first meeting.

certain sense. Here we study basic properties of this notion. The main result (Theorem 5.8) will be proved in the next talk.

6) Solid abelian groups II, [Sch1, Lecture I].
Proof of Theorem 5.8 on the category of solid abelian groups; maybe some details have to be omitted. Completed tensor product of solid abelian groups, calculation of some completed tensor products (Example 6.4).

7) Analytic rings, [Sch1, Lecture VII].
An analytic ring $\mathcal{A}$ consists of an underlying condensed ring $\underline{\mathcal{A}}$ and a notion of complete free modules with such properties that the existence of a category of solid $\mathcal{A}$-modules with properties similar to those of solid abelian groups is a consequence; this can be seen as an axiomatic extension of Theorem 5.8. For every discrete ring A there is an analytic ring $(A, \mathbb{Z})_{\blacksquare}$.

8) Solid A-modules, [Sch1, Lecture VIII].
Theorem: For every finitely generated $\mathbb{Z}$-algebra A there is a natural analytic ring $A_{\blacksquare}$. This allows to define a functor cohomology with compact support $f_! : D(A_{\blacksquare}) \to D((A, \mathbb{Z})_{\blacksquare}) \to D(\mathbb{Z}_{\blacksquare})$ where the first arrow is an extension by zero. The right adjoint functor $f^!$ will give the dualising complex of coherent duality in the affine case.

9) Globalization, [Sch1, Lecture IX]. The globalization of the preceding constructions uses discrete adic spaces, which are introduced here. For more on general adic spaces one can consult [Wed], but the lecture is mostly self-contained. The correct formulation of the relevant glueing of solid modules (Theorem 9.8) requires derived categories (explain why), necessarily in the ∞-version as glueing is involved.

10?) [optional] Stable ∞-categories.
The aim of this talk would be to introduce the necessary background to understand the precise meaning of Theorem 9.8 and its proof in the next lecture (in particular Proposition 10.5).

I1) Lecture X: Globalization II, [Sch1, Lecture X].
Proof of Theorem 9.8. Most part of the proof are concrete arguments with adic spaces (no ∞-categories).

12) Coherent duality, [Sch1, Lecture XI]. The aim is to construct the 6-functor formalism in coherent duality, based on solid modules. Some proofs are missing; here one can consult [Ma, Section 2.9], in particular Proposition 2.9.31. Note that [Ma] works throughout with animated rings (the ∞-category version of rings).

References

[CS] Dustin Clausen, Peter Scholze, Lectures on Condensed Mathematics and Complex Geometry, https://people.mpim-bonn.mpg.de/scholze/Complex.pdf, 2022

[JC] Joaquín Rodrigues Jacinto and Juan Esteban Rodríguez Camargo, Solid locally analytic representations of p-adic lie groups, Representation theory, Volume 26, Pages 962–1024 (2022). https://arxiv.org/abs/2110.11916

[Ma] Lucas Mann, A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry, https://arxiv.org/pdf/2206.02022.pdf, 2022

[Sch1] Peter Scholze, Lectures on Condensed Mathematics, results joint with Dustin Clausen, people.mpim-bonn.mpg.de/scholze/Condensed.pdf, 2019

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[Stacks] The Stacks project, https://stacks.math.columbia.edu

[Wed] Torsten Wedhorn, Adic Spaces, https://arxiv.org/abs/1910.05934